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On the global attractivity of difference equation of higher order

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ABSTRACT.
In this paper we investigate the global convergence result and boundedness of solutions of the recursive sequence

$$x_{n+1} = \frac{ax_n^p + b\prod_{r=1}^{p} x_{n-r}}{cx_n^p + d\prod_{r=1}^{p} x_{n-r}}, \quad n = 0, 1, ...$$

where the parameters a, b, c and d are positive real numbers and the initial values $x_{-p}, x_{-p+1}, \ldots, x_{-1}$ and x_0 are arbitrary positive numbers.

1. INTRODUCTION

Our goal in this paper is to investigate the global stability character and boundedness of solutions of the recursive sequence

$$x_{n+1} = \frac{ax_n^p + b\prod_{r=1}^{p} x_{n-r}}{cx_n^p + d\prod_{r=1}^{p} x_{n-r}}, \tag{1.1}$$

where a, b, c and $d \in (0, \infty)$ with the initial values $x_{-p}, x_{-p+1}, ..., x_{-1}$ and $x_0 \in (0, \infty)$, where p is a positive integer.

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I$$

be a continuously differentiable function. Then for every set of initial values $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [11].

A point $\overline{x} \in I$ is called an equilibrium point of (1.2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of (1.2), or equivalently, $\overline{x}$ is a fixed point of F.

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Definition 1.1. The difference equation (1.2) is said to be *persistent* if there exist numbers m and M with $0 < m \leq M < \infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in (0, \infty)$ there exists a positive integer N which depends on the initial conditions such that

$$m \leq x_n \leq M \quad \text{for all } n \geq N.$$

Definition 1.2. (Stability) Let I be some interval of real numbers.

(i) The equilibrium point $\overline{x}$ of (1.2) is *locally stable* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \varepsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of (1.2) is *locally asymptotically stable* if $\overline{x}$ is locally stable solution of (1.2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}$, $x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of (1.2) is a *global attractor* if for all $x_{-k}, x_{-k+1}, ..., x_{-1}$, $x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of (1.2) is *globally asymptotically* stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of (1.2).

(v) The equilibrium point $\overline{x}$ of (1.2) is *unstable* if $\overline{x}$ is not locally stable.

The linearized equation of (1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, \ldots, \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.3}$$

Theorem A. [10] *Assume that* $p, q \in \mathbb{R}$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ... \,.$$

Remark 1.1. Theorem A can be easily extended to a general linear equation of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ... \tag{1.4}$$

where $p_1, p_2, \ldots, p_k \in \mathbb{R}$ and $k \in \{1, 2, ...\}$. Then (1.4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Recently there has been a lot of interest in studying the global attractivity, boundedness character and the periodic nature of nonlinear difference equations. For some results in this area, see for example [10–15].

Many researchers have investigated the behavior of the solution of difference equations for example: Camouzis et al. [1] investigated the behavior of solutions of the rational recursive sequence

$$x_{n+1} = \frac{\beta x_n^2}{1 + x_{n-1}^2}.$$

In [2] Elabbasy et al. investigated the global stability character, boundedness and the periodicity of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{A x_n + B x_{n-1} + C x_{n-2}}.$$

Elabbasy et al. [4] investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Elabbasy et al. [5] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{d x_{n-l} x_{n-k}}{c x_{n-s} - b} + a.$$

Grove, Kulenovic and Ladas [8] presented a summary of a recent work and a large of open problems and conjectures on the third order rational recursive sequence of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}}.$$

Kalabusic and Kulenovic [9] investigated the global character of solutions of the nonlinear, third order, rational difference equation

$$x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1} + D x_{n-2}}.$$

In [14] Kulenovic, G. Ladas and W. Sizer studied the global stability character and the periodic nature of the recursive sequence

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{\gamma x_n + \delta x_{n-1}}.$$

Kulenovic and Ladas [15] studied the second-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}.$$

Other related results on rational difference equations can be found in refs. [3], [6-7].

2. Local stability of the equilibrium point

In this section we study the local stability character of the solutions of (1.1). Equation (1.1) has a unique positive equilibrium point and is given by

$$\overline{x} = \frac{a+b}{c+d}.$$

Let $f : (0,\infty)^{p+1} \longrightarrow (0,\infty)$ be a continuous function defined by

$$f(u_0,u_1,u_2,...,u_p) = \frac{au_0^p + bu_1u_2...u_p}{cu_0^p + du_1u_2...u_p}. \tag{2.5}$$

We have

$$\frac{\partial f(u_0,u_1,u_2,\ldots,u_p)}{\partial u_0} = \frac{pu_0^{p-1}(u_1u_2\ldots u_p)(ad-bc)}{(cu_0^p + du_1u_2\ldots u_p)^2}$$

$$\frac{\partial f(u_0,u_1,u_2,\ldots,u_p)}{\partial u_1} = \frac{u_0^p(u_2u_3\ldots u_p)(bc-ad)}{(cu_0^p + du_1u_2\ldots u_p)^2}$$

$$\frac{\partial f(u_0,u_1,u_2,...,u_p)}{\partial u_2} = \frac{u_0^p(u_1u_3...u_p)(bc-ad)}{(cu_0^p + du_1u_2...u_p)^2}$$

$$\ldots$$

$$\ldots$$

$$\ldots$$

$$\frac{\partial f(u_0,u_1,u_2,...,u_p)}{\partial u_p} = \frac{u_0^p(u_1u_2...u_{p-1})(bc-ad)}{(cu_0^p + du_1u_2...u_p)^2}.$$

Then we see that

$$\frac{\partial f(\overline{x},\overline{x},\overline{x},...,\overline{x})}{\partial u_0} = \frac{p\overline{x}^{p-1}\overline{x}^p(ad-bc)}{(c\overline{x}^p + d\overline{x}^p)^2} = \frac{p(ad-bc)}{(c+d)^2\,\overline{x}} = \frac{p(ad-bc)}{(c+d)(a+b)}$$

$$\frac{\partial f(\overline{x},\overline{x},\overline{x},...,\overline{x})}{\partial u_1} = \frac{\overline{x}^p\overline{x}^{p-1}(bc-ad)}{(c+d)^2\,\overline{x}^{2p}} = \frac{(bc-ad)}{(c+d)(a+b)}$$

$$\frac{\partial f(\overline{x},\overline{x},\overline{x},...,\overline{x})}{\partial u_2} = \frac{\overline{x}^p\overline{x}^{p-1}(bc-ad)}{(c+d)^2\,\overline{x}^{2p}} = \frac{(bc-ad)}{(c+d)(a+b)},$$

$$\ldots$$

$$\ldots$$

$$\ldots$$

$$\frac{\partial f(\overline{x},\overline{x},\overline{x},...,\overline{x})}{\partial u_p} = \frac{\overline{x}^p\overline{x}^{p-1}(bc-ad)}{(c+d)^2\,\overline{x}^{2p}} = \frac{(bc-ad)}{(c+d)(a+b)},$$

and therefore the linearized equation of (1.1) about $\overline{x}$ is

$$y_{n+1} + \sum_{i=0}^{p} d_i y_{n-i} = 0, \tag{2.6}$$

where $d_i = -f_{u_i}(\overline{x}, \overline{x}, \ldots, \overline{x})$ for $i = 0, 1, \ldots, p$, whose characteristic equation is

$$\lambda^{p+1} + \sum_{i=0}^{p} d_i \lambda^i = 0. \tag{2.7}$$

Theorem 2.1. *Assume that*

$$2p\,|ad - bc| < (c + d)\,(a + b).$$

Then the positive equilibrium point of (1.1) *is locally asymptotically stable.*

Proof. It follows from Remark 1.1 that (1.4) is asymptotically stable if all roots of (2.7) lie in the open disc $|\lambda| < 1$ that is if

$$\left|\frac{p\,(ad - bc)}{(c + d)\,(a + b)}\right| + \left|\frac{(bc - ad)}{(c + d)\,(a + b)}\right| + \left|\frac{(bc - ad)}{(c + d)\,(a + b)}\right| + \cdots + \left|\frac{(bc - ad)}{(c + d)\,(a + b)}\right| < 1$$

$$\frac{p\,|ad - bc|}{(c + d)\,(a + b)} + \frac{p\,|bc - ad|}{(c + d)\,(a + b)} < 1,$$

or

$$\frac{2p\,|ad - bc|}{(c + d)\,(a + b)} < 1.$$

This completes the proof. ☐

3. Boundedness of solutions

Here we study the permanence of (1.1).

Theorem 3.2. *Every solution of* (1.1) *is bounded and persists.*

Proof. Let $\{x_n\}_{n=-p}^{\infty}$ be a solution of (1.1). It follows from (1.1) that

$$x_{n+1} = \frac{ax_n^p + b\prod_{r=1}^{p} x_{n-r}}{cx_n^p + d\prod_{r=1}^{p} x_{n-r}} = \frac{ax_n^p}{cx_n^p + d\prod_{r=1}^{p} x_{n-r}} + \frac{b\prod_{r=1}^{p} x_{n-r}}{cx_n^p + d\prod_{r=1}^{p} x_{n-r}}$$

$$\leq \frac{ax_n^p}{cx_n^p} + \frac{b\prod_{r=1}^{p} x_{n-r}}{d\prod_{r=1}^{p} x_{n-r}}.$$

Then

$$x_n \leq \frac{a}{c} + \frac{b}{d} = M \quad \text{for all} \quad n \geq 1. \tag{3.8}$$

Now we wish to show that there exists $m > 0$ such that

$$x_n \geq m \quad \text{for all} \quad n \geq 1.$$

The transformation

$$x_n = \frac{1}{y_n},$$

will reduce equation (1.1) to the equivalent form

$$\frac{1}{y_{n+1}} = \frac{\dfrac{a}{y_n^p} + \dfrac{b}{\prod\limits_{r=1}^{p} y_{n-r}}}{\dfrac{c}{y_n^p} + \dfrac{d}{\prod\limits_{r=1}^{p} y_{n-r}}} = \frac{a\prod\limits_{r=1}^{p} y_{n-r} + by_n^p}{c\prod\limits_{r=1}^{p} y_{n-r} + dy_n^p},$$

or

$$y_{n+1} = \frac{dy_n^p + c\prod\limits_{r=1}^{p} y_{n-r}}{by_n^p + a\prod\limits_{r=1}^{p} y_{n-r}} = \frac{dy_n^p}{by_n^p + a\prod\limits_{r=1}^{p} y_{n-r}} + \frac{c\prod\limits_{r=1}^{p} y_{n-r}}{by_n^p + a\prod\limits_{r=1}^{p} y_{n-r}}.$$

It follows that

$$y_{n+1} \leq \frac{d}{b} + \frac{c}{a} = \frac{bc+ad}{ab} = H \quad \text{for all} \quad n \geq 1.$$

Thus we get

$$x_n = \frac{1}{y_n} \geq \frac{1}{H} = \frac{ab}{bc+ad} = m \quad \text{for all} \quad n \geq 1. \tag{3.9}$$

From (3.8) and (3.9) we see that

$$m \leq x_n \leq M \quad \text{for all} \quad n \geq 1.$$

Therefore every solution of (1.1) is bounded and persists. □

4. Global stability

In this section we investigate the global asymptotic stability of (1.1).

Lemma 4.1. *(a) If $\frac{a}{c} > \frac{b}{d}$ then the function $f(u_0, u_1, u_2, ..., u_p)$ is non-decreasing in the first variable u_0 and non-increasing in all others variables.*

(b) If $\frac{a}{c} < \frac{b}{d}$ then the function $f(u_0, u_1, u_2, ..., u_p)$ is non-increasing in the first variable u_0 and non-decreasing in all others variables.

Proof. The proof follows from the calculations after formula (2.5). □

Theorem 4.3. *The equilibrium point $\overline{x}$ is a global attractor of (1.1) if one of the following statements holds*

$$(1) \quad ad \geq bc \ \text{ and } \ 2pc + d \geq (2p-1)c\left[\frac{ad}{bc}\right]^{2p-1}. \tag{4.10}$$

$$(2) \quad ad \leq bc \ \text{ and } \ 2pd + c \geq (2p-1)d\left[\frac{bc}{ad}\right]^{2p-1}. \tag{4.11}$$

Proof. Let $\{x_n\}_{n=-p}^{\infty}$ be a solution of (1.1) and again let f be a function defined by (2.5).

We will prove the theorem when Case (1) is true and since the proof of Case (2) is similar it is left to the reader.

Assume that (3.9) is true, then using Lemma 4.1, part (a); we obtain

$$x_{n+1} = \frac{ax_n^p + b\prod_{r=1}^{p} x_{n-r}}{cx_n^p + d\prod_{r=1}^{p} x_{n-r}} \le \frac{ax_n^p + b(0)}{cx_n^p + d(0)} = \frac{a}{c}\,.$$

Then

(4.12) $$x_n \le \frac{a}{c} = H \quad \text{for all} \quad n \ge 1.$$

(4.13) $$x_{n+1} = \frac{ax_n^p + b\prod_{r=1}^{p} x_{n-r}}{cx_n^p + d\prod_{r=1}^{p} x_{n-r}} \ge \frac{a(0) + b\prod_{r=1}^{p} x_{n-r}}{c(0) + d\prod_{r=1}^{p} x_{n-r}} = \frac{b}{d} = h \quad \text{for all} \quad n \ge 1.$$

Then from (4.12) and (4.13), we see that

$$0 < h = \frac{b}{d} \le x_n \le \frac{a}{c} = H \quad \text{for all} \quad n \ge 1.$$

Let $\{x_n\}_{n=0}^{\infty}$ solution of (1.1) with

$$I := \lim_{n\to\infty} \inf x_n \quad \text{and} \quad S := \lim_{n\to\infty} \sup x_n.$$

It suffices to show that $I = S$.

Now it follows from (1.1) that

$$I \ge f(I, S, S, .., S),$$

or

$$I \ge \frac{aI^p + bS^p}{cI^p + dS^p},$$

and so

$$aI^p + bS^p - cI^{p+1} \le dS^p I,$$

or

(4.14) $$aI^{2p-1} + bS^p I^{p-1} - cI^{2p} \le dS^p I^p.$$

Similarly, we see from Eq.(1) that

$$S \le f(S, I, I, ..., I),$$

or,

$$S \le \frac{aS^p + bI^p}{cS^p + dI^p},$$

and so

$$aS^p + bI^p - cS^{p+1} \ge dSI^p,$$

or

(4.15) $$aS^{2p-1} + bI^p S^{p-1} - cS^{2p} \ge dS^p I^p.$$

Therefore it follows from (4.14) and (4.15) that

$$aS^{2p-1} + bI^pS^{p-1} - cS^{2p} \geq dS^pI^p \geq aI^{2p-1} + bS^pI^{p-1} - cI^{2p}$$

$$c(I^{2p} - S^{2p}) + bS^{p-1}I^{p-1}(I - S) - a(I^{2p-1} - S^{2p-1}) \geq 0,$$

which is equivalent to

$$c(I-S)(I^{2p-1} + I^{2p-2}S+I^{2p-3}S^2 + ... + S^{2p-2}I+S^{2p-1})+bS^{p-1}I^{p-1}(I-S)$$
$$-a(I - S)(I^{2p-2} + I^{2p-3}S + ... + S^{2p-3}I + S^{2p-2}) \geq 0,$$

which holds if and only if

$$(I - S)\left[\begin{array}{c} c(I^{2p-1} + I^{2p-2}S + ... + S^{2p-2}I + S^{2p-1}) \\ +bS^{p-1}I^{p-1} - a(I^{2p-2} + I^{2p-3}S + ... + S^{2p-3}I + S^{2p-2}) \end{array}\right] \geq 0,$$

and so

$$I \geq S \quad \text{if} \quad \left[\begin{array}{c} c(I^{2p-1} + I^{2p-2}S + ... + S^{2p-2}I + S^{2p-1}) \\ +bS^{p-1}I^{p-1} - a(I^{2p-2} + I^{2p-3}S + ... + S^{2p-3}I + S^{2p-2}) \end{array}\right] \geq 0.$$

Now, we know by (4.10) that

$$\begin{aligned} 2pc + d &\geq (2p-1)c\left[\frac{ad}{bc}\right]^{2p-1} \\ 2pc\left[\frac{b}{d}\right]^{2p-1} + d\left[\frac{b}{d}\right]^{2p-1} &\geq (2p-1)c\left[\frac{a}{c}\right]^{2p-1} \\ 2pc\left[\frac{b}{d}\right]^{2p-1} + b\left[\frac{b}{d}\right]^{2p-2} &\geq (2p-1)a\left[\frac{a}{c}\right]^{2p-2} \end{aligned}$$

$$c(I^{2p-1}+I^{2p-2}S + ... + S^{2p-2}I+S^{2p-1})+bS^{p-1}I^{p-1} \geq 2pc\left[\frac{b}{d}\right]^{2p-1}+b\left[\frac{b}{d}\right]^{2p-2}$$

$$\geq (2p-1)a\left[\frac{a}{c}\right]^{2p-2} \geq a(I^{2p-2} + I^{2p-3}S + ... + S^{2p-3}I + S^{2p-2})$$

$$c(I^{2p-1} + I^{2p-2}S + ... + S^{2p-2}I + S^{2p-1}) + bS^{p-1}I^{p-1}$$
$$- a(I^{2p-2} + I^{2p-3}S + ... + S^{2p-3}I + S^{2p-2}) \geq 0,$$

and so it follows that

$$I \geq S.$$

Therefore

$$I = S.$$

This completes the proof. □

Remark 4.2. It follows from (1.1), when $\frac{a}{c} = \frac{b}{d}$ that $x_{n+1} = \lambda$ for all $n \geq -p$ and for some constant λ.

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