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System of integral equations with modified argument

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ABSTRACT.
For a system of nonlinear integral equations, we will use the Perov's theorem fixed point, general data dependence theorem and fiber generalized contraction theorem given by I. A. Rus, in order to establish the conditions concerning existence and uniqueness of the solution in Banach space $C([a,b],R^m)$, the continuous data dependence of the solution and a theorem of differentiability of the solution with respect to parameters a and b. Two examples are also given here.

1. INTRODUCTION

We consider the system of nonlinear integral equations

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \tag{1.1}$$

where $t \in [a,b]$, $K \in C\left([a,b] \times [a,b] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m\right)$, $f \in C([a,b],\mathbb{R}^m)$, $g \in C([a,b],[a,b])$.

This type of nonlinear integral equations or system of nonlinear integral equations have been studied by M. Dobriţoiu in [1], [2], [3] and [4].

For this system of integral equations, we will study the existence and uniqueness of the solution in Banach space $C([a,b],R^m)$, the data dependence of the solution and the differentiability of the solution with respect to parameters a and b.

Let X be a nonempty set, d a metric on X and $A : X \to X$ an operator. In this paper we shall use the following notation:

$F_A := \{x \in X |\ A(x) = x\}$ - the fixed point set of A.

Also, we will use the Banach space

$C([a,b],\mathbb{R}^m) = \{f : [a,b] \longrightarrow \mathbb{R}^m | f - continuous\ function\}$,

endowed with the generalized Chebyshev norm on $C([a,b],\mathbb{R}^m)$, defined by the relation:

$$\|x\| := \begin{pmatrix} \|x_1\|_C \\ \dots \\ \|x_m\|_C \end{pmatrix}, \tag{1.2}$$

where

$$\|x_k\|_C := \max_{t\in[a,b]} |x_k(t)|\,. \tag{1.3}$$

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Definition 1.1. (Rus [9] or [11]) Let (X, d) be a metric space. An operator $A : X \to X$ is *Picard operator* (*PO*) if there exists $x^* \in X$ such that:

(a) $F_A = \{x^*\}$;
(b) the sequence $(A^n(x_0))_{n\in N}$ converges to x^*, for all $x_0 \in X$.

Definition 1.2. (Rus [9] or [11]) Let (X, d) be a metric space. An operator $A : X \to X$ is *weakly Picard operator* (*WPO*) if the sequence $(A^n(x_0))_{n\in N}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.

In the section 2 we need the Perov's fixed point theorem (see [7]) to study the existence and uniqueness of the solution of the system of integral equations (1.1).

Theorem 1.1. (Perov) *Let (X, d), with $d(x, y) \in \mathbb{R}^m$ be a complete generalized metric space and $A : X \longrightarrow X$ an operator. We suppose that there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$ such that*

(i) *$d(A(x), A(y)) \le Qd(x, y)$, for all $x, y \in X$;*
(ii) *$Q^n \longrightarrow 0$ as $n \longrightarrow \infty$.*

Then

(a) $F_A = \{x^*\}$;
(b) $A^n(x) \longrightarrow x^*$ as $n \longrightarrow \infty$ and

$$d(A^n(x), x^*) \le (I - Q)^{-1} Q^n d(x_0, A(x_0)).$$

In section 3 we need the general data dependence theorem (see [8], [10] and [13]).

Theorem 1.2. (General data dependence theorem). *Let (X, d) be a complete metric space and $A, B : X \to X$ two operators. We suppose that:*

(i) *A is an α-contraction;*
(ii) *$x^*_B \in F_B$;*
(iii) *there exists $\eta > 0$ such that*

$$d(A(x), B(x)) < \eta$$

for all $x \in X$.

In these conditions we have

$$d(x^*_A, x^*_B) \le \frac{\eta}{1-\alpha},$$

where x^*_A is the unique fixed point of A.

In section 4 we need the Fiber Picard operators theorem and also some other results (see [9], [10], [11], [12]).

Theorem 1.3. (Fiber Picard operators theorem). *Let (X, d) and (Y, ρ) be two metric spaces and $A = (B, C)$ a triangular operator. We suppose that:*

(i) *(Y, ρ) is a complete metric space;*
(ii) *the operator $B : X \longrightarrow X$ is WPO;*
(iii) *there exists $\alpha \in [0, 1)$, such that $C(x, \cdot)$ is an α-contraction, for all $x \in X$;*
(iv) *if $(x^*, y^*) \in F_A$, then $C(\cdot, y^*)$ is continuous in x^*.*

Then the operator A is WPO. Moreover, if B is a Picard operator, then A is a Picard operator, too.

Definition 1.3. (Rus [7]) A matrix $Q \in M_{mm}(\mathbb{R})$ converges to zero if Q^k converges to the zero matrix as $k \longrightarrow \infty$.

Theorem 1.4. (Rus [12]) *Let (X, d) be a metric space (generalized or not) and (Y, ρ) be a complete generalized metric space ($\rho(x, y) \in \mathbb{R}^m$). Let $A : X \times Y \longrightarrow X \times Y$ be a continuous operator. We suppose that:*

(i) *$A(x, y) = (B(x), C(x, y))$, for all $x \in X$, $y \in Y$;*
(ii) *$B : X \longrightarrow X$ is a weakly Picard operator;*
(iii) *There exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$ converging to zero, such that*

$$\rho\left(C(x, y_1), C(x, y_2)\right) \leq Q\rho\left(y_1, y_2\right),$$

for all $x \in X$, y_1 and $y_2 \in Y$.

Then, the operator A is weakly Picard operator. Moreover, if B is a Picard operator, then A is a Picard operator, too.

2. Existence and uniqueness in the space $C([a, b], \mathbb{R}^m)$

For the system of integral equations (1.1), we have the following existence and uniqueness theorem:

Theorem 2.5. *We suppose that:*

(i) *$K \in C\left([a, b] \times [a, b] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m\right)$;*
(ii) *$f \in C([a, b], \mathbb{R}^m)$;*
(iii) *$g \in C([a, b], [a, b])$;*
(iv) *there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$ such that*

$$\begin{pmatrix} |K_1(t, s, u_1, u_2, u_3, u_4) - K_1(t, s, v_1, v_2, v_3, v_4)| \\ \cdots\cdots\cdots \\ |K_m(t, s, u_1, u_2, u_3, u_4) - K_m(t, s, v_1, v_2, v_3, v_4)| \end{pmatrix}$$
$$\leq Q \begin{pmatrix} |u_{11} - v_{11}| + |u_{21} - v_{21}| + |u_{31} - v_{31}| + |u_{41} - v_{41}| \\ \cdots\cdots\cdots \\ |u_{1m} - v_{1m}| + |u_{2m} - v_{2m}| + |u_{3m} - v_{3m}| + |u_{4m} - v_{4m}| \end{pmatrix},$$

for all $t, s \in [a, b]$, $u_i, v_i \in \mathbb{R}^m$, $i = \overline{1, 4}$.

(v) *$[4(b - a)Q]^n \longrightarrow 0$, as $n \longrightarrow \infty$.*

Then the system of integral equations (1.1) has a unique solution $x^* \in C([a, b], \mathbb{R}^m)$. This solution can be obtained by the successive approximations method, starting at any element from $C([a, b], \mathbb{R}^m)$, and if x_n is the n^{th} successive approximation, then we have the following estimation:

$$\|x^* - x_n\|_{\mathbb{R}^m} \leq [4(b - a)Q]^n \cdot [I - 4(b - a)Q]^{-1} \cdot \|x_0 - x_1\|_{\mathbb{R}^m}. \tag{2.2}$$

Proof. We consider the operator $A : C([a, b], \mathbb{R}^m) \longrightarrow C([a, b], \mathbb{R}^m)$, defined by

$$A(x)(t) := \int_a^b K(t, s, x(s), x(g(s)), x(a), x(b))ds + f(t), \quad t \in [a, b]. \tag{2.2}$$

Using the conditions (i), (ii), (iii), we obtain that the operator A is well defined.

The set of the solutions of the system of integral equations (1.1) coincides with the set of fixed points of the operator A.

From the condition (iv) it results that the function K satisfies a Lipschitz condition with respect to last four arguments, with a matrix $Q \in M_{mm}(\mathbb{R}_+)$, and therefore, for the operator A we have:

$$|A(x)(t) - A(y)(t)| = \begin{pmatrix} |A_1(x)(t) - A_1(y)(t)| \\ \dots \\ |A_m(x)(t) - A_m(y)(t)| \end{pmatrix}$$

$$\leq \begin{pmatrix} \left|\int_a^b [K_1(t,s,x(s),x(g(s)),x(a),x(b)) - K_1(t,s,y(s),y(g(s)),y(a),y(b))]\, ds\right| \\ \dots \\ \left|\int_a^b [K_m(t,s,x(s),x(g(s)),x(a),x(b)) - K_m(t,s,y(s),y(g(s)),y(a),y(b))]\, ds\right| \end{pmatrix}$$

$$\leq \begin{pmatrix} \int_a^b |K_1(t,s,x(s),x(g(s)),x(a),x(b)) - K_1(t,s,y(s),y(g(s)),y(a),y(b))|\, ds \\ \dots \\ \int_a^b |K_m(t,s,x(s),x(g(s)),x(a),x(b)) - K_m(t,s,y(s),y(g(s)),y(a),y(b))|\, ds \end{pmatrix}.$$

Using the condition (iv) and the generalized Chebyshev norm on $C([a,b],\mathbb{R}^m)$, defined by the relations (1.2) and (1.3), we obtain

$$\|A(x) - A(y)\|_{\mathbb{R}^m} \leq 4(b-a)Q \cdot \|x - x\|_{\mathbb{R}^m} \ .$$

By the condition (v) it results that the operator A is a generalized contraction with the matrix $4(b-a)Q$, $Q \in M_{mm}(\mathbb{R}_+)$.

Now, by the conditions $(i) - (v)$ and using Perov's fixed point theorem, it results that the system of nonlinear integral equations with modified argument (1.1) has a unique solution $x^* \in C([a,b],\mathbb{R}^m)$. □

3. Data dependence

In what follows, we will study the dependence of the solution of the system of integral equations (1.1), with respect to K and f.

Now, we consider the perturbed system of integral equations

$$(3.3) \qquad y(t) = \int_a^b H(t,s,y(s),y(g(s)),y(a),y(b))ds + h(t), \;\; t \in [a,b]\,,$$

where $H \in C([a,b]\times[a,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m)$, $h \in C([a,b],\mathbb{R}^m)$, $g \in C([a,b],[a,b])$.

We have the following data dependence theorem:

Theorem 3.6. *We suppose that:*

(i) *the conditions of the Theorem* 2.1 *are satisfied and we denote by* x^* *the unique solution of the system of integral equations* (1.1);

(ii) *there exists* $T_1, T_2 \in M_{m1}(\mathbb{R}_+)$ *such that*

$$\|K(t,s,u_1,u_2,u_3,u_4) - H(t,s,u_1,u_2,u_3,u_4)\|_{\mathbb{R}^m} \leq T_1,$$

for all $t,s \in [a,b]$ *,we* $u_i \in \mathbb{R}^m$, $i = \overline{1,4}$, *and*

$$|f(t) - h(t)| \leq T_2\,,$$

for all $t \in [a,b]$.

Under these assumptions, if $y^* \in C([a,b],\mathbb{R}^m)$ *is a solution of the perturbed system of integral equations* (3.3)*, then we have*

$$\|x^* - y^*\|_{\mathbb{R}^m} \le [I - 4(b-a)Q]^{-1} \cdot [(b-a)T_1 + T_2]\,. \tag{3.4}$$

Proof. We consider the operator $A : C([a,b],\mathbb{R}^m) \longrightarrow C([a,b],\mathbb{R}^m)$ defined by the relation (2.2).

Let $B : C([a,b],\mathbb{R}^m) \longrightarrow C([a,b],\mathbb{R}^m)$ be the operator defined by

$$B(y)(t) := \int_a^b H(t,s,y(s),y(g(s)),y(a),y(b))ds + h(t), \quad t \in [a,b]. \tag{3.5}$$

We have

$$\begin{aligned}|A(x)(t) - B(x)(t)| &= |\int_a^b [K(t,s,x(s),x(g(s)),x(a),x(b)) \\ &\quad - H(t,s,x(s),x(g(s)),x(a),x(b))]ds + [f(t)-h(t)]| \\ &\le \int_a^b |K(t,s,x(s),x(g(s)),x(a),x(b)) \\ &\quad - H(t,s,x(s),x(g(s)),x(a),x(b))|ds + |f(t)-h(t)|.\end{aligned}$$

Using the condition (ii) and the generalized Chebyshev norm on $C([a,b],\mathbb{R}^m)$, defined by the relations (1.2) and (1.3), we obtain

$$\|A(x) - B(x)\|_{\mathbb{R}^m} \le T_1(b-a) + T_2.$$

Now, the conditions of the *General data dependence theorem* (Theorem 1.2) being satisfied, we obtain the estimation (3.4) and the proof is complete. □

4. Differentiability with respect to a parameter

In this section we will study the differentiability of the solution of the system of integral equations (1.1) with respect to parameters a and b

$$x(t;a,b) = \int_a^b K(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))ds\, + f(t), \tag{4.6}$$

$t \in [\alpha,\beta]$, where $\alpha,\beta \in \mathbb{R}$, $\alpha \le \beta$, $a,b \in [\alpha,\beta]$ and
$K \in C\left([\alpha,\beta]\times[\alpha,\beta]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m\right)$, $f \in C([\alpha,\beta],\mathbb{R}^m)$,
$g \in C([a,b],[a,b])$ and $x \in C([\alpha,\beta],\mathbb{R}^m)$.

We have the following theorem:

Theorem 4.7. *We suppose that there exists a matrix* $Q \in M_{mm}(\mathbb{R}_+)$ *such that*

(i) $[4(b-a)Q]^n \longrightarrow 0$ *, as* $n \longrightarrow \infty$ *;*

$$(ii)\quad \begin{pmatrix} |K_1(t,s,u_1,u_2,u_3,u_4) - K_1(t,s,v_1,v_2,v_3,v_4)| \\ \cdots\cdots\cdots \\ |K_m(t,s,u_1,u_2,u_3,u_4) - K_m(t,s,v_1,v_2,v_3,v_4)| \end{pmatrix}$$
$$\le Q \begin{pmatrix} |u_{11}-v_{11}| + |u_{21}-v_{21}| + |u_{31}-v_{31}| + |u_{41}-v_{41}| \\ \cdots\cdots\cdots \\ |u_{1m}-v_{1m}| + |u_{2m}-v_{2m}| + |u_{3m}-v_{3m}| + |u_{4m}-v_{4m}| \end{pmatrix},$$

for all $t,s \in [\alpha,\beta]$*,* $u_i, v_i \in \mathbb{R}^m$*,* $i = \overline{1,4}$*.*

Then:

(a) *the system of integral equations* (4.6) *has a unique solution* $x^*(\cdot; a, b) \in C([\alpha, \beta], \mathbb{R}^m)$;

(b) *for all* $x_0 \in C([\alpha, \beta], \mathbb{R}^m)$ *the sequence* $(x^n)_{n\in\mathbb{N}}$, *defined by*

$$x^{n+1}(t; a, b) = \int_a^b K(t, s, x^n(s; a, b), x^n(g(s); a, b), x^n(a; a, b), x^n(b; a, b))\, ds + f(t)$$

converges uniformly to x^* *for all* $t, a, b \in [\alpha, \beta]$, *and*

$$\begin{pmatrix} |x_1^n(t; a, b) - x_1^*(t; a, b)| \\ \cdots\cdots\cdots \\ |x_m^n(t; a, b) - x_m^*(t; a, b)| \end{pmatrix}$$

$$\leq [I - 4(\beta - \alpha) Q]^{-1} [4(\beta - \alpha) Q]^n \begin{pmatrix} \left|x_1^0(t; a, b) - x_1^1(t; a, b)\right| \\ \cdots\cdots\cdots \\ \left|x_m^0(t; a, b) - x_m^1(t; a, b)\right| \end{pmatrix};$$

(c) *the function*

$$x^* : [\alpha, \beta] \times [\alpha, \beta] \times [\alpha, \beta] \longrightarrow \mathbb{R}^m, \quad (t; a, b) \longmapsto x^*(t; a, b)$$

is continuous;

(d) *if*

$$K(t, s, \cdot, \cdot, \cdot, \cdot) \in C^1(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m) \quad for \ \ all \ \ t, s \in [\alpha, \beta],$$

then

$$x^*(t; \cdot, \cdot) \in C^1([\alpha, \beta] \times [\alpha, \beta], \mathbb{R}^m), \quad for \ \ all \ \ t \in [\alpha, \beta].$$

Proof. $(a) + (b) + (c)$. We consider the generalized Chebyshev norm on $X := C([\alpha, \beta]^3, \mathbb{R}^m)$, defined by the relations (1.2) and (1.3).

Let we consider the operator $B : X \to X$ defined by

$$B(x)(t; a, b) := \int_a^b K(t, s, x(s; a, b), x(g(s); a, b), x(a; a, b), x(b; a, b)) ds$$

for all $t, a, b \in [\alpha, \beta]$.

Using the conditions (i), (ii) and the Perov's fixed point theorem, we obtain the conclusions $(a) + (b) + (c)$.

(d) Let we prove that there exists $\dfrac{\partial x^*}{\partial a}$ and $\dfrac{\partial x^*}{\partial a} \in X$.

If we suppose that there exists $\dfrac{\partial x^*}{\partial a}$, then from (1.1) we have

$$\begin{aligned} \frac{\partial x^*(t; a, b)}{\partial a} &= -K(t, a, x^*(a; a, b), x^*(g(a); a, b), x^*(a; a, b), x^*(b; a, b)) \\ &+ \int_a^b \Bigg[\left(\frac{\partial K_j(t, s, x^*(s; a, b), x^*(g(s); a, b), x^*(a; a, b), x^*(b; a, b))}{\partial x_i} \right) \frac{\partial x^*(s; a, b)}{\partial a} \\ &+ \left(\frac{\partial K_j(t, s, x^*(s; a, b), x^*(g(s); a, b), x^*(a; a, b), x^*(b; a, b))}{\partial x_i} \right) \frac{\partial x^*(g(s); a, b)}{\partial a} \\ &+ \left(\frac{\partial K_j(t, s, x^*(s; a, b), x^*(g(s); a, b), x^*(a; a, b), x^*(b; a, b))}{\partial x_i} \right) \frac{\partial x^*(a; a, b)}{\partial a} \\ &+ \left(\frac{\partial K_j(t, s, x^*(s; a, b), x^*(g(s); a, b), x^*(a; a, b), x^*(b; a, b))}{\partial x_i} \right) \frac{\partial x^*(b; a, b)}{\partial a} \Bigg] ds. \end{aligned}$$

This relation suggest to consider the following operator

$$C : X \times X \to X,$$

defined by

$$(4.7)\quad \begin{aligned} C(x,y)(t;a,b) &:= -K(t,a,x(a;a,b),x(g(a);a,b),x(a;a,b),x(b;a,b)) \\ &+ \int_a^b \Big[\Big(\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))}{\partial x_i} \Big) y(s;a,b) \\ &+ \Big(\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x(b;a,b))}{\partial x_i} \Big) y(g(s);a,b) \\ &+ \Big(\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x^*(b;a,b))}{\partial x_i} \Big) y(a;a,b) \\ &+ \Big(\frac{\partial K_j(t,s,x(s;a,b),x(g(s);a,b),x(a;a,b),x^*(b;a,b))}{\partial x_i} \Big) y(b;a,b) \Big] ds. \end{aligned}$$

From (ii), we remark that

$$(4.8)\quad \left(\left| \frac{\partial K_j(t,s,u,v,w,z)}{\partial x_i} \right| \right) \leq Q$$

for all $t, s \in [\alpha, \beta]$ and $u, v, w, z \in \mathbb{R}^m$.

From (4.7) and (4.8) it follows that

$$\|C(x,y_1) - C(x,y_2)\| \leq (\beta - \alpha)Q,$$

for all $x, y_1, y_2 \in X$.

If we take the operator

$$A : X \times X \to X \times X, \quad A = (B, C),$$

then we are in the conditions of Theorem 1.4. From this theorem, it results that the operator A is a Picard operator and that the sequence

$$x^{n+1}(t;a,b) = \int_a^b K(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))ds + f(t)$$

$$\begin{aligned} y^{n+1}(t;a,b) &:= -K(t,a,x^n(a;a,b),x^n(g(a);a,b),x^n(a;a,b),x^n(b;a,b)) \\ &+ \int_a^b \Big[\Big(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))}{\partial x_i} \Big) y^n(s;a,b) \\ &+ \Big(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))}{\partial x_i} \Big) y^n(g(s);a,b) \\ &+ \Big(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))}{\partial x_i} \Big) y^n(a;a,b \\ &+ \Big(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))}{\partial x_i} \Big) y^n(b;a,b) \Big] ds \end{aligned}$$

converges uniformly (with respect to $t, a, b \in [\alpha, \beta]$) to $(x^*, y^*) \in F_A$, for all $x^0, y^0 \in X$.

If we take $x^0 = y^0 = 0$, then $y^1 = \dfrac{\partial x^1}{\partial a}$. By induction we prove that $y^n = \dfrac{\partial x^n}{\partial a}$. Thus

$$x^n \stackrel{unif.}{\longrightarrow} x^* \text{ as } n \to \infty,$$

$$\frac{\partial x^n}{\partial a} \stackrel{unif.}{\longrightarrow} y^* \text{ as } n \to \infty.$$

These imply that there exists $\dfrac{\partial x^*}{\partial a}$ and $\dfrac{\partial x^*}{\partial a} = y^*$. In a similar way we prove that there exists $\dfrac{\partial x^*}{\partial b}$. □

5. Examples

Two examples are given in this section.

Example 5.1. In what follows, we consider the system of integral equations with modified argument

$$\left\{\begin{aligned} x_1(t) &= \int_0^1 \left[\frac{t+2}{15}x_1(s) + \frac{2t+1}{15}x_1(s/2) + \frac{t}{5}x_1(0) + \frac{t}{5}x_1(1)\right] ds + 2t + 1 \\ x_2(t) &= \int_0^1 \left[\frac{t+2}{21}x_2(s) + \frac{2t+1}{21}x_2(s/2) + \frac{t}{7}x_2(0) + \frac{t}{7}x_2(1)\right] ds + \sin t, \end{aligned}\right. \tag{5.9}$$

$t \in [0,1]$, where $K \in C\left([0,1] \times [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2\right)$, $K = (K_1, K_2)$,

$$K_1 = \frac{t+2}{15}x_1(s) + \frac{2t+1}{15}x_1(s/2) + \frac{t}{5}x_1(0) + \frac{t}{5}x_1(1),$$

$$K_2 = \frac{t+2}{21}x_2(s) + \frac{2t+1}{21}x_2(s/2) + \frac{t}{7}x_2(0) + \frac{t}{7}x_2(1),$$

$f \in C\left([0,1], \mathbb{R}^2\right)$, $f = (f_1, f_2)$, $f_1(t) = 2t+1$, $f_2(t) = \sin t$, $g \in C([0,1],[0,1])$, $g(s) = s/2$, and $x \in C\left([0,1], \mathbb{R}^2\right)$, and the perturbed system

$$\left\{\begin{aligned} y_1(t) &= \int_0^1 \left[\frac{s+3}{15}y_1(s) + \frac{2s+3}{15}y_1(s/2) + \frac{t}{5}y_1(0) + \frac{t}{5}y_1(1) - 3\right] ds + 2t - 1 \\ y_2(t) &= \int_0^1 \left[\frac{s+3}{21}y_2(s) + \frac{2s+3}{21}y_2(s/2) + \frac{t}{7}y_2(0) + \frac{t}{7}y_2(1) - 1\right] ds + \cos t, \end{aligned}\right. \tag{5.10}$$

$t \in [0,1]$, where $H \in C\left([0,1] \times [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2\right)$, $H = (H_1, H_2)$,

$$H_1 = \frac{s+3}{15}y_1(s) + \frac{2s+3}{15}y_1(s/2) + \frac{t}{5}y_1(0) + \frac{t}{5}y_1(1) - 3,$$

$$H_2 = \frac{s+3}{21}y_2(s) + \frac{2s+3}{21}y_2(s/2) + \frac{t}{7}y_2(0) + \frac{t}{7}y_2(1) - 1,$$

$h \in C\left([0,1], \mathbb{R}^2\right)$, $h = (h_1, h_2)$, $h_1(t) = 2t - 1$, $h_2(t) = \cos t$, $g \in C([0,1],[0,1])$, $g(s) = s/2$, and $x \in C\left([0,1], \mathbb{R}^2\right)$.

The operator $A : C\left([0,1],\mathbb{R}^2\right) \longrightarrow C\left([0,1],\mathbb{R}^2\right)$, $A=(A_1,A_2)$, attached to the system (5.9), defined by

$$\begin{aligned} A_1(x)(t) \quad &: \quad = \int_0^1 \left[\frac{t+2}{15}x_1(s) + \frac{2t+1}{15}x_1\left(s/2\right) + \frac{t}{5}x_1(0) + \frac{t}{5}x_1(1)\right] ds + 2t + 1 \\ A_2(x)(t) \quad &: \quad = \int_0^1 \left[\frac{t+2}{21}x_2(s) + \frac{2t+1}{21}x_2\left(s/2\right) + \frac{t}{7}x_2(0) + \frac{t}{7}x_2(1)\right] ds + \sin t \end{aligned}$$

satisfies the Lipschitz condition with the matrix $Q = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/7 \end{pmatrix}$. The matrix $4(b-a)Q = \begin{pmatrix} 4/5 & 0 \\ 0 & 4/7 \end{pmatrix}$ converges to zero and therefore, the operator A is a contraction with this matrix.

The conditions of the Theorem 2.1 being satisfied, it results that the system (5.9) has a unique solution $x^* \in C\left([0,1],\mathbb{R}^2\right)$.

We have

$$\|K(t,s,x(s),x(s/2),x(0),x(1)) - H(t,s,x(s),x(s/2),x(0),x(1))\|_{\mathbb{R}^2} \le \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

for all $t,s \in [0,1]$, and

$$\|f(t)-h(t)\|_{\mathbb{R}^2} \le \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{for all } \ t \in [0,1].$$

Under these conditions, if $y^* \in C\left([0,1],\mathbb{R}^2\right)$ is a solution of the perturbed system (5.10), then by theorem 3.1, the following estimation is hold:

$$\|x^*-y^*\|_{\mathbb{R}^2} \le \begin{pmatrix} 1/5 & 0 \\ 0 & 3/7 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 25 \\ 14/3 \end{pmatrix}.$$

Example 5.2. We will apply Theorem 4.1 in order to study the differentiability of the solution with respect to parameters a and b, for the following system of integral equations:

$$\text{(5.11)} \quad \begin{cases} x_1(t) = \displaystyle\int_a^b \left[\frac{t+s}{10}x_1(s) + \frac{1}{5}x_1\left(s/2\right) + \frac{2t+1}{15}x_1(a) + \frac{t+2}{15}x_1(b)\right] ds + 1 - \cos t \\ x_2(t) = \displaystyle\int_a^b \left[\frac{1}{2}x_1(s) + \frac{2t+s}{24}x_2(s) + \frac{1}{8}x_2(s) + \frac{2t+1}{24}x_2(a) + \frac{t+2}{24}x_2(b)\right] ds + \sin t, \end{cases}$$

where $t,a,b \in [0,1]$, $K \in C\left([0,1]\times[0,1]\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2,\mathbb{R}^2\right)$, $K=(K_1,K_2)$,

$$\begin{aligned} K_1 &= \frac{t+s}{10}x_1(s) + \frac{1}{5}x_1\left(s/2\right) + \frac{2t+1}{15}x_1(a) + \frac{t+2}{15}x_1(b), \\ K_2 &= \frac{1}{2}x_1(s) + \frac{2t+s}{24}x_2(s) + \frac{1}{8}x_2(s) + \frac{2t+1}{24}x_2(a) + \frac{t+2}{24}x_2(b), \end{aligned}$$

$f \in C\left([0,1],\mathbb{R}^2\right)$, $f=(f_1,f_2)$, $f_1(t) = 1-\cos t$, $f_2(t) = \sin t$, $g \in C([0,1],[0,1])$, $g(s) = s/2$.

From the condition (ii) of Theorem 4.1, we have

$$\begin{pmatrix} \left|K_1(t,s,x(s),x\left(\frac{s}{2}\right),x(a),x(b)) - K_1(t,s,z(s),z\left(\frac{s}{2}\right),z(a),z(b))\right| \\ \left|K_2(t,s,x(s),x\left(\frac{s}{2}\right),x(a),x(b)) - K_2(t,s,z(s),z\left(\frac{s}{2}\right),z(a),z(b))\right| \end{pmatrix}$$

$$\leq \begin{pmatrix} \frac{1}{5} & 0 \\ \frac{1}{2} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} |x_1(s)-z_1(s)|+\left|x_1\left(\frac{s}{2}\right)-z_1\left(\frac{s}{2}\right)\right| + |x_1(a)-z_1(a)| + |x_1(b)-z_1(b)| \\ |x_2(s)-z_2(s)|+\left|x_2\left(\frac{s}{2}\right)-z_2\left(\frac{s}{2}\right)\right|+|x_2(a)-z_2(a)|+|x_2(b)-z_2(b)| \end{pmatrix}.$$

The matrix $4(b-a)Q = (b-a)\begin{pmatrix} 4/5 & 0 \\ 2 & 1/2 \end{pmatrix}$, $0 < b-a < 1$, $Q \in M_{22}(\mathbb{R}_+)$, converges to zero.

Therefore, the conditions of Theorem 4.1 being satisfied, it results that:

1) the system (5.9) has in $C([0,1],\mathbb{R}^2)$ a unique solution $x^*(\cdot,a,b)$;

2) for all $x^0 \in C([0,1],\mathbb{R}^2)$, the sequence $(x^n)_{n\in\mathbb{N}}$, defined by

$$x^{n+1}(t;a,b)=\int_a^b K(t,s,x^n(s;a,b),x^n(g(s);a,b),x^n(a;a,b),x^n(b;a,b))ds+f(t)$$

converges uniformly to x^*, for all $t,a,b,\in[0,1]$ and we have

$$\begin{pmatrix} |x_1^n(t;a,b) - x_1^*(t;a,b)| \\ |x_2^n(t;a,b) - x_2^*(t;a,b)| \end{pmatrix} \leq$$

$$\leq \begin{pmatrix} 5 & 0 \\ 20 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4/5 & 0 \\ 2 & 1/2 \end{pmatrix}^n \cdot \begin{pmatrix} \left|x_1^0(t;a,b) - x_1^1(t;a,b)\right| \\ \left|x_2^0(t;a,b) - x_2^1(t;a,b)\right| \end{pmatrix};$$

3) the function

$$x^* : [0,1]\times[0,1]\times[0,1] \longrightarrow \mathbb{R}^2\ , \qquad (t,a,b)\longmapsto x^*(t;a,b)$$

is continuous;

4) if

$$K(t,s,\cdot,\cdot,\cdot,\cdot)\in C^1(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2,\mathbb{R}^2)\ , \quad \text{for all } \ t,s\in[0,1]\ ,$$

then

$$x^*(t;\cdot,\cdot)\in C^1([0,1]\times[0,1],\mathbb{R}^2)\ , \quad \text{for all } \ t\in[0,1]\ .$$

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