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Positive solutions for nonlinear integral equations of Hammerstein type

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ABSTRACT.
We apply a variant of Krasnoselskii's compression-expansion theorem for nonlinear operators which satisfy a compact condition of Mönch type. Our approach makes possible to establish conditions which ensure the existence of positive solutions of abstract integral equations of Hammerstein type.

1. INTRODUCTION

Let X be a real Banach space, $\mathbb{R}_+ = [0, \infty)$ be the set of positive real numbers and $h > 0$.

The goal of this paper is to establish sufficient conditions for the existence of nonnegative solutions to the nonlinear integral equation of Hammerstein type

$$u(t) = \int_0^h k(t, s) F(u(s)) ds, \quad t \in [0, h], \tag{1.1}$$

where $k : [0, h] \times [0, h] \to \mathbb{R}_+$ and $F : U \subset X \to X$ is Bochner integrable on $[0, h]$.

Let X be endowed with the norm $|\cdot|$ and $K \subset X$ be a cone of X which induces a partial order on X, i.e., "$x \leq y$" if and only if $y - x \in K$. We say that the norm $|\cdot|$ is increasing with respect to K if $|x| \leq |y|$ whenever $0 \leq x \leq y$. For $0 < r < R$ we use the notation $\Omega_r = \{x \in X : |x| < r\}$, $K_r = \{x \in K : |x| < r\}$, $S_r = \{x \in K : |u| = r\}$, $K_{r,R} = \{x \in K : r \leq |x| \leq R\}$. We observe that $K_r = K \cap \Omega_r$ and $K_{r,R} = K \cap \left(\overline{\Omega_R} \backslash \Omega_r\right)$.

In this paper, we introduce the new notion of $\mu 2$-bounded map. If we want to localize a positive solution of (1.1) in a positive cone K, then we must be sure that $\int_0^h k(\cdot, s) F(u(s)) ds$ is an element of K. This condition is implied by the hypothesis that k is $\mu 2$-bounded. In fact, the maps which are $\mu 2$-bounded ensure that the values of some integral operators are situated in a positive cone of a Banach space.

To localize o positive solution of (1.1) we use the compression-expansion fixed point theorem of Krasnoselskii's type. This technique has been applied in the literature to scalar equations, when $X = \mathbb{R}$, see [10, 12, 13], and recently to nonlinear equations in Banach spaces, see [3, 16, 4]. In all this works, the nonlinear integral equations were studied assuming that the associated operator is compact

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or completely continuous. Our existence result do not require completely continuity of T and is based upon the continuation theorem of Mönch [14] and the corresponding compression theorem, stated in the following:

Theorem 1.1 (A. Horvat-Marc [9]). *Let X be a real Banach space, endowed with the norm $|\cdot|$, K be a cone in X, $0<r<R$ and the continuous operator $T : K\cap\left(\overline{\Omega}_R\backslash\Omega_r\right)\to K$. Assume that*

MK1) *the norm $|\cdot|$ is increasing with respect to K,*

MK2) *there exist $x_0 \in K \cap \left(\overline{\Omega}_R\backslash\Omega_r\right)$ and $C \subset K \cap \left(\overline{\Omega}_R\backslash\Omega_r\right)$ such that*

$$C \subset \overline{co}\left(\{x_0\} \cup T\left(C\right)\right) \text{ implies } \overline{C} \text{ compact.} \tag{1.2}$$

MK3) *T is such that*

$$|T\left(u\right)| \leq |u| \text{ on } K \cap \Omega_R \text{ and } |T\left(u\right)| \geq |u| \text{ on } K \cap \Omega_r.$$

Then T has at least one fixed point in $K \cap \left(\overline{\Omega}_R\backslash\Omega_r\right)$.

The proof of this result may be found in [9] and some examples of operators which satisfy MK2) are presented in [5, 6]. In fact, if an operator T satisfies MK2), we say that T is operator of Mönch type.

2. Preliminary results

In what follows we introduce the notion of $\mu 2$-bounded map.

Definition 2.1. Let $\mu \in (0,1)$, $\kappa : [a,b] \to \mathbb{R}_+$ and $[a',b'] \subset [a,b]$. We say that the map $k : [a,b] \times [a,b] \to \mathbb{R}_+$ is $\mu 2$-*bonded* on $[a,b]$ with respect to κ and $[a',b']$ if

i) for every $t \in [a,b]$ we have

$$k\left(t,s\right) \leq \kappa\left(s\right) \text{ for all } s \in [a,b]\,, \tag{2.3}$$

ii) for every $t' \in [a',b']$ the inequality

$$\mu\kappa\left(s\right) \leq k\left(t',s\right) \text{ for all } s \in [a,b]\,. \tag{2.4}$$

holds.

The next lemmas give some examples of $\mu 2$-bounded maps.

Lemma 2.1. *Let $k : [a,b] \times [a,b] \to \mathbb{R}_+$ be a map and $[c,d] \subset [a,b]$. Assume that:*

Li) *for all $s \in [a,b]$ the map $k\left(\cdot,s\right) : [a,b] \to \mathbb{R}_+$ is concave on $[c,d]$, i.e. for any $s \in [a,b]$ and $t_1,t_2 \in [c,d]$ we have*

$$k\left(\left(1-\lambda\right)t_1 + \lambda t_2, s\right) \geq \left(1-\lambda\right)k\left(t_1,s\right) + \lambda k\left(t_2,s\right), \quad \lambda \in [0,1]\,;$$

Lii) *for all $s \in [a,b]$ the map $k\left(\cdot,s\right) : [a,b] \to \mathbb{R}_+$ is increasing on $[a,b]$, i.e. for any $t_1,t_2 \in [a,b]$ with $t_1 < t_2$ and $s \in [a,b]$ we have*

$$k\left(t_1,s\right) \leq k\left(t_2,s\right).$$

Then k is $\mu 2$-bounded on $[a,b]$ with respect to κ and $[c,d]$, where $\mu = \frac{c-a}{b}$ and $\kappa\left(s\right) = k\left(b,s\right)$ for all $s \in [a,b]$.

Proof. From Lii) we have

$$k(t,s) \le k(b,s) = \kappa(s) \text{ for all } t,s \in [a,b]. \tag{2.5}$$

So, (2.3) holds.

Let $t^* \in [c,d]$. We can consider $t^* = \frac{c-a}{b} \cdot b + \left(1 - \frac{c-a}{b}\right) \frac{t^* - (c-a)}{b-(c-a)} \cdot b$, where $\frac{c-a}{b} \in (0,1)$ and $\frac{t^* - (c-a)}{b-(c-a)} \cdot b \in [a,b]$. Now, from Li) we obtain that

$$\begin{aligned} k(t^*,s) &= k\left(\frac{c-a}{b} \cdot b + \left(1 - \frac{c-a}{b}\right) \frac{t^* - (c-a)}{b-(c-a)} \cdot b, s\right) \\ &\ge \frac{c-a}{b} k(b,s) + \left(1 - \frac{c-a}{b}\right) k\left(\frac{t^* - (c-a)}{b-(c-a)} \cdot b, s\right), \end{aligned}$$

for all $s \in [a,b]$. Then for every $t^* \in [c,d]$ we have

$$k(t^*,s) \ge \frac{c-a}{b} k(b,s) \text{ for all } s \in [a,b]. \tag{2.6}$$

Hence, Li) guarantees ii). □

Lemma 2.2. *Let* $G : [0,1] \times [0,1] \to \mathbb{R}_+$ *defined by*

$$G(t,s) = \begin{cases} \dfrac{(C+D-Ct)(B+As)}{CB+AC+AD}, & 0 \le s \le t \le 1 \\ \dfrac{(C+D-Cs)(B+At)}{CB+AC+AD}, & 0 \le t \le s \le 1. \end{cases} \tag{2.7}$$

Then G *is* $\mu 2$*-bounded with respect to* κ *and* I*, where* $\kappa \in C[0,1]$ *with* $\kappa(s) = G(s,s)$ *for* $s \in [0,1]$, $I = \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$ *for every* $\varepsilon \in \left(0, \frac{1}{2}\right)$ *and*

$$\mu = \min\left\{\frac{C(1-2\varepsilon)+2D}{2(C+D)}, \frac{A(1-2\varepsilon)+2B}{2(A+B)}\right\}. \tag{2.8}$$

Proof. Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. We prove that (2.3) and (2.4) are satisfied for $[a,b] = [0,1]$, $[a',b'] = \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$, $k = G$, $\kappa(s) = G(s,s)$ with $s \in [0,1]$ and $\mu \in (0,1)$ given by (2.8).

If $0 \le s \le t \le 1$, then $C + D - Cs \ge C + D - Ct$, so

$$G(s,s) = \frac{(C+D-Cs)(B+As)}{CB+AC+AD} \ge \frac{(C+D-Ct)(B+As)}{CB+AC+AD} = G(t,s). \tag{2.9}$$

If $0 \le t \le s \le 1$, then $B + As \ge B + At$, so

$$G(s,s) = \frac{(C+D-Cs)(B+As)}{CB+AC+AD} \ge \frac{(C+D-Cs)(B+At)}{CB+AC+AD} = G(t,s). \tag{2.10}$$

From (2.9) and (2.10) we obtain

$$G(t,s) \le G(s,s) = \kappa(s) \text{ for every } t,s \in [0,1]. \tag{2.11}$$

If $t \in \left[0, \varepsilon + \frac{1}{2}\right]$, then

$$C + D - Ct \geq C + D - C\left(\varepsilon + \frac{1}{2}\right) = \frac{C(1-2\varepsilon) + 2D}{2}.$$

Hence for $s \geq 0$ we have

$$C + D - Ct \geq \frac{C(1-2\varepsilon) + 2D}{2} \cdot \frac{C + D - Cs}{C + D} \geq \mu(C + D - Cs).$$

It results that

$$G(t,s) \geq \mu G(s,s), \quad 0 \leq s \leq t \leq \varepsilon + \frac{1}{2}. \tag{2.12}$$

If $t \in \left[\frac{1}{2} - \varepsilon, 1\right]$ then

$$B + At \geq B + A\left(\frac{1}{2} - \varepsilon\right) = \frac{A(1-2\varepsilon) + 2B}{2}.$$

Hence for $s \leq 1$ we have

$$B + At \geq \frac{A(1-2\varepsilon) + 2B}{2} \cdot \frac{B + As}{B + A} \geq \mu(B + As).$$

It results that

$$G(t,s) \geq \mu G(s,s), \quad \frac{1}{2} - \varepsilon \leq t \leq s \leq 1. \tag{2.13}$$

From (2.12) and (2.13) we obtain

$$\mu G(s,s) \leq G(t,s) \text{ for every } t \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right], s \in [0,1]. \tag{2.14}$$

Now, (2.11) and (2.14) guarantee that G is $\mu 2$-bounded. □

The function G considered in Lemma 2.2 is the Green function associated to the boundary values problem

$$\begin{cases} y'' = 0 \\ Ay(0) - By'(0) = 0 \\ Cy(1) + Dy'(1) = 0, \end{cases} \tag{2.15}$$

with $AC + BD + AD > 0$.

The next example of $\mu 2$-bounded map is a Green function associated to a boundary values problem arising in chemical reactor theory, see [1, 2, 7].

Lemma 2.3. *Let $G : [0,1] \times [0,1] \to [0,1]$ be the map defined by*

$$G(t,s) = \begin{cases} 1 & \text{if} \quad 0 \leq s \leq t \leq 1, \\ e^{\frac{t-s}{\xi}} & \text{if} \quad 0 \leq t \leq s \leq 1. \end{cases} \tag{2.16}$$

If $\mu = e^{\frac{a-2}{\xi}}$ and $\kappa : [0,1] \to (0,\infty)$ is such that

$$\kappa(s) = e^{\frac{s}{\xi}}, \quad s \in [0,1],$$

then G is $\mu 2$-bounded on $[0,1]$ with respect to κ and every $[a,b]\subset[0,1]$.

Proof. We have $G(t,s)\leq 1\leq\kappa(s)$ for all $t,s\in[0,1]$. So, (2.3) from *i)* holds. Now, let $[a,b]\subset[0,1]$ and $t^*\in[a,b]$. If $s\in[0,t^*]$, then

$$G(t^*,s)=1\geq e^{\frac{a+s-2}{\xi}}=\mu\kappa(s)\,. \tag{2.17}$$

If $s\in[t^*,1]\subset[a,1]$, then $t^*-s\geq a+s-2$ and

$$G(t^*,s)=e^{\frac{t-s}{\xi}}\geq e^{\frac{a+s-2}{\xi}}=\mu\kappa(s)\,. \tag{2.18}$$

From (2.17) and (2.18) we obtain that for any $t^*\in[a,b]$

$$G(t^*,s)\geq\mu\kappa(s)\,,\ \text{for any } s\in[0,1]\,.$$

holds So, Lii) is satisfied and this complete the proof. □

In what follows, we show how the maps which are $\mu 2$-bounded ensure that the values of some integral operators are situated in a positive cone of a Banach space.

We denote by $C(a,b;K)$ the set of all continuous maps from $[a,b]$ to K.

Lemma 2.4. *Let X be a Banach space, $K\subset X$ be a cone in X and let $\leq$ be the order relation on X induced by the cone K. Let $\mu\in(0,1)$, $[c,d]\subset[a,b]$, $t'\in[c,d]$, $k:[a,b]\times[a,b]\to\mathbb{R}$ and the cone*

$$K'_\mu=\{u\in C(a,b;K)\,;\ \ \mu u(t)\leq u(t')\,,t\in[a,b]\}\,.$$

If the map k is $\mu 2$-bounded with respect to κ and $[c,d]$, then for all $v\in C(a,b;K)$ we have

$$\int_a^b k(\cdot,s)\,v(s)\,ds\in K'_\mu. \tag{2.19}$$

Proof. Let $v\in C(a,b;K)$ and $u(t)=\int_a^b k(t,s)\,v(s)\,ds$, for $t\in[a,b]$. It is obvious that $u\in C(a,b;K)$. Since, the map k is uppermajorated by κ on $[a,b]$, we have

$$\mu u(t)=\mu\int_a^b k(t,s)\,v(s)\,ds\leq\int_a^b\mu\kappa(s)\,v(s)\,ds,\ \ t\in[a,b]\,.$$

Now, because $\mu\kappa(s)\leq k(t',s)$ for all $s\in[a,b]$ and $t'\in[a',b']$, we obtain

$$\mu u(t)\leq\int_a^b k(t',s)\,v(s)\,ds=u(t')\,,\ \ t\in[a,b]\,.$$

So, $\mu u(t)\leq u(t')$ for all $t\in[a,b]$ and this ensure that $u\in K'_\mu$. □

Lemma 2.5. *Let X be a Banach space, $K\subset X$ be a cone in X, which induces on X the order relation $\leq$, and let $1\leq p\leq\infty$. Let $\mu\in(0,1)$, $[c,d]\subset[a,b]$, $t'\in[c,d]$, $k:[a,b]\times[a,b]\to\mathbb{R}$ and the cone*

$$K'_{p,\mu}=\{u\in L^p(a,b;K)\,;\ \ \mu u(t)\leq u(t')\ \textit{a.e. on}\ t\in[a,b]\}\,.$$

If the map k is $\mu 2$-bounded with respect to κ and $[c,d]$, then for all $v \in C(a,b;K)$ we have

(2.20) $$\int_a^b k(\cdot,s)\,v(s)\,ds \in K'_{p,\mu}.$$

Proof. Let $v \in L^p(a,b;K)$ and $u(t) = \int_a^b k(t,s)\,v(s)\,ds$ a.e. on $[a,b]$. It is obvious that $u \in L^p(a,b;K)$. Since, the map k is a.e. uppermajorated by κ on $[a,b]$, we have

$$\mu u(t) = \mu \int_a^b k(t,s)\,v(s)\,ds \le \int_a^b \mu\kappa(s)\,v(s)\,ds, \text{ a.e. on } [a,b].$$

Now, because $\mu\kappa(s) \le k(t',s)$ a.e. $s \in [a,b]$ and $t' \in [a',b']$, we obtain

$$\mu u(t) \le \int_a^b k(t',s)\,v(s)\,ds = u(t'), \text{ a.e. on } [a,b].$$

So, $\mu u(t) \le u(t')$ a.e. on $[a,b]$ and this ensure that $u \in K'_{p,\mu}$. □

3. Main result

In this section we establish conditions which guarantee the localization of positive solution of (1.1) in $K_{r,R}$. We say that u is a positive solution to (1.1) if $u \in C(0,h;K)$ and satisfies (1.1).

The Banach space $C(0,h;K)$ is endowed with the norm

$$|u|_\infty = \max_{t\in[0,h]} |u(t)|, \quad u \in C(0,h;K).$$

We remind that $u \le y$ in $C(0,h;K)$ if and only if for any $t \in [0,h]$ we have $u(t) \le y(t)$ in K. For $M \subset X$ a bounded set we denote by $\alpha(M)$ the Kuratowskii measure of noncompactness on X, i.e.,

$$\alpha(M) = \inf\{\varepsilon > 0; M \text{ admits a finite cover by sets of diameter } \le \varepsilon\}.$$

To attain our goal we introduce the following hypotheses:

H1) the map $F : K \to K$ is L^1-Carathéodory, $q \ge 1$ and
$$F(x) \le F(y) \text{ for every } x, y \in K \text{ with } x \le y;$$

H2) for any $t \in [0,h]$, we have $k(t,\cdot) \in L^p[0,h]$, where $\frac{1}{p} + \frac{1}{q} = 1$;

H3) the map $t \longmapsto k(t,\cdot)$ is continuous from $[0,h]$ to $L^p[0,h]$;

H4) there exists $\omega : [0,2R] \to \mathbb{R}_+$ a L^p-Carathéodory map such that
  (a) $\alpha(F(M)) \le \omega(\alpha(M))$ for any bounded set $M \subset C(a,b;K)$;
  (b) the unique solution of
  $$\phi(t) \le 2\int_0^h k(t,s)\,\omega(\phi(s))\,ds, \quad t \in [0,h]$$
  is $\phi \equiv 0$.

H5) there exist $\mu \in (0,1)$, $\kappa \in L^p(0,h;\mathbb{R}_+)$ and $[a,b] \subset [0,1]$ such that k is $\mu 2$-bounded a.e. on $[0,h]$ with respect to κ and $[a,b]$;
H6) there exist $m_k, M_k > 0$ with $m_k \le \kappa(s)$ a.e. $s \in [a,b]$ and $\kappa(s) \le M_k$ a.e. $s \in [0,h]$.

Theorem 3.2. *Assume that* H1)-H6) *are satisfied and*

(h1) *there exist* $r > 0$ *such that*

$$\inf\{|F(x)|\,; x \in K, |x| = \mu r\} > \frac{r}{\mu m_k (b-a)}; \tag{3.21}$$

(h2) *there exist* $R > 0$ *such that*

$$\sup\{|F(x)|\,; x \in K, |x| = R\} \le \frac{R}{h M_k}. \tag{3.22}$$

Then the integral equation of Hammerstein type (1.1) *has at least one positive solution* $u \in C(0,h;K)$ *for which we have*

$$\begin{cases} 0 < r \le |u|_\infty \le R, \\ \mu u(t) \le u(t') \text{ for any } t \in [0,h] \text{ and } t' \in [a,b]. \end{cases}$$

Proof. Let be the cone

$$K_\mu = \{u \in C(0,h;K)\,;\ \ \mu u(t) \le u(t'), t \in [0,h], t' \in [a,b]\}$$

and the radial shell

$$K_{r,R} = \{u \in K_\mu;\ \ r \le |u|_\infty \le R\}.$$

If we consider the operator $T : K_{r,R} \to K_\mu$ defined by

$$T(u)(t) = \int_0^h k(t,s) F(u(s))\, ds,\ \ t \in [0,h],$$

then by Lemma 2.4, from H5) we have $T(u) \in K_\mu$.

The existence of a positive solution to (1.1) is equivalent to the existence of a fixed point to T. We will apply Theorem 1.1, so we will prove that MK2) and MK3) hold. Hypotheses H1)-H4) guarantee that T is an opertor of Mönch type, see [15]. So, MK2) holds.

Let $u \in K_\mu$ with $|u|_\infty = R$. By H6), for any $t \in [0,h]$ we have

$$|T(u)(t)| = \left|\int_0^h k(t,s) F(u(s))\, ds\right| \le \int_0^h k(t,s) |F(u(s))|\, ds$$
$$\le \int_0^h \kappa(s) |F(u(s))|\, ds \le M_\kappa \int_0^h |F(u(s))|\, ds.$$

Results

$$|T(u)|_\infty \le h M_\kappa \cdot \sup\{|F(x)|\,;\ \ x \in K, |x| \le R\}.$$

From (h2) we obtain

$$|T(u)|_\infty \le |u|_\infty. \tag{3.23}$$

Let $u \in K_\mu$ with $|u|_\infty = r$ and $t^* \in [0,h]$, $|u(t^*)| = r$. Hence $u \in K_\mu$, we can assume that $\mu u(t^*) \leq u(s)$ for any $s \in [a,b]$. From H1), we have

$$F(\mu u(t^*)) \leq F(u)(t) \text{ for any } t \in [a,b]. \tag{3.24}$$

Now, for $t \in [0,h]$ by H6) holds

$$T(u)(t) = \int_0^h k(t,s) F(u(s)) ds \geq \int_a^b \mu\kappa(s) F(u(s)) ds \tag{3.25}$$
$$\geq \mu F(\mu u(t^*)) \int_a^b \kappa(s) ds \geq \mu m_\kappa (b-a) \cdot F(\mu u(t^*)).$$

Since the norm $|\cdot|_\infty$ is increasing with respect to cone $C(0,h;K)$, we have

$$|T(u)|_\infty \geq \mu m_\kappa (b-a) \cdot \inf\{|F(x)|;\ \ x \in K, |x| = \mu r\}.$$

From (h1) we obtain

$$|T(u)|_\infty > |u|_\infty. \tag{3.26}$$

Now (3.23) and (3.26) guarantee that the inequalities from MK3) are satisfied and this complete the proof. □

Theorem 3.2 is an useful tool to study the various problems from the theory of abstract ordinary differential equations or abstract nonlinear integral equations.

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