CARPATHIAN J. MATH.
24 (2008), No. 2, 37 - 44

Some results in $\mathcal{L}$-fuzzy metric spaces

HAKAN EFE

ABSTRACT.

The purpose of this paper to prove Baire's theorem and uniform limit theorem for $\mathcal{L}$-fuzzy metric spaces. Also, we show every separable $\mathcal{L}$-fuzzy metric spaces are second countable and subspace of a separable $\mathcal{L}$-fuzzy metric space is separable.

1. INTRODUCTION

Since the introduction of the concept of fuzzy set by Zadeh [17] in 1965, many authors have introduced the concept of fuzzy metric space in different ways [2, 5, 9, 11, 12]. George and Veeramani [6, 7] modified the concept of fuzzy metric space introduced by Kromosil and Michalek [12] and defined a Hausdorff topology on this fuzzy metric space. Using to idea of $\mathcal{L}$-fuzzy sets [8], Saadati et al. [15] introduced the notion of $\mathcal{L}$-fuzzy metric spaces with the help of continuous t-norms as a generalization of fuzzy metric space due to George and Veeramani [6] and intuitionistic fuzzy metric space due to Park and Saadati [13, 14]. Recently, Saadati [16] proved some known results of metric spaces including Uniform continuity theorem and Ascoli–Arzela theorem for $\mathcal{L}$ -fuzzy metric spaces. He also proved that every $\mathcal{L}$ -fuzzy metric space has a countably locally finite basis and used this result to conclude that every $\mathcal{L}$-fuzzy metric space is metrizable.

In this paper we show that every $\mathcal{L}$-fuzzy metric space is Hausdorff. We also show every compact subset of an $\mathcal{L}$-fuzzy metric space is $\mathcal{L}F$-strongly bounded. Then we prove Baire's theorem for $\mathcal{L}$-fuzzy metric spaces. Furthermore we show that separable $\mathcal{L}$-fuzzy metric spaces are second countable and subspace of a separable $\mathcal{L}$-fuzzy metric space is separable. Finally, we prove uniform limit theorem for $\mathcal{L}$-fuzzy metric spaces.

2. PRELIMINARIES

Definition 2.1. [7] Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice, and U a non-empty set called universe. An $\mathcal{L}$-**fuzzy set** $\mathcal{A}$ on U is defined as a mapping $\mathcal{A} : U \to L$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u satisfies $\mathcal{A}$.

Lemma 2.1. [4] *Consider the set L^* and operation $\leq_{L^*}$ defined by $L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2$ and $x_1 + x_2 \leq 1\}$, $(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1$ and $x_2 \geq y_2$, for every (x_1, x_2), $(y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.*

Received: 14.02.2008; In revised form: 21.04.2008; Accepted: 30.09.2008
2000 *Mathematics Subject Classification.* 06B23, 46S40.
Key words and phrases. *$\mathcal{L}$-fuzzy metric space, separable metric space, compactness, boundedness.*

Definition 2.2. [1] An **intuitionistic fuzzy set** $\mathcal{A}_{\zeta,\eta}$ on a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\zeta,\eta}$, and furthermore satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

Classically, a triangular norm T on $([0,1], \leq)$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(x,1) = x$, for all $x \in [0,1]$. These definitions can be straightforwardly extended to any lattice $\mathcal{L} = (L, \leq_L)$. Define first $0_{\mathcal{L}} = \inf L$ and $1_{\mathcal{L}} = \sup L$.

Definition 2.3. A **triangular norm** (**t-norm**) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:

(i) $(\forall x \in L)(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$; (boundary condition)
(ii) $(\forall (x,y) \in L^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x)$; (commutativity)
(iii) $(\forall (x,y,z) \in L^3)(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y), z)$; (associativity)
(iv) $(\forall (x,x',y,y') \in L^4)(x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x,y) \leq_L \mathcal{T}(x',y'))$ (monotonicity).

A t-norm $\mathcal{T}$ on $\mathcal{L}$ is said to be **continuous** if for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y we have $\lim_n \mathcal{T}(x_n, y_n) = \mathcal{T}(x,y)$.

For example, $\mathcal{T}(x,y) = \min(x,y)$ and $\mathcal{T}(x,y) = xy$ are two continuous t-norms on $[0,1]$.

A t-norm can also be defined recursively as an $(n+1)$-ary operation ($n \in \mathbb{N}$) by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x_1, ..., x_{n+1}) = \mathcal{T}(\mathcal{T}^{n-1}(x_1, ..., x_n), x_{n+1})$$

for $n \geq 2$ and $x_i \in L$.

Definition 2.4. [3] A t-norm $\mathcal{T}$ on L^* is called t-**representable** if and only if there exist a t-norm T and a t-conorm S on $[0,1]$ such that, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x,y) = (T(x_1, y_1), S(x_2, y_2)).$$

Definition 2.5. A **negation** on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for $x \in L$, then $\mathcal{N}$ is called an involutive negation.

The negation N_s on $([0,1], \leq)$ defined as, for all $x \in [0,1]$, $N_s(x) = 1 - x$, is called the standard negation on $([0,1], \leq)$. We show $(N_s(x), x) = \mathcal{N}_s(x)$.

Definition 2.6. [15] The 3-tuple $(X, \mathcal{M}, \mathcal{T})$ is said to be an $\mathcal{L}$-**fuzzy metric space** if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2 \times (0, +\infty)$ satisfying the following conditions for every x, y, z in X and t, s in $(0, +\infty)$:

(a) $\mathcal{M}(x,y,t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{M}(x,y,t) = 1_{\mathcal{L}}$ for all $t > 0$ if and only if $x = y$;
(c) $\mathcal{M}(x,y,t) = \mathcal{M}(y,x,t)$;
(d) $\mathcal{T}(\mathcal{M}(x,y,t), \mathcal{M}(y,z,s)) \leq_L \mathcal{M}(x,z,t+s)$;
(e) $\mathcal{M}(x,y,\cdot) : (0,+\infty) \to L$ is continuous.

In this case $\mathcal{M}$ is called an $\mathcal{L}$-**fuzzy metric**. If $\mathcal{M} = \mathcal{M}_{M,N}$ is an intuitionistic fuzzy set (see Definition 2.2) then the 3-tuple $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be an **intuitionistic fuzzy metric space**.

Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. For $t \in (0, +\infty)$, we define the open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in L \backslash \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, as

$$B(x, r, t) = \{y \in X : \mathcal{M}(x, y, t) >_L \mathcal{N}(r)\}.$$

A subset $A \subseteq X$ is called open if for each $x \in A$, there exist $t > 0$ and $r \in L \backslash \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x, r, t) \subseteq A$. Let $\tau_{\mathcal{M}}$ denote the family of all open subsets of X. Then $\tau_{\mathcal{M}}$ is called the $\mathcal{L}$-fuzzy topology induced by the $\mathcal{L}$-fuzzy metric $\mathcal{M}$.

Example 2.1. Let (X,d) be a metric space. Define $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$and $b = (b_1, b_2)$ in L^* and let M and N be fuzzy sets on $X^2 \times (0, +\infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{ht^n}{ht^n + md(x, y)}, \frac{md(x, y)}{ht^n + md(x, y)} \right),$$

for all $t, h, m, n \in \mathbb{R}^+$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space. If $h = m = n = 1$ then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is a standard intuitionistic fuzzy metric space. Also, if we define

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{t}{t + md(x, y)}, \frac{d(x, y)}{t + d(x, y)} \right),$$

where $m > 1$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space in which $\mathcal{M}_{M,N}(x, x, t) = 1_{L^*}$ and $\mathcal{M}_{M,N}(x, y, t) <_{L^*} 1_{L^*}$ for $x \neq y$.

Example 2.2. Let $X = \mathbb{N}$. Define $\mathcal{T}(a, b) = (\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2 b_2)$ for all $a = (a_1, a_2)$and $b = (b_1, b_2)$ in L^* and let M and N be fuzzy sets on $X^2 \times (0, +\infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \begin{cases} \left(\frac{x}{y}, \frac{y - x}{y} \right) & \text{if} \quad x \leq y, \\ \left(\frac{y}{x}, \frac{x - y}{y} \right) & \text{if} \quad y \leq x, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Lemma 2.2. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then, $\mathcal{M}(x, y, t)$ is nondecreasing with respect to t, for all $x, y \in X$.*

Definition 2.7. [16] A sequence $\{x_n\}$ in an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ is called a **Cauchy sequence**, if for each $\varepsilon \in L \backslash \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m \geq n \geq n_0$ $(n \geq m \geq n_0)$

$$\mathcal{M}(x_m, x_n, t) >_L \mathcal{N}(\varepsilon).$$

The sequence $\{x_n\}$ is said to be **convergent** to $x \in X$ in the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ (denoted by $x_n \xrightarrow{\mathcal{M}} x$) if $\mathcal{M}(x_n, x, t) = \mathcal{M}(x, x_n, t) \to 1_{\mathcal{L}}$ as $n \to \infty$ for every $t > 0$. An $\mathcal{L}$-fuzzy metric space is said to be **complete** iff every Cauchy sequence is convergent.

Henceforth, we assume that $\mathcal{T}$ is a continuous t-norm on lattice $\mathcal{L}$ such that for every $\mu \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, there is a $\lambda \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that

$$\mathcal{T}^{n-1}(\mathcal{N}(\lambda), ..., \mathcal{N}(\lambda)) >_L \mathcal{N}(\mu).$$

3. Main Results

Theorem 3.1. *Every $\mathcal{L}$-fuzzy metric space is Hausdorff.*

Proof. Let $(X, \mathcal{M}, \mathcal{T})$ be the given $\mathcal{L}$-fuzzy metric space. Let x, y be two distinct points of X. Then $\mathcal{M}(x, y, t) \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$. Let $\mathcal{M}(x, y, t)=\mathcal{N}(r)$, $r\in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$. Then for each r_0, $r_0 <_L r$, we can find a $r_1 \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{T}(\mathcal{N}(r_1), \mathcal{N}(r_1)) >_L \mathcal{N}(r_0)$. Now consider the open balls $B(x, r_1, t/2)$ and $B(y, r_1, t/2)$. Then $B(x, r_1, t/2) \cap B(y, r_1, t/2) = \varnothing$. For if there exists $z \in B(x, r_1, t/2) \cap B(y, r_1, t/2)$ then,

$$\begin{aligned} \mathcal{N}(r) &= \mathcal{M}(x, y, t) \geq_L \mathcal{T}(\mathcal{M}(x, z, t/2), \mathcal{M}(z, y, t/2)) \\ &\geq {}_L\mathcal{T}(\mathcal{N}(r_1), \mathcal{N}(r_1)) >_L \mathcal{N}(r_0) >_L \mathcal{N}(r), \end{aligned}$$

which is a contradiction. □

Definition 3.8. [10] Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space and $A \subseteq X$. The $\mathcal{L}$-**fuzzy diameter** of a set A is defined by

$$\delta_A = \sup_{t>0} \inf_{x,y\in A} \sup_{\varepsilon<t} \mathcal{M}(x, y, \varepsilon).$$

If $\delta_A = 1_{\mathcal{L}}$ then we say that the set A is $\mathcal{L}F$-strongly bounded.

Lemma 3.3. [10] *The set $A \subseteq X$ is $\mathcal{L}F$-strongly bounded if and only if for arbitrary negation $\mathcal{N}(r)$ and each $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ there exists $t > 0$ such that $\mathcal{M}(x, y, t) >_L \mathcal{N}(r)$ for all $x, y \in A$.*

Theorem 3.2. *Every compact subset A of a $\mathcal{L}$-fuzzy metric space X is $\mathcal{L}F$-strongly bounded.*

Proof. Let A be a compact subset of X. Fix $t > 0$ and $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$. Consider an open cover $\{B(x, r, t) : x\in A\}$ of A. Since A is compact, there exist $x_1, x_2, ..., x_n \in A$ such that $A \subseteq \cup_{i=1}^{n} B(x_i, r, t)$. Let $x, y \in A$. Then $x \in B(x_i, r, t)$ and $y \in B(x_j, r, t)$ for some i, j. Thus we have $\mathcal{M}(x, x_i, t) >_L \mathcal{N}(r)$ and $\mathcal{M}(y, x_j, t) >_L \mathcal{N}(r)$. Now let $\alpha = \min\{\mathcal{M}(x_i, x_j, t) : 1 \leq i, j \leq n\}$. Then $\alpha \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and there exists $s \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{T}^2(\mathcal{N}(r), \mathcal{N}(r), \alpha) >_L \mathcal{N}(s)$. Therefore,

$$\begin{aligned} \mathcal{M}(x, y, t) &\geq {}_L\mathcal{T}^2\left(\mathcal{M}\left(x, x_i, \frac{t}{3}\right), \mathcal{M}\left(x_i, x_j, \frac{t}{3}\right), \mathcal{M}\left(x_j, y, \frac{t}{3}\right)\right) \\ &\geq {}_L\mathcal{T}^2(\mathcal{N}(r), \mathcal{N}(r), \alpha) >_L \mathcal{N}(s) \end{aligned}$$

for all $x, y \in A$. Hence A is $\mathcal{L}F$-strongly bounded. □

Remark 3.1. In an $\mathcal{L}$-fuzzy metric space every compact set closed and $\mathcal{L}F$-strongly bounded.

Lemma 3.4. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. If $t > 0$ and $r, s \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{T}(\mathcal{N}(s), \mathcal{N}(s)) \geq_L \mathcal{N}(r)$, then $\overline{B(x, s, t/2)} \subset B(x, r, t)$.*

Proof. Let $y \in \overline{B(x,s,t/2)}$ and let $B(y,s,t/2)$ be an open ball with center $x \in X$ and radius $s \in L\backslash\{0_{\mathcal{L}},1_{\mathcal{L}}\}$. Since $B(y,s,t/2) \cap B(y,s,t/2) \neq \varnothing$, there exists $z \in B(y,s,t/2) \cap B(x,s,t/2)$. Then we have

$$\begin{aligned}\mathcal{M}(x,y,t) &\geq {}_L\mathcal{T}\left(\mathcal{M}\left(x,z,\frac{t}{2}\right),\mathcal{M}\left(y,z,\frac{t}{2}\right)\right)\\ &> {}_L\mathcal{T}(\mathcal{N}(s),\mathcal{N}(s)) \geq_L \mathcal{N}(r).\end{aligned}$$

Hence $z \in B(x,r,t)$ and thus $\overline{B(x,s,t/2)} \subset B(x,r,t)$. □

Theorem 3.3. *A subset A of an $\mathcal{L}$-fuzzy metric space $(X,\mathcal{M},\mathcal{T})$ is nowhere dense if and only if every nonempty open set in X contain an open ball whose closure is disjoint from A.*

Proof. Let U be a nonempty open subset of X. Then there exists a nonempty open set V such that $V \subset U$ and $V \cap \overline{A} \neq \varnothing$. Let $x \in V$. Then there exist $t > 0$ and $r \in L\backslash\{0_{\mathcal{L}},1_{\mathcal{L}}\}$ such that $B(x,r,t) \subset V$. Choose $s \in L\backslash\{0_{\mathcal{L}},1_{\mathcal{L}}\}$ such that $\mathcal{T}(\mathcal{N}(s),\mathcal{N}(s)) \geq_L \mathcal{N}(r)$. By Lemma 3.4 $\overline{B(x,s,t/2)} \subset B(x,r,t)$. Thus $B(x,s,t/2) \subset U$ and $\overline{B(x,s,t/2)} \cap A = \varnothing$.

Conversely, suppose A is not nowhere dense. Then $int(\overline{A}) \neq \varnothing$, so there exists a nonempty set U such that $U \subset \overline{A}$. Let $B(x,r,t)$ be an open ball such that $B(x,r,t) \subset U$. Then $\overline{B(x,r,t)} \cap A \neq \varnothing$. This is a contradiction. □

Theorem 3.4. (Baire's Theorem) *Let $\{U_n : n \in \mathbb{N}\}$ be a sequence of dense open subsets of a complete $\mathcal{L}$-fuzzy metric space $(X,\mathcal{M},\mathcal{T})$. Then $\cap_{n\in\mathbb{N}}U_n$ is also dense in X.*

Proof. Let V be a nonempty open set of X. Since U_1 is dense in X, $V \cap U_1 \neq \varnothing$. Let $x_1 \in V \cap U_1$. Since $V \cap U_1$ is open, there exist $r_1 \in L\backslash\{0_{\mathcal{L}},1_{\mathcal{L}}\}$ and $t_1 > 0$ such that $B(x_1,r_1,t_1) \subset V \cap U_1$. Choose $r'_1 <_L r_1$ and $t'_1 = \min\{t_1, 1\}$ such that $\overline{B(x_1,r'_1,t'_1)} \subset V \cap U_1$. Since U_2 is dense in X, $B(x_1,r'_1,t'_1) \cap U_2 \neq \varnothing$. Let $x_2 \in B(x_1,r'_1,t'_1) \cap U_2$. Since $B(x_1,r'_1,t'_1) \cap U_2$ is open, there exist $r_2 \in L\backslash\{0_{\mathcal{L}},1_{\mathcal{L}}\}$ and $t_2 > 0$ such that $B(x_2,r_2,t_2) \subset B(x_1,r'_1,t'_1) \cap U_2$. Choose $r'_2 <_L r_2$ and $t'_2 = \min\{t_2, 1/2\}$ such that $\overline{B(x_2,r'_2,t'_2)} \subset B(x_1,r'_1,t'_1) \cap U_2$. Contiuning in this manner we obtain a sequence $\{x_n\}$ in X and a sequence $\{t'_n\}$ such that $0 < t'_n < \frac{1}{n}$ and $\overline{B(x_n,r'_n,t'_n)} \subset B(x_{n-1},r'_{n-1},t'_{n-1}) \cap U_n$ in which $r'_n \longrightarrow 0_{\mathcal{L}}$.

Now we claim that $\{x_n\}$ is a Cauchy sequence. For a given $t > 0$ and $\varepsilon \in L\backslash\{0_{\mathcal{L}},1_{\mathcal{L}}\}$, choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < t$ and $\frac{1}{n_0} <_L \varepsilon$. Then for $n \geq n_0$ and $m \geq n$,

$$\mathcal{M}(x_n,x_m,t) \geq_L \mathcal{M}\left(x_n,x_m,\frac{1}{n}\right) \geq_L \mathcal{N}(\frac{1}{n}) >_L \mathcal{N}(\varepsilon).$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $x_n \longrightarrow x$. Since $x_k \in B(x_n,r'_n,t'_n)$ for $k \geq n$, we obtain $x \in \overline{B(x_n,r'_n,t'_n)}$. Hence $x \in \overline{B(x_n,r'_n,t'_n)} \subset B(x_{n-1},r'_{n-1},t'_{n-1}) \cap U_n$ for all $n \in \mathbb{N}$. Therefore $V \cap (\cap_{n\in\mathbb{N}}U_n) \neq \varnothing$. Hence $\cap_{n\in\mathbb{N}}U_n$ is dense in X. □

Remark 3.2. Since any complete $\mathcal{L}$-fuzzy metric space cannot be represented as the union of a sequence of nowhere dense sets, it is not of the first category.

Definition 3.9. Let $(X,\mathcal{M},\mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. A collection of sets $\{F_n\}_{n\in I}$ is said have $\mathcal{L}$-fuzzy diameter zero if and only if for arbitrary negation $\mathcal{N}(r)$ and each $r\in L\backslash\{0_\mathcal{L},1_\mathcal{L}\}$ and $t>0$, there exists $n\in I$ such that $\mathcal{M}(x,y,t)>_L \mathcal{N}(r)$ for all $x,y\in F_n$.

Remark 3.3. A nonempty subset F of an $\mathcal{L}$-fuzzy metric space X has $\mathcal{L}$-fuzzy diameter zero if and only if F is a singleton set.

Theorem 3.5. *An $\mathcal{L}$-fuzzy metric space $(X,\mathcal{M},\mathcal{T})$ is complete if and only if every nested sequence of nonempty closed sets $\{F_n\}_{n=1}^{\infty}$ with $\mathcal{L}$-fuzzy diameter zero we have nonempty intersection.*

Proof. First suppose that the given condition is satisfied. We claim that $(X,\mathcal{M},\mathcal{T})$ is complete. Let $\{x_n\}$ be a Cauchy sequence in X. Take $A_n=\{x_n,x_{n+1},x_{n+2},...\}$ and $F_n=\overline{A_n}$, then we claim that $\{F_n\}$ has $\mathcal{L}$-fuzzy diameter zero. For given $s\in L\backslash\{0_\mathcal{L},1_\mathcal{L}\}$, $t>0$, we can find a $r\in L\backslash\{0_\mathcal{L},1_\mathcal{L}\}$, such that $\mathcal{T}^2(\mathcal{N}(r),\mathcal{N}(r),\mathcal{N}(r))>_L \mathcal{N}(s)$. Since $\{x_n\}$ is a Cauchy sequence, for $r\in L\backslash\{0_\mathcal{L},1_\mathcal{L}\}$, $t>0$, there exists $n_0\in\mathbb{N}$ such that $\mathcal{M}\left(x_m,x_n,\frac{t}{3}\right)>_L \mathcal{N}(r)$ for all $m\geq n\geq n_0$ $(n\geq m\geq n_0)$. Therefore, $\mathcal{M}\left(x,y,\frac{t}{3}\right)>_L \mathcal{N}(r)$ for all $x,y\in A_{n_0}$. Let $x,y\in F_{n_0}$. Then there exist sequences $\{x_n'\}$ and $\{y_n'\}$ in A_{n_0} such that $x_n'\longrightarrow x$ and $y_n'\longrightarrow y$. Hence $x_n'\in B\left(x,r,\frac{t}{3}\right)$ and $y_n'\in B\left(y,r,\frac{t}{3}\right)$ for sufficiently large n. Now

$$\begin{aligned}\mathcal{M}(x,y,t) &\geq {}_L\mathcal{T}^2\left(\mathcal{M}\left(x,x_n',\frac{t}{3}\right),\mathcal{M}\left(x_n',y_n',\frac{t}{3}\right),\mathcal{M}\left(y_n',y,\frac{t}{3}\right)\right)\\ &> {}_L\mathcal{T}^2(\mathcal{N}(r),\mathcal{N}(r),\mathcal{N}(r))\\ &> {}_L\mathcal{N}(s).\end{aligned}$$

Therefore, $\mathcal{M}(x,y,t)>_L\mathcal{N}(s)$ for all $x,y\in F_{n_0}$. Thus $\{F_n\}$ has $\mathcal{L}$-fuzzy diameter zero. Hence by hypothesis $\cap_{n=1}^{\infty}F_n$ is nonempty.

Take $x\in\cap_{n=1}^{\infty}F_n$. Then for $r\in L\backslash\{0_\mathcal{L},1_\mathcal{L}\}$, $t>0$, there exists $n_1\in\mathbb{N}$ such that $\mathcal{M}(x_n,x,t)>_L\mathcal{N}(r)$ for all $n\geq n_1$. Therefore, for each $t>0$, $\mathcal{M}(x_n,x,t)\rightarrow 1_\mathcal{L}$ as $n\rightarrow\infty$. Hence $x_n\rightarrow x$. Therefore, $(X,\mathcal{M},\mathcal{T})$ is a complete $\mathcal{L}$-fuzzy metric space.

Conversely, suppose that $(X,\in\mathcal{M},\mathcal{T})$ is $\mathcal{L}$-fuzzy complete and $\{F_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty closed sets with $\mathcal{L}$-fuzzy diameter zero.
Let $x_n\in F_n$, $n=1,2,....$ Since $\{F_n\}$ has $\mathcal{L}$-fuzzy diameter zero, for $r\in L\backslash\{0_\mathcal{L},1_\mathcal{L}\}$, $t>0$, there exists $n_0\in\mathbb{N}$ such that $\mathcal{M}(x,y,t)>_L\mathcal{N}(r)$ for all $x,y\in F_{n_0}$. Therefore, $\mathcal{M}(x_n,x_m,\frac{t}{3})>_L\mathcal{N}(r)$ for all $m\geq n\geq n_0$ $(n\geq m\geq n_0)$. Since $x_n\in F_n\subset F_{n_0}$ and $x_n\in F_m\subset F_{n_0}$, $\{x_n\}$ is a Cauchy sequence. But $(X,\mathcal{M},\mathcal{T})$ a complete $\mathcal{L}$-fuzzy metric space and hence x_n converges to x for some $x\in X$. Now for each fixed n, $x_k\in F_n$ for all $k\geq n$. Therefore, $x\in\overline{F_n}=F_n$ for every n, and hence $x\in\cap_{n=1}^{\infty}F_n$. This completes the proof. □

Remark 3.4. The element $x\in\cap_{n=1}^{\infty}F_n$ is unique. For if there are two elements $x,y\in\cap_{n=1}^{\infty}F_n$, since $\{F_n\}_{n=1}^{\infty}$ has $\mathcal{L}$-fuzzy diameter zero, for each fixed $t>0$, $\mathcal{M}(x,y,t)>_L\mathcal{N}(\frac{1}{n})$ for each n. This implies $\mathcal{M}(x,y,t)=1_\mathcal{L}$ and hence $x=y$.

Theorem 3.6. *Every separable $\mathcal{L}$-fuzzy metric space is second countable.*

Proof. Let $(X,\mathcal{M},\mathcal{T})$ be the given separable $\mathcal{L}$-fuzzy metric space. Let $\mathcal{A}=\{a_n : n\in\mathbb{N}\}$ be a countable dense subset of X. Consider $\mathcal{B}=\{B(a_j,r_k,1/k): j,k\in\mathbb{N}\}$

where $r_k \longrightarrow 0_{\mathcal{L}}$. Then B is countable. We claim that B is a base for the family of all open sets in X. Let G be an arbitrary open set in X. Let $x \in G$, then there exist $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, $t > 0$ such that $B(x, r, t) \subset G$. Since $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, we can find $s <_L r$ such that $\mathcal{T}(\mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r)$. Choose $m \in \mathbb{N}$ such that $1/m <_L \min\{s, t/2\}$. Since A is dense in X, there exists $a_j \in A$ such that $a_j \in B(x, r_m, 1/m)$ where $r_m \longrightarrow 0_{\mathcal{L}}$. Now if $y \in B(a_j, r_m, 1/m)$ then

$$\begin{aligned} \mathcal{M}(x, y, t) &\geq {}_L\mathcal{T}\left(\mathcal{M}\left(x, a_j, \frac{t}{2}\right), \mathcal{M}\left(y, a_j, \frac{t}{2}\right)\right) \\ &\geq {}_L\mathcal{T}\left(\mathcal{M}\left(x, a_j, \frac{1}{m}\right), \mathcal{M}\left(y, a_j, \frac{1}{m}\right)\right) \\ &\geq {}_L\mathcal{T}\left(\mathcal{N}(\frac{1}{m}), \mathcal{N}(\frac{1}{m})\right) \\ &\geq {}_L\mathcal{T}\left(\mathcal{N}(s), \mathcal{N}(s)\right) >_L \mathcal{N}(r). \end{aligned}$$

Thus, $y \in B(x, r, t)$ and hence $\mathcal{B}$ is a base. □

Remark 3.5. Every subspace of separable $\mathcal{L}$-fuzzy metric space is separable.

Proof. Let $(X, \mathcal{M}, \mathcal{T})$ be the given separable $\mathcal{L}$-fuzzy metric space and Y be a subspace of X. Let $A = \{x_n : n \in \mathbb{N}\}$ be a countable dense subset of X. For arbitrary but fixed $n, k \in \mathbb{N}$, if there are points $x \in X$ such that $\mathcal{M}(x_n, y, 1/k) >_L \mathcal{N}(\frac{1}{k})$, choose one of them and denote it by x_{n_k}. Let $B = \{x_{n_k} : n, k \in \mathbb{N}\}$, then B is countable. Now we claim that $Y \subset \overline{B}$. Let $y \in Y$. Given $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $t > 0$, we can find a $k \in \mathbb{N}$ such that $\mathcal{T}\left(\mathcal{N}(\frac{1}{k}), \mathcal{N}(\frac{1}{k})\right) >_L \mathcal{N}(r)$. Since A is dense in X, there exists an $m \in \mathbb{N}$ such that $\mathcal{M}(x_m, y, 1/k) >_L \mathcal{N}(\frac{1}{k})$. But by definition of B, there exists $x_{m_k} \in A$ such that $\mathcal{M}(x_{m_k}, x_m, 1/k) >_L \mathcal{N}(\frac{1}{k})$. Now

$$\begin{aligned} \mathcal{M}(x_{mk}, y, t) &\geq {}_L\mathcal{T}\left(\mathcal{M}\left(x_{m_k}, x_m, \frac{t}{2}\right), \mathcal{M}\left(x_m, y, \frac{t}{2}\right)\right) \\ &\geq {}_L\mathcal{T}\left(\mathcal{M}\left(x_{m_k}, x_m, \frac{1}{k}\right), \mathcal{M}\left(x_m, y, \frac{1}{k}\right)\right) \\ &\geq {}_L\mathcal{T}\left(\mathcal{N}(\frac{1}{k}), \mathcal{N}(\frac{1}{k})\right) >_L \mathcal{N}(r). \end{aligned}$$

Thus $y \in \overline{B}$ and hence Y is separable. □

Definition 3.10. Let X be any nonempty set and $(Y, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then a sequence $\{f_n\}$ of functions from X to Y is said to be **converge uniformly** to a function f from X to Y if for given $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}(f_n(x), f(x), t) >_L \mathcal{N}(r)$ for all $n \geq n_0$ and for all $x \in X$.

Theorem 3.7. (Uniform Limit Theorem) *Let $f_n : X \longrightarrow Y$ be a sequence of continuous functions from a topological space X to an $\mathcal{L}$-fuzzy metric space Y. If $\{f_n\}$ converges uniformly to f then f is continuous.*

Proof. Let X be a given topological space and $(Y, \mathcal{M}, \mathcal{T})$ be the given $\mathcal{L}$-fuzzy metric space. For any open set V in Y, let $x_0 \in f^{-1}(V)$ and let $y_0 \in f(x_0)$. Since

V is open, we can find $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $t > 0$ such that $B(y_0, r, t) \subset V$. Since $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, we can find $s <_L r$ such that $\mathcal{T}^2(\mathcal{N}(s), \mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r)$.

Since $\{f_n\}$ converges uniformly to f, given $s \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}(f_n(x), f(x), t/3) >_L \mathcal{N}(s)$ for all $n \geq n_0$. Since, for all $n \in \mathbb{N}$, f_n is continuous we can find a neighborhood U of x_0, for a fixed $n \geq n_0$, such that $f_n(U) \subset B(f_n(x_0), s, t/3)$. Hence $\mathcal{M}(f_n(x), f_n(x_0), t/3) >_L \mathcal{N}(s)$ for all $x \in U$. Now

$$\begin{aligned} \mathcal{M}(f(x), f(x_0), t) &\geq {}_L\mathcal{T}^2 \begin{pmatrix} \mathcal{M}\left(f(x), f_n(x), \frac{t}{3}\right), \mathcal{M}\left(f_n(x), f_n(x_0), \frac{t}{3}\right), \\ \mathcal{M}\left(f_n(x_0), f(x_0), \frac{t}{3}\right) \end{pmatrix} \\ &\geq {}_L\mathcal{T}^2(\mathcal{N}(s), \mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r). \end{aligned}$$

Thus, $f(x) \in B(f(x_0), r, t) \subset V$ for all $x \in U$. Hence $f(U) \subset V$ and then f is continuous. □

References

[1] Atanassov, K. T., *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20** (1986), 87-96

[2] Deng, Z. K., *Fuzzy pseudo-metric spaces*, J. Math. Anal. Appl. **86** (1982), 74-95

[3] G. Deschrijver, G., Cornelis, C., Kerre, E. E., *On the representation of intuitionistic fuzzy t-norms and t -conorms*, IEEE Trans. Fuzzy Systems **12** (2004), 45-61

[4] Deschrijver, G., Kerre, E. E., *On the relationship between some extensions of fuzzy set theory*, Fuzzy Sets and Systems **133** (2003), 227-235

[5] Erceg, M. A., *Metric spaces in fuzzy set theory*, J. Math. Anal. Appl. **69** (1979), 205-230

[6] George, A. and Veeramani, P., *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems **64** (1994), 395-399

[7] George, A. and Veeramani, P., *On some results of analysis for fuzzy metric spaces*, Fuzzy Sets and Systems **90** (1997), 365-368

[8] Goguen, J., *$\mathcal{L}$-fuzzy sets*, J. Math. Anal. Appl. **18** (1967), 145-174

[9] Grabiec, M., *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems **27** (1988), 385-389

[10] Ješić, S. N. and Babačev, N. A., *Common fixed point theorems in intuitionistic fuzzy metric spaces and $\mathcal{L}$-fuzzy metric spaces with nonlinear contractive condition*, Chaos, Solitons & Fractals **37** (2008), 675-687

[11] Kaleva, O. and Seikkala, S., *On fuzzy metric spaces*, Fuzzy Sets and Systems **12** (1984), 225-229

[12] Kramosil, O. and Michalek, J., *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326-334

[13] Park, J. H., *Intuitionistic fuzzy metric spaces*, Chaos, Solitons & Fractals **22** (2004), 1039-1046

[14] Saadati, R., Park, J. H., *On the intuitionistic topological spaces*, Chaos, Solitons & Fractals **27** (2006), 331-344

[15] Saadati, R., Razani, A., Adibi, H., *A common fixed point theorem in $\mathcal{L}$-fuzzy metric spaces*, Chaos, Solitons & Fractals **33** (2007), 358-363

[16] Saadati, R., *On the $\mathcal{L}$-fuzzy topological spaces* Chaos, Solitons & Fractals, **37** (2008), 1419-1426

[17] Zadeh, L. A., *Fuzzy sets*, Inform. and Control **8** (1965), 338-353

Department of Mathematics
Faculty of Science and Arts
Gazi University
teknikokullar, 06500 Ankara, Turkey
E-mail address: hakanefe@gazi.edu.tr