Positive solutions for nonlinear integral equations of Hammerstein type

Andrei Horvat-Marc

Abstract.

We apply a variant of Krasnoselskii’s compression-expansion theorem for nonlinear operators which satisfy a compact condition of Mönch type. Our approach makes possible to establish conditions which ensure the existence of positive solutions of abstract integral equations of Hammerstein type.

1. Introduction

Let $X$ be a real Banach space, $\mathbb{R}_+ = [0, \infty)$ be the set of positive real numbers and $h > 0$.

The goal of this paper is to establish sufficient conditions for the existence of nonnegative solutions to the nonlinear integral equation of Hammerstein type

$$u(t) = \int_0^h k(t, s) F(u(s)) \, ds, \quad t \in [0, h],$$

where $k : [0, h] \times [0, h] \to \mathbb{R}_+$ and $F : U \subset X \to X$ is Bochner integrable on $[0, h]$.

Let $X$ be endowed with the norm $|\cdot|$ and $K \subset X$ be a cone of $X$ which induces a partial order on $X$, i.e., $x \leq y$ if and only if $y - x \in K$. We say that the norm $|\cdot|$ is increasing with respect to $K$ if $|x| \leq |y|$ whenever $0 \leq x \leq y$. For $0 < r < R$ we use the notation $\Omega_r = \{x \in X : |x| < r\}$, $K_r = \{x \in K : |x| < r\}$, $S_r = \{x \in K : |x| = r\}$, $K_{r,R} = \{x \in K : r \leq |x| \leq R\}$. We observe that $K_r = K \cap \Omega_r$ and $K_{r,R} = K \cap (\Omega_R \setminus \Omega_r)$.

In this paper, we introduce the new notion of $\mu$2-bounded map. If we want to localize a positive solution of (1.1) in a positive cone $K$, then we must be sure that $\int_0^h k(t, s) F(u(s)) \, ds$ is an element of $K$. This condition is implied by the hypothesis that $k$ is $\mu$2-bounded. In fact, the maps which are $\mu$2-bounded ensure that the values of some integral operators are situated in a positive cone of a Banach space.

To localize a positive solution of (1.1) we use the compression-expansion fixed point theorem of Krasnoselskii’s type. This technique has been applied in the literature to scalar equations, when $X = \mathbb{R}$, see [10, 12, 13], and recently to nonlinear equations in Banach spaces, see [3, 16, 4]. In all this works, the nonlinear integral equations were studied assuming that the associated operator is compact

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or completely continuous. Our existence result do not require completely continuity of $T$ and is based upon the continuation theorem of Mönch [14] and the corresponding compression theorem, stated in the following:

**Theorem 1.1** (A. Horvat-Marc [9]). Let $X$ be a real Banach space, endowed with the norm $|\cdot|$, $K$ be a cone in $X$, $0 < r < R$ and the continuous operator $T : K \cap (\overline{\Omega_R} \setminus \Omega_r) \to K$. Assume that

MK1) the norm $|\cdot|$ is increasing with respect to $K$,
MK2) there exist $x_0 \in K \cap (\overline{\Omega_R} \setminus \Omega_r)$ and $C \subset K \cap (\overline{\Omega_R} \setminus \Omega_r)$ such that

\[
C \subset \overline{\partial} \{ x_0 \cup T(C) \} \implies \overline{C} \text{ compact}
\]

MK3) $T$ is such that

\[
|T(u)| \leq |u| \text{ on } K \cap \Omega_R \text{ and } |T(u)| \geq |u| \text{ on } K \cap \Omega_r.
\]

Then $T$ has at least one fixed point in $K \cap (\overline{\Omega_R} \setminus \Omega_r)$.

The proof of this result may be found in [9] and some examples of operators which satisfy MK2) are presented in [5, 6]. In fact, if an operator $T$ satisfies MK2), we say that $T$ is operator of Mönch type.

### 2. Preliminary results

In what follows we introduce the notion of $\mu 2$-bounded map.

**Definition 2.1.** Let $\mu \in (0, 1), \kappa : [a, b] \to \mathbb{R}_+$ and $[a', b'] \subset [a, b]$. We say that the map $k : [a, b] \times [a, b] \to \mathbb{R}_+$ is $\mu 2$-bounded on $[a, b]$ with respect to $\kappa$ and $[a', b']$ if

i) for every $t \in [a, b]$ we have

\[
k(t, s) \leq \kappa(s) \text{ for all } s \in [a, b],
\]

ii) for every $t' \in [a', b']$ the inequality

\[
\mu \kappa(s) \leq k(t', s) \text{ for all } s \in [a, b].
\]

holds.

The next lemmas give some examples of $\mu 2$-bounded maps.

**Lemma 2.1.** Let $k : [a, b] \times [a, b] \to \mathbb{R}_+$ be a map and $[c, d] \subset [a, b]$. Assume that:

Li) for all $s \in [a, b]$ the map $k(\cdot, s) : [a, b] \to \mathbb{R}_+$ is concave on $[c, d]$, i.e. for any $s \in [a, b]$ and $t_1, t_2 \in [c, d]$ we have

\[
k((1 - \lambda) t_1 + \lambda t_2, s) \geq (1 - \lambda) k(t_1, s) + \lambda k(t_2, s), \quad \lambda \in [0, 1];
\]

Lii) for all $s \in [a, b]$ the map $k(\cdot, s) : [a, b] \to \mathbb{R}_+$ is increasing on $[a, b]$, i.e. for any $t_1, t_2 \in [a, b]$ with $t_1 < t_2$ and $s \in [a, b]$ we have

\[
k(t_1, s) \leq k(t_2, s).
\]

Then $k$ is $\mu 2$-bounded on $[a, b]$ with respect to $\kappa$ and $[c, d]$, where $\mu = \frac{c-a}{b}$ and $\kappa(s) = k(b, s)$ for all $s \in [a, b]$. 

Proof. From (ii) we have
\begin{equation}
\tag{2.5}
k(t, s) \leq k(b, s) = \kappa(s) \quad \text{for all } t, s \in [a, b].
\end{equation}
So, (2.3) holds.

Let \( t^* \in [c, d] \). We can consider \( t^* = \frac{c-a}{b} \cdot b + \left( 1 - \frac{c-a}{b} \right) \frac{t^* - (c-a)}{b - (c-a)} \cdot b \), where \( \frac{c-a}{b} \in (0, 1) \) and \( \frac{t^* - (c-a)}{b - (c-a)} \cdot b \in [a, b] \). Now, from Li) we obtain that
\[
k(t^*, s) = k \left( \frac{c-a}{b} \cdot b + \left( 1 - \frac{c-a}{b} \right) \frac{t^* - (c-a)}{b - (c-a)} \cdot b, s \right) \\
\geq \frac{c-a}{b} k(b, s) + \left( 1 - \frac{c-a}{b} \right) k \left( \frac{t^* - (c-a)}{b - (c-a)} \cdot b, s \right),
\]
for all \( s \in [a, b] \). Then for every \( t^* \in [c, d] \) we have
\begin{equation}
\tag{2.6}
k(t^*, s) \geq \frac{c-a}{b} k(b, s) \quad \text{for all } s \in [a, b].
\end{equation}
Hence, Li) guarantees ii). \( \square \)

Lemma 2.2. Let \( G : [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+ \) defined by
\begin{equation}
\tag{2.7}
G(t, s) = \begin{cases}
\frac{(C + D - Ct) (B + As)}{CB + AC + AD}, & 0 \leq t \leq 1 \\
\frac{(C + D - Cs) (B + At)}{CB + AC + AD}, & 0 \leq t \leq 1.
\end{cases}
\end{equation}
Then \( G \) is \( \mu \)-bounded with respect to \( \kappa \) and I, where \( \kappa \in C[0, 1] \) with \( \kappa(s) = G(s, s) \)

for \( s \in [0, 1] \), I = \( \left[ \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon \right] \) for every \( \varepsilon \in \left( 0, \frac{1}{2} \right) \) and
\begin{equation}
\tag{2.8}
\mu = \min \left\{ \frac{C (1 - 2 \varepsilon) + 2D}{2(C + D)}, \frac{A (1 - 2 \varepsilon) + 2B}{2(A + B)} \right\}
\end{equation}
Proof. Let \( \varepsilon \in \left( 0, \frac{1}{2} \right) \). We prove that (ii.3) and (ii.4) are satisfied for \( [a, b] = [0, 1] \),
\( [a', b'] = \left[ \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon \right] \), \( k = G, \kappa(s) = G(s, s) \) with \( s \in [0, 1] \) and \( \mu \in (0, 1) \) given by (2.8).

If \( 0 \leq s \leq t \leq 1 \), then \( C + D - Cs \geq C + D - Ct \), so
\begin{equation}
\tag{2.9}
G(s, s) = \frac{(C + D - Cs) (B + As)}{CB + AC + AD} \geq \frac{(C + D - Ct) (B + As)}{CB + AC + AD} = G(t, s).
\end{equation}
If \( 0 \leq t \leq s \leq 1 \), then \( B + As \geq B + At \), so
\begin{equation}
\tag{2.10}
G(s, s) = \frac{(C + D - Cs) (B + As)}{CB + AC + AD} \geq \frac{(C + D - Cs) (B + At)}{CB + AC + AD} = G(t, s).
\end{equation}
From (2.9) and (2.10) we obtain
\begin{equation}
\tag{2.11}
G(t, s) \leq G(s, s) = \kappa(s) \quad \text{for every } t, s \in [0, 1].
\end{equation}
If \( t \in \left[0, \varepsilon + \frac{1}{2}\right] \), then
\[
C + D - Ct \geq C + D - C \left(\varepsilon + \frac{1}{2}\right) = \frac{C(1 - 2\varepsilon) + 2D}{2}.
\]
Hence for \( s \geq 0 \) we have
\[
C + D - Ct \geq \frac{C(1 - 2\varepsilon) + 2D}{2} \cdot \frac{C + D - Cs}{C + D} \geq \mu(C + D - Cs).
\]
It results that
\[
G(t, s) \geq \mu G(s, s), \quad 0 \leq s \leq \varepsilon + \frac{1}{2}.
\]
If \( t \in \left[\frac{1}{2} - \varepsilon, 1\right] \) then
\[
B + At \geq B + A \left(\frac{1}{2} - \varepsilon\right) = \frac{A(1 - 2\varepsilon) + 2B}{2}.
\]
Hence for \( s \leq 1 \) we have
\[
B + At \geq \frac{A(1 - 2\varepsilon) + 2B}{2} \cdot \frac{B + As}{B + A} \geq \mu(B + As).
\]
It results that
\[
G(t, s) \geq \mu G(s, s), \quad \frac{1}{2} - \varepsilon \leq t \leq s \leq 1.
\]
From (2.12) and (2.13) we obtain
\[
\mu G(s, s) \leq G(t, s) \text{ for every } t \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right], s \in [0, 1].
\]
Now, (2.11) and (2.14) guarantee that \( G \) is \( \mu_2 \)-bounded.

The function \( G \) considered in Lemma 2.2 is the Green function associated to the boundary values problem
\[
\begin{align*}
y'' &= 0 \\
Ay(0) - By'(0) &= 0 \\
Cy(1) + Dy'(1) &= 0,
\end{align*}
\]
with \( AC + BD + AD > 0 \).

The next example of \( \mu_2 \)-bounded map is a Green function associated to a boundary values problem arising in chemical reactor theory, see [1, 2, 7].

**Lemma 2.3.** Let \( G : [0, 1] \times [0, 1] \to [0, 1] \) be the map defined by
\[
G(t, s) = \begin{cases} 
1 & \text{if } 0 \leq s \leq t \leq 1, \\
\frac{t-s}{e^{\kappa(s)}} & \text{if } 0 \leq t \leq s \leq 1.
\end{cases}
\]
If \( \mu = e^{\frac{\kappa - 2}{4}} \) and \( \kappa : [0, 1] \to (0, \infty) \) is such that
\[
\kappa(s) = e^{s^2}, \quad s \in [0, 1],
\]
then \( G \) is \( \mu_2 \)-bounded on \([0, 1]\) with respect to \( \kappa \) and every \([a, b] \subset [0, 1]\).

**Proof.** We have \( G(t, s) \leq 1 \leq \kappa(s) \) for all \( t, s \in [0, 1] \). So, (2.13) from (2) holds. Now, let \([a, b] \subset [0, 1]\) and \( t^* \in [a, b] \). If \( s \in [0, t^*] \), then

\[
G(t^*, s) = 1 \geq e^{\frac{a+s-2}{t-s}} = \mu\kappa(s).
\]

If \( s \in [t^*, 1] \subset [a, 1] \), then \( t^* - s \geq a + s - 2 \) and

\[
G(t^*, s) = e^{\frac{t^* - s}{t-s}} \geq e^{\frac{a+s-2}{t-s}} = \mu\kappa(s).
\]

From (2.17) and (2.18) we obtain that for any \( t^* \in [a, b] \)

\[
G(t^*, s) \geq \mu\kappa(s), \text{ for any } s \in [0, 1].
\]

holds. So, (ii) is satisfied and this complete the proof.

In what follows, we show how the maps which are \( \mu_2 \)-bounded ensure that the values of some integral operators are situated in a positive cone of a Banach space.

We denote by \( C([a, b]; K) \) the set of all continuous maps from \([a, b]\) to \( K \).

**Lemma 2.4.** Let \( X \) be a Banach space, \( K \subset X \) be a cone in \( X \) and let \( \leq \) be the order relation on \( X \) induced by the cone \( K \). Let \( \mu \in (0, 1) \), \( [c, d] \subset [a, b] \), \( t' \in [c, d] \), \( k : [a, b] \times [a, b] \to \mathbb{R} \) and the cone

\[
K'_\mu = \{ u \in C([a, b]; K) ; \; \mu u(t) \leq u(t') \text{, } t \in [a, b] \}.
\]

If the map \( k \) is \( \mu_2 \)-bounded with respect to \( \kappa \) and \([c, d]\), then for all \( v \in C([a, b]; K) \) we have

\[
\int_a^b k(t, s) v(s) ds \in K'_\mu.
\]

**Proof.** Let \( v \in C([a, b]; K) \) and \( u(t) = \int_a^b k(t, s) v(s) ds \), for \( t \in [a, b] \). It is obvious that \( u \in C([a, b]; K) \). Since, the map \( k \) is uppermajorated by \( \kappa \) on \([a, b]\), we have

\[
\mu u(t) = \mu \int_a^b k(t, s) v(s) ds \leq \int_a^b \mu\kappa(s) v(s) ds, \quad t \in [a, b].
\]

Now, because \( \mu\kappa(s) \leq k(t', s) \) for all \( s \in [a, b] \) and \( t' \in [a', b'] \), we obtain

\[
\mu u(t) \leq \int_a^b k(t', s) v(s) ds = u(t'), \quad t \in [a, b].
\]

So, \( \mu u(t) \leq u(t') \) for all \( t \in [a, b] \) and this ensure that \( u \in K'_\mu \).

**Lemma 2.5.** Let \( X \) be a Banach space, \( K \subset X \) be a cone in \( X \), which induces on \( X \) the order relation \( \leq \), and let \( 1 \leq p \leq \infty \). Let \( \mu \in (0, 1) \), \([c, d] \subset [a, b] \), \( t' \in [c, d] \), \( k : [a, b] \times [a, b] \to \mathbb{R} \) and the cone

\[
K'_{p, \mu} = \{ u \in L^p([a, b]; K) ; \; \mu u(t) \leq u(t') \text{ a.e. on } t \in [a, b] \}.
\]
If the map \( k \) is \( \mu_2 \)-bounded with respect to \( \kappa \) and \([c, d]\), then for all \( v \in C \langle a, b; K \rangle \) we have

\[
(2.20) \quad \int_a^b k(\cdot, s) v(s) \, ds \in K'_{p, \mu}.
\]

**Proof.** Let \( v \in L^p \langle a, b; K \rangle \) and \( u(t) = \int_a^b k(t, s) v(s) \, ds \) a.e. on \([a, b]\). It is obvious that \( u \in L^p \langle a, b; K \rangle \). Since, the map \( k \) is a.e. uppermajorated by \( \kappa \) on \([a, b]\), we have

\[
\mu u(t) = \mu \int_a^b k(t, s) v(s) \, ds \leq \int_a^b \mu \kappa(s) v(s) \, ds, \text{ a.e. on } [a, b].
\]

Now, because \( \mu \kappa(s) \leq k(t', s) \) a.e. \( s \in [a, b] \) and \( t' \in [a', b'] \), we obtain

\[
\mu u(t) \leq \int_a^b k(t', s) v(s) \, ds = u(t'), \text{ a.e. on } [a, b].
\]

So, \( \mu u(t) \leq u(t') \) a.e. on \([a, b]\) and this ensure that \( u \in K'_{p, \mu} \). \hfill \Box

### 3. Main result

In this section we establish conditions which guarantee the localization of positive solution of (1.1) in \( K_{r,R} \). We say that \( u \) is a positive solution to (1.1) if \( u \in C \langle 0, h; K \rangle \) and satisfies (1.1).

The Banach space \( C \langle 0, h; K \rangle \) is endowed with the norm

\[
|u|_\infty = \max_{t \in [0, h]} |u(t)|, \quad u \in C \langle 0, h; K \rangle.
\]

We remind that \( u \leq y \) in \( C \langle 0, h; K \rangle \) if and only if for any \( t \in [0, h] \) we have \( u(t) \leq y(t) \) in \( K \). For \( M \subset X \) a bounded set we denote by \( \alpha(M) \) the Kuratowski measure of noncompactness on \( X \), i.e.,

\[
\alpha(M) = \inf \{ \varepsilon > 0; M \text{ admits a finite cover by sets of diameter } \leq \varepsilon \}.\]

To attain our goal we introduce the following hypotheses:

**H1)** the map \( F : K \rightarrow K \) is \( L^1 \)-Carathéodory, \( q \geq 1 \) and

\[
F(x) \leq F(y) \text{ for every } x, y \in K \text{ with } x \leq y;
\]

**H2)** for any \( t \in [0, h] \), we have \( k(t, \cdot) \in L^p \langle 0, h \rangle \), where \( \frac{1}{p} + \frac{1}{q} = 1 \);

**H3)** the map \( t \mapsto k(t, \cdot) \) is continuous from \([0, h]\) to \( L^p \langle 0, h \rangle \);

**H4)** there exists \( \omega : [0, 2R] \rightarrow \mathbb{R}_+ \) a \( L^p \)-Carathéodory map such that

(a) \( \alpha(F(M)) \leq \omega(\alpha(M)) \) for any bounded set \( M \subset C \langle a, b; K \rangle \);

(b) the unique solution of

\[
\phi(t) \leq 2 \int_0^h k(t, s) \omega(\phi(s)) \, ds, \quad t \in [0, h]
\]

is \( \phi \equiv 0 \).
H5) there exist $\mu \in (0, 1)$, $\kappa \in L^p(0, h; \mathbb{R}_+)$ and $[a, b] \subset [0, 1]$ such that $k$ is $\mu^2$-bounded a.e. on $[0, h]$ with respect to $\kappa$ and $[a, b]$;
H6) there exist $m_k, M_k > 0$ with $m_k \leq \kappa(s)$ a.e. $s \in [a, b]$ and $\kappa(s) \leq M_k$ a.e. $s \in [0, h]$.

**Theorem 3.2.** Assume that H1)-H6) are satisfied and

(h1) there exist $r > 0$ such that

$$\inf \{|F(x)| ; x \in K, |x| = \mu r\} > \frac{r}{\mu m_k (b - a)};$$

(h2) there exist $R > 0$ such that

$$\sup \{|F(x)| ; x \in K, |x| = R\} \leq \frac{R}{h M_k}.$$

Then the integral equation of Hammerstein type (1.1) has at least one positive solution $u \in C(0, h; K)$ for which we have

$$\begin{cases} 0 < r \leq |u|_{\infty} \leq R, \\ \mu u(t) \leq u(t') \text{ for any } t \in [0, h] \text{ and } t' \in [a, b]. \end{cases}$$

**Proof.** Let be the cone

$$K_{\mu} = \{u \in C(0, h; K) ; \mu u(t) \leq u(t'), t \in [0, h], t' \in [a, b]\}$$

and the radial shell

$$K_{r, R} = \{u \in K_{\mu} ; r \leq |u|_{\infty} \leq R\}.$$

If we consider the operator $T : K_{r, R} \to K_{\mu}$ defined by

$$T(u)(t) = \int_0^h k(t, s) F(u(s)) \, ds, \quad t \in [0, h],$$

then by Lemma 2.4 from H5) we have $T(u) \in K_{\mu}$.

The existence of a positive solution to (1.1) is equivalent to the existence of a fixed point to $T$. We will apply Theorem 1.1 so we will prove that MK2) and MK3) hold. Hypotheses H1)-H4) guarantee that $T$ is an operator of Mönch type, see [15]. So, MK2) holds.

Let $u \in K_{\mu}$ with $|u|_{\infty} = R$. By H6), for any $t \in [0, h]$ we have

$$|T(u)(t)| = \left| \int_0^h k(t, s) F(u(s)) \, ds \right| \leq \int_0^h k(t, s) |F(u(s))| \, ds \leq \int_0^h \kappa(s) |F(u(s))| \, ds \leq M_k \int_0^h |F(u(s))| \, ds.$$

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$$|T(u)|_{\infty} \leq h M_k \cdot \sup \{|F(x)| ; x \in K, |x| \leq R\}.$$

From (h2) we obtain

$$|T(u)|_{\infty} \leq |u|_{\infty}. \quad (3.23)$$
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Let $u \in K_\mu$ with $|u|_\infty = r$ and $t^* \in [0, h], |u(t^*)| = r$. Hence $u \in K_\mu$, we can assume that $\mu u(t^*) \leq u(s)$ for any $s \in [a, b]$. From H1), we have

$$F(\mu u(t^*)) \leq F(u(t)) \text{ for any } t \in [a, b].$$

Now, for $t \in [0, h]$ by H6) holds

$$T(u)(t) = \int_0^h k(t, s) F(u(s)) \, ds \geq \int_a^b \mu \kappa(s) F(u(s)) \, ds$$

$$\geq \mu F(\mu u(t^*)) \int_a^b \kappa(s) \, ds \geq \mu m_\kappa (b - a) \cdot F(\mu u(t^*)).$$

Since the norm $|\cdot|_\infty$ is increasing with respect to cone $C(0, h; K)$, we have

$$|T(u)|_\infty \geq \mu m_\kappa (b - a) \cdot \inf \{|F(x)|; x \in K, |x| = \mu r\}.$$ 

From (h1) we obtain

$$|T(u)|_\infty > |u|_\infty.$$ 

Now (3.23) and (3.26) guarantee that the inequalities from MK3) are satisfied and this complete the proof. 

Theorem 3.2 is an useful tool to study the various problems from the theory of abstract ordinary differential equations or abstract nonlinear integral equations.

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**North University of Baia Mare**
**Department of Mathematics and Informatics**
**Str Victoriei 76, 430122, Baia Mare, Romania**
**E-mail address:** andrei.horvatmarc@gmail.com