CARPATHIAN J. MATH.
24 (2008), No. 2, 10 - 19

General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces

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ABSTRACT.
Existence and existence and uniqueness results for fixed points of single-valued Ćirić almost contractions as well as convergence theorems for Picard iteration to these fixed points are proved. The Ćirić type almost contraction condition appear to be one of the most general metrical condition for which the set of fixed points is not a singleton but the fixed points can be approximated by means of Picard iteration. Our results unify, generalize and extend most of the fundamental metrical fixed point theorems in literature (Banach, Kannan, Bianchini, Reich, Rus, Chatterjea, Rhoades, Hardy and Rogers, Zamfirescu, Ćirić etc.) from the case of a unique fixed point to the case of non unique fixed points.

1. INTRODUCTION

The classical Banach's contraction principle is one of the most useful results in nonlinear analysis. In a metric space setting the full statement of the contraction mapping principle is given by the next theorem.

Theorem B. *Let (X, d) be a complete metric space and $T : X \to X$ a map satisfying*

$$d(Tx, Ty) \le a\, d(x, y)\,, \quad \text{for all } x, y \in X\,, \tag{1.1}$$

where $0 \le a < 1$ is constant. Then:
$(p1)$ T has a unique fixed point x^* in X;
$(p2)$ The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \dots \tag{1.2}$$

converges to x^*, for any $x_0 \in X$.
$(p3)$ The following estimate holds:

$$d(x_{n+i-1}, x^*) \le \frac{a^i}{1-a} d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \dots; i = 1, 2, \dots \tag{1.3}$$

$(p4)$ The rate of convergence of Picard iteration is given by

$$d(x_n, x^*) \le a\, d(x_{n-1}, x^*)\,, \quad n = 1, 2, \dots \tag{1.4}$$

Remark 1.1. A map satisfying $(p1)$ and $(p2)$ is said to be a *Picard operator*, see [37], [38].

Received: 03.06.2008; In revised form: 31.07.2008; Accepted: 30.09.2008
2000 *Mathematics Subject Classification.* 47H10; 54H25.
Key words and phrases. *Metric space, quasi contraction, Ćirić almost contraction, fixed point, Picard iteration, error estimate.*

Theorem B has many applications in solving nonlinear equations. Its merit is not only to state the existence and uniqueness of the fixed point of the strict contraction T but also to show that the fixed point can be approximated by means of Picard iteration (1.2). Moreover, for this iterative method, the *a priori*

$$d(x_n, x^*) \leq \frac{a^n}{1-a}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \ldots$$

and *a posteriori*

$$d(x_n, x^*) \leq \frac{a}{1-a}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots$$

error estimates are available, which are both obtained from the compact estimate (1.3), suggested by a similar result in [26]. On the other hand, the inequality (1.4) shows that the rate of convergence of Picard iteration is linear.

Despite these important features, Theorem B suffers from one drawback - the contractive condition (1.1) forces T be continuous on X.

It was then natural to ask if there exist or not weaker contractive conditions which do not imply the continuity of T. This was answered in the affirmative by R. Kannan [21] in 1968, who proved a fixed point theorem which extends Theorem B to mappings that need not be continuous on X (but are continuous *at* their fixed point, see [33]), by considering instead of (1.1) the next contractive condition: there exists a constant $b \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big], \quad \text{for all } x, y \in X\,. \tag{1.5}$$

Following the Kannan's theorem, a lot of papers were devoted to obtaining fixed point or common fixed point theorems for various classes of contractive type conditions that do not require the continuity of T, see, for example, [36], [39], [6] and the references therein.

One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [11], is based on a condition similar to (1.5): there exists a constant $c \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big], \quad \text{for all } x, y \in X. \tag{1.6}$$

For a presentation and comparison of such kind of fixed point theorems, see [31], [32], [23] and [6].

On the other hand, in 1972, Zamfirescu [42] obtained a very interesting fixed point theorem which gather together all three contractive conditions mentioned above, i.e., conditions (1.1) of Banach, (1.5) of Kannan and (1.6) of Chatterjea, in a rather unexpected way: if T is such that, for any $x, y \in X$, at least one of the conditions (1.1), (1.5) and (1.6) holds, then T has a unique fixed point. Note that considering conditions (1.1), (1.5) and (1.6) all together is not trivial since, as shown later by Rhoades [31], see also [16], the contractive conditions (1.1), (1.5) and (1.6), are independent to each other.

One of the most general contraction condition of this kind for which the map satisfying it is still a Picard operator, has been obtained in 1974 by Ciric [15]: there exists $0 \leq h < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq h \cdot \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{1.7}$$

Remark 1.2. A mapping satisfying (1.7) is commonly called *quasi contraction*. It is obvious that each of the conditions (1.1), (1.5) and (1.6) does imply (1.7). As shown in [3], any quasi contraction with $h < 1/2$ is an almost contraction, too.

The Zamfirescu fixed point theorem [42] has been further extended in [5] to *almost contractions*, a class of contractive type mappings which exhibits new features with respect to the ones of the particular results incorporated, i.e., an almost contraction does not have in general a unique fixed point, but the fixed point(s) can be approximated by Picard iteration, as illustrated by the next theorem, see also the papers [3], [4] and [6].

Theorem 1.1. ([5], Theorem 2.1) *Let (X, d) be a complete metric space and $T : X \to X$ an almost contraction, that is, a mapping for which there exist a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx), \quad \textit{for all } \ x, y \in X. \tag{1.8}$$

Then

1) $Fix\,(T) = \{x \in X : Tx = x\} \neq \emptyset$;

2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) converges to some $x^* \in Fix\,(T)$;*

3) *The following estimate holds*

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta} d(x_n, x_{n-1}), \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots \tag{1.9}$$

Note that in Theorem 1.1 we can have $\delta = 0$, provided that, in this case, we automatically consider $L = 0$. On the other hand, let us notice that any quasi-contraction has a *unique* fixed point, while an almost contraction needs not have a unique fixed point, see Examples 1-2.

The main aim of this paper is to extend Theorem 1.1 by combining the Ćirić and almost contractive conditions. The fixed point theorems thus obtained are indeed very important, are genuine generalizations, and unify and extend the most important fixed point theorems of this kind in literature, amongst which we mention the results due to Banach [2], Kannan [21], Chatterjea [11], Zamfirescu [42], and Ćirić [15].

2. Contracting orbital mappings

In order to prove our main results in this paper we need the following concepts and results from [15].

Definition 2.1. Let (X, d) be a metric space and $T : X \to X$ a self map. Denote $O(x, n) = \{x, Tx, \ldots, T^n x\}$, for any positive integer n. The set $O(x, \infty) = \{x, Tx, \ldots, T^n x, \ldots\}$ is called the *orbit* of T at x. The metric space X is said to be T*-orbitally complete* iff every Cauchy sequence which is contained in $O(x, \infty)$ for some $x \in X$, converges in X.

Clearly, any complete metric space is T-orbitally complete, but the reverse is not true.

We can state now the Ćirić's fixed point theorem for the sake of completness and in view of its generalization.

Theorem C. (Ciric [15], Theorem 1) *Let T be a quasi-contraction on a metric space (X, d) and let X be T-orbitally complete metric space. Then*
(a) T has a unique fixed point u in X;
(b) $\lim_{n\to\infty} T^n x = u$, and

(c) $d(T^n x, u) \leq (h^n/(1-h))d(x, Tx)$ for every $x \in X$.

In order to prove Theorem C, Ćirić used the following two lemmas which will also be essential for us in proving our main results.

Lemma 2.1. (Ciric [15]) *Let T be a quasi-contraction on X and let n be any positive integer. Then, for each $x \in X$ and all positive integers i and j, $i, j \in \{1, 2, \ldots, n\}$ implies*

$$d(T^i x, T^j x) \leq h \cdot \delta[O(x, n)],$$

where for $A \subset X$, $\delta(A) = \sup\{d(a, b) : a, b \in A\}$.

Remark 2.3. Note that, in view of Lemma 2.1, for each n, there exist $k \leq n$ such that

$$d(x, T^k x) = \delta[O(x, n)].$$

Lemma 2.2. (Ciric [15]) *If T is a quasi-contraction on X, then the inequality*

$$\delta[O(x, n)] \leq \frac{1}{1-h} d(x, T^k x)$$

holds for all $x \in X$.

3. Main results

We state and prove our main results in a metric space setting, although they could be easily stated and proved in the more general case of a T-orbitally complete metric space.

Theorem 3.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be a Ćirić almost contraction, that is, a mapping for which there exist a constant $\alpha \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \alpha \cdot M(x, y) + L\, d(y, Tx), \quad \text{for all } x, y \in X, \tag{3.10}$$

where

$$M(x, y) = \max\left\{d(x, y),\, d(x, Tx),\, d(y, Ty),\, d(x, Ty),\, d(y, Tx)\right\}.$$

Then

1) *$Fix\,(T) = \{x \in X : Tx = x\} \neq \emptyset$;*

2) *For any $x_0 = x \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) converges to some $x^* \in Fix\,(T)$;*

3) *The following estimate holds*

$$d(x_n, x^*) \leq \frac{\alpha^n}{(1-\alpha)^2} d(x, Tx), \quad n = 1, 2, \ldots \tag{3.11}$$

Proof. We shall prove that T has at least a fixed point in X. To this end, let $x \in X$ be arbitrary and let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by (1.2) with $x_0 = x$. By taking $x := x_{n-1}$, $y := x_n$ in (3.10), we obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha \cdot M(x_{n-1}, x_n)\,,$$

since $d(x_n, Tx_{n-1}) = 0$. Then, for $n \geq 1$, by Lemma 2.1 we have

$$d(T^n x, T^{n+1}x) = d(TT^{n-1}x, T^2T^{n-1}x) \leq \alpha \cdot \delta[O(T^{n-1}x, 2)]\,.$$

Now by Remark 2.3, there exist a positive integer k_1, $1 \leq k_1 \leq 2$ such that

$$\delta[O(T^{n-1}x, 2)] = d(T^{n-1}x, T^{k_1}T^{n-1}x)$$

and therefore

$$d(x_n, x_{n+1}) \leq \alpha d(T^{n-1}x, T^{k_1}T^{n-1}x).$$

By applying once again Lemma 2.1, we have, for $n \geq 2$,

$$d(T^{n-1}x, T^{k_1}T^{n-1}x) = d(TT^{n-2}x, T^{k_1+1}T^{n-2}x) \leq$$
$$\leq \alpha \cdot \delta[O(T^{n-2}x, k_1+1)] \leq \alpha \cdot \delta[O(T^{n-2}x, 3)].$$

Therefore

$$d(T^n x, T^{n+1}x) \leq \alpha \cdot \delta[O(T^{n-1}x, 2)] \leq \alpha^2 \cdot \delta[O(T^{n-2}x, 3)].$$

Continuing in this manner we get

$$d(T^n x, T^{n+1}x) \leq \alpha \cdot \delta[O(T^{n-1}x, 2)] \leq \cdots \leq \alpha^n \cdot \delta[O(x, n+1)]. \tag{3.12}$$

On the other hand, by Lemma 2.2 we have that

$$\delta[O(x, n+1)] \leq \delta[O(x, \infty] \leq \frac{1}{1-\alpha} d(x, Tx)$$

which by (3.12) yields

$$d(T^n x, T^{n+1}x) \leq \frac{\alpha^n}{1-\alpha} d(x, Tx). \tag{3.13}$$

Now, using the triangle inequality, it is a simple task to derive from (3.13) the following estimate

$$d(T^n x, T^{n+p}x) \leq \frac{\alpha^n}{1-\alpha} \cdot \frac{1-\alpha^p}{1-\alpha}\, d(x, Tx). \tag{3.14}$$

which, in view of the fact that $0 \leq \alpha < 1$, shows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and hence it is convergent. Let us denote

$$x^* = \lim_{n\to\infty} x_n\,.$$

Then

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) = d(T^{n+1}x, x^*) + d(T^n x, Tx^*) \\ &\leq d(T^{n+1}x, x^*) + \alpha \max\{d(T^n x, u), d(T^n x, T^{n+1}x), d(x^*, Tx^*), \end{aligned}$$

$$\begin{aligned} d(T^n x, Tx^*), d(T^{n+1}x, x^*)\} &+ L\, d(x^*, Tx_n) \\ &\leq d(T^{n+1}x, x^*) + \alpha \left(d(T^n x, u) + d(T^n x, T^{n+1}x) + d(x^*, Tx^*)\right. \\ &\left. + d(T^{n+1}x, x^*)\right) + L\, d(x^*, Tx_n) \end{aligned}$$

from which we obtain

$$d(x^*,Tx^*) \leq \frac{1}{1-\alpha}\big[(1+\alpha)d(T^{n+1}x,x^*) + (\alpha+L)d(x^*,Tx_n)+\alpha d(T^n x,T^{n+1}x)\big]. \tag{3.15}$$

Letting $n\to\infty$ in (3.15) we obtain

$$d(x^*,Tx^*)=0$$

i.e. x^* is a fixed point of T.

The estimate (3.11) is obtained from (3.14) by letting $p\to\infty$. □

Remark 3.4. 1) Note that, like in the case of Theorem 1.1, in Theorem 3.2 we can have $\alpha=0$ provided that, in this case, we automatically consider $L=0$. This ensure the fact that Theorem 3.2 includes Ćirić fixed point theorem in its complete form as a particular case. As we already mentioned, Theorem 3.2 is a significant extension of Theorem B, Kannan's fixed point theorem, Chatterjea's fixed point theorem, Zamfirescu's fixed point theorem, as well as of Theorem 1.1 and of many other related results based on related similar contractive conditions.

2) Note that, although the first four fixed point theorems mentioned at 1) actually ensures the uniqueness of the fixed point, the Ćirić almost contractions need not have a unique fixed point, as shown by Example 1 below.

3) Recall, see [37], [38], that an operator $T: X\to X$ is said to be a *weakly Picard operator* if the sequence $\{T^n x_0\}_{n=0}^{\infty}$ converges for all $x_0\in X$ and the limits are fixed points of T. So, Theorem 3.2 provides a very large class of weakly Picard operators. Note also that the fixed point x^* attained by the Picard iteration depends on the initial guess $x_0\in X$. However, the a priori estimate (3.11) we obtained in Theorem 3.2 is not so good as (1.3) or the estimate given in Theorem C.

It is possible to force the uniqueness of the fixed point of a Ćirić almost contraction, like in the case of simple almost contractions [5], by imposing an additional contractive condition, quite similar to (3.10), as shown by the next theorem.

Theorem 3.3. *Let (X,d) be a complete metric space and let $T: X\to X$ be a Ćirić almost contraction for which there exist $\theta\in[0,1)$ and some $L_1\geq 0$ such that*

$$d(Tx,Ty)\leq \theta\cdot d(x,y)+L_1\cdot d(x,Tx)\,,\quad \textit{for all}\;\; x,y\in X\,. \tag{3.16}$$

Then

1) *T has a unique fixed point, i.e. $Fix\,(T)=\{x^*\}$;*
2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) converges to x^*, for any $x_0\in X$;*
3) *The a priori error estimate (3.11) holds.*
4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n,x^*)\leq \theta\, d(x_{n-1},x^*)\,,\quad n=1,2,\ldots \tag{3.17}$$

Proof. Assume T has two distinct fixed points $x^*,y^*\in X$. Then by (3.16), with $x:=x^*$, $y:=y^*$ we get

$$d(x^*,y^*)\leq \theta\cdot d(x^*,y^*)\Longleftrightarrow (1-\theta)\,d(x^*,y^*)\leq 0\,,$$

so contradicting $d(x^*,y^*)>0$.

Now letting $y := x_n$, $x := x^*$ in (3.16), we obtain the estimate (3.17). The rest of proof follows by Theorem 3.2. □

A stronger but simpler contractive condition that ensures the uniqueness of the fixed point and which actually unifies (3.10) and (3.16), has been very recently obtained by Babu et al. [1]. We state the fixed point theorem corresponding to this uniqueness condition.

Theorem 3.4. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping for which there exist $\alpha \in [0, 1)$ and some $L \geq 0$ such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \alpha \cdot M(x, y) + L \min \{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}\ , \tag{3.18}$$

where

$$M(x, y) = \max \big\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ d(x, Ty),\ d(y, Tx)\big\}.$$

Then

1) *T has a unique fixed point, i.e., $Fix\,(T) = \{x^*\}$;*
2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) converges to x^*, for any $x_0 \in X$;*
3) *The a priori error estimate (3.11) holds.*

4. Particular cases, examples and concluding remarks

1. If in Theorems 3.2-3.4 we have $L = L_1 = 0$, then we obtain the well known Ćirić's fixed point theorem for single valued mappings given in [15]. As previously mentioned in this paper, the Ćirić's contractive condition is one of the most general metrical condition that ensures a unique fixed point that can be approximated by means of Picard iteration. Our contractive condition (3.10), which does not assume a unique fixed point but **still** ensures that the fixed points could be approximated by means of Picard iteration, is very general, indeed.

2. If

$$\max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} = d(x, y),$$

for all $x, y \in X$, then by Theorem 3.2 we obtain Theorem 2.1 in [5], while by Theorem 3.3 we obtain Theorem 2.2 in the same paper [5]. Under the same assumptions, from Theorem 3.4 we obtain Theorem 2.3 in [1].

3. As the contractive conditions (1), (4), (5), (7), (9), (11), (12), (14), (18) and (19) in Rhoades' classification [31] do imply the Ćirić contractive condition, numbered in [31] as condition (24), our Theorems 3.2-3.4 extend and unify all the corresponding fixed point theorems established for these contractive conditions.

4. Let us now consider the Meszaros' alternative classification in various papers [23], in which Banach's contractive condition (1.8) is denoted by (D.1.1), Kannan's contractive condition (1.5) is (D.2.1), Bianchini's contractive condition [7] is (D.3.1), Reich-Rus contractive condition [29]-[30], [35] is (D.4.1), Chatterjea's contractive condition (1.6) is (D.6.1), Hardy and Rogers's contractive condition [20] is (D.10.1), Zamfirescu's contractive condition [42] is (D.11.1), while Ćirić's condition (1.7) is numbered (D.13.1). Note that all these conditions assume coefficients less than 1 for each of the displacements $d(x, y), d(x, Tx), d(y, Ty), d(x, Ty)$, $d(y, Tx)$, while our contractive conditions (1.8) and (3.10) allow the coefficient of $d(y, Tx)$, $L \geq 0$, to be arbitrary large.

As condition (1.8) is obtained from (1.1) by adding the extra term $L\,d(y,Tx)$, while (3.10) is obtained from (1.7) in the same manner, we can obtain similar general contractive conditions corresponding to the remaining 11 contractive conditions in Meszaros' classification. The challenging problem would be now to compare all these new contractive conditions, which do not ensure uniqueness of the fixed point, in the same way as the original corresponding conditions were compared in the papers [31], [32], [23], [16], [22].

5. Note that we can extend further Theorems 3.2-3.4, similarly to the results obtained in [15], by considering a certain iterate T^k instead of T or by considering a multi-valued mapping. All these will be approached in future works. We note, in passing, that our results in this paper open new directions for theoretical investigations as well as for applications of fixed point theorems that extend the contraction mapping principle. For some recent results that consider such generalizations and applications of the contraction mapping principle we refer to [8]-[10], [12], [17]-[19], [24]-[26], [27] and [34].

We end this paper with two examples that have an illustrative purpose. The first one presents a Ćirić type almost contraction possessing exactly two fixed points, while the second example shows a mapping with a reacher set of fixed points, i.e., the entire interval $[0,1]$. As the mapping in the second example is also nonexpansive, this shows the close connection between Ćirić type almost contractions and nonexpansive mappings. These examples also give an idea on how general and important our contractive condition (3.10) is. Indeed, this condition is general enough to allow the set of fixed points to contain more than one fixed point and, simultaneously, sharp enough to ensure that any fixed point may be effectively constructed by using the simplest fixed point iterative method, i.e., the Picard iteration.

Example 4.1. Let $[0,1]$ be the unit interval with the usual norm and let $T:[0,1]\to[0,1]$ be given by $Tx=\dfrac{1}{2}$ for $x\in[0,2/3)$ and $Tx=1$, for $x\in[2/3,1]$.

As T has two fixed points, that is, $Fix\,(T)=\left\{\dfrac{1}{2},1\right\}$, it does not satisfy neither Ćirić's condition (1.7), nor Banach, Kannan, Chatterjea and Zamfirescu contractive conditions but T satisfies the contraction condition (1.8), and therefore the contractive condition (3.10), too. Indeed, for $x,y\in[0,2/3)$ or $x,y\in[2/3,1]$, (1.8) is obvious. For $x\in[0,2/3)$ and $y\in[2/3,1]$ or $y\in[0,2/3)$ and $x\in[2/3,1]$ we have $d(Tx,Ty)=1/2$ and $d(y,Tx)=|y-1/2|\in[1/6,1/2]$, in the first case, and $d(y,Tx)=|y-1|\in[1/3,1]$, in the second case, which shows that it suffices to take $L=3$ in order to ensure that (1.8) holds for all $x,y\in X$.

Example 4.2. Let $[0,1]$ be the unit interval with the usual norm and let $T:[0,1]\to[0,1]$ be the identity map, i.e., $Tx=x$, for all $x\in[0,1]$. Then

1) T does not satisfy the Ciric's contractive condition (1.7), since $M(x,y)=|x-y|$ and

$$|x-y|>h\cdot|x-y|\,,\quad \text{for all } \ x\neq y \quad \text{and} \quad 0<h<1\,.$$

2) T satisfies condition (1.8) with $\delta \in (0,1)$ arbitrary and $L \geq 1-\delta$. Indeed condition (1.8) is equivalent to

$$|x-y| \leq \delta|x-y| + L\cdot|y-x|$$

which is true for all $x, y \in [0,1]$ if we take $\delta \in (0,1)$ arbitrary and $L \geq 1-\delta$. It is easy to check that T also satisfy the contractive condition (3.10)

3) The set of fixed points of T is the entire interval $[0,1]$. i.e., $Fix\,(T) = [0,1]$.

Acknowledgements. The author thanks CNRS and Université Pierre et Marie Curie (Paris VI), where he was a visitor during the writing of this paper, in the framework of the project PICS franco-roumain 3450 ”Mathématiques et applications” funded by CNRS. The research was also partially supported by the CEEX Grant 2532 of the Romanian Ministry of Education and Research.

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