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Random coincidence points of subcompatible multivalued maps with applications

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ABSTRACT.
The notion of subcompatible multivalued mapping is introduced. We present some random coincidence point and invariant approximation results for subcompatible random operators. Our work extends most of the important known results in the current literature to a new class of noncommuting multivalued mappings. We also develop random coincidence results for maps satisfying a more general contractive condition introduced by Ćirić, Ume and Jesic [5].

1. INTRODUCTION AND PRELIMINARIES

Let M be a nonempty subset of a normed space $(X, \|\cdot\|)$. We denote by 2^M, $C(M)$, $CB(M)$ and $K(M)$, the families of all nonempty, nonempty closed, nonempty closed bounded and nonempty compact subsets of M, respectively. On $CB(M)$, we define the Hausdorff metric H, by setting for $A, B \in CB(M)$,

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

where $d(a,B) = \inf\{\|a-x\| : x \in B\}$.

We define $P_M(x) = \{y \in M : \|y-x\| = d(x,M)\}$ as the set of best approximants to $x \in X$ out of M. Let $f : M \to M$ be a mapping. A mapping $T : M \to CB(M)$ is called f-Lipschitz if, for any $x, y \in M$, there exists $k \geq 0$ such that $H(Tx,Ty) \leq kd(fx,fy)$. If $k < 1$ (resp. $k = 1$), then T is called f-contraction (resp. f-nonexpansive). A point $x \in M$ is a coincidence point (common fixed point) of f and T if $fx \in Tx$ $(x = fx \in Tx)$. $F(T)$ denotes the set of fixed points of T. The pair $\{f,T\}$ is called:

(1) commuting if $Tfx = fTx$ for all $x \in M$;

(2) R-weakly commuting if for all $x \in M$, $fTx \in CB(M)$ and there exists $R > 0$ such that $H(fTx,Tfx) \leq Rd(fx,Tx)$;

(3) compatible (cf. [1], p. 319) if $fTx \in CB(M)$ for all $x \in M$ and $\lim_n H(fTx_n,Tfx_n) = 0$ whenever $\{x_n\}$ is a sequence in M such that $\lim_n fx_n = t \in \lim_n Tx_n = A \in CB(M)$;

(4) weakly compatible if f and T commute at their coincidence points.

The set M is called starshaped with respect to $q \in M$ if for all $x \in M$, the segment $[q,x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to x, is contained in M.

Suppose that M is q-starshaped with $q \in F(f)$, the set of fixed points of f. Then the pair $\{f,T\}$ is called:

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(5) R-subcommuting on M (see [20]) if for all $x \in M$, $fTx \in CB(M)$ and there exists a real number $R > 0$ such that $H(fTx, Tfx) \leq \frac{R}{k}\|y_k - fx\|$ for each $k \in (0, 1]$ and $y_k \in T_k x$, where $T_k x = (1-k)q + kTx$;

(6) R-subweakly commuting on M (see [10, 24]) if for all $x \in M$, $fTx \in CB(M)$ and there exists a real number $R > 0$ such that $H(fTx, Tfx) \leq Rd(fx, T_k x)$ for each $k \in [0, 1]$.

A map $T : M \to CB(X)$ is said to be demiclosed at y if for every sequence $\{x_n\}$ in M and $y_n \in T(x_n), n = 1, 2, \ldots$, such that $\{x_n\}$ converges weakly to x and $\{y_n\}$ converges to $y \in X$, then we have $y \in T(x)$. We say $T : M \to CB(X)$ is upper (lower) semicontinuous if for any closed (open) subset B of X, $T^{-1}(B) = \{x \in M : T(x) \cap B \neq \emptyset\}$ is closed (open); if T is both upper and lower semicontinuous, then T is continuous. In case $Tx \in K(X)$ for all $x \in M$, then T is continuous if and only if T is continuous from M into the metric space $(K(X), H)$, where H is the Hausdorff metric induced by the metric d. A mapping $f : X \to X$ is called weakly continuous if $\{x_n\}$ converges weakly to x implies that $\{f(x_n)\}$ converges weakly to $f(x)$. A mapping f on a q-starshaped set M is called affine if $f(tx + (1-t)q) = tf(x) + (1-t)f(q)$ for all $x \in M$ and $0 \leq t \leq 1$. If M is convex, then the mapping $T : M \to CB(M)$ is said to be semiconvex if for any $x, y \in M$, $z = tx + (1-t)y$, where $0 \leq t \leq 1$, and any $x_1 \in T(x)$, $y_1 \in T(y)$, there exists $z_1 \in T(z)$ such that $\|z_1\| \leq \max\{\|x_1\|, \|y_1\|\}$. A Banach space X satisfies Opial's condition if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \neq x$.

A mapping $T : \Omega \to CB(X)$ is called measurable if for any open subset C of X,

$$T^{-1}(C) = \{\omega \in \Omega : T(\omega) \cap C \neq \phi\} \in \sum .$$

A mapping $\xi : \Omega \to X$ is said to be a measurable selector of a measurable mapping $T : \Omega \to CB(X)$ if ξ is measurable and for any $\omega \in \Omega$, $\xi(\omega) \in T(\omega)$. A mapping $T : \Omega \times X \to CB(X)($ resp. $f : \Omega \times X \to X)$ is called a random operator if for any $x \in X, T(., x)($ resp. $f(., x))$ is measurable. A measurable mapping $\xi : \Omega \to X$ is called a random fixed point of a random operator $T : \Omega \times X \to CB(X)($ resp. $f : \Omega \times X \to X)$ if for every $\omega \in \Omega$, $\xi(\omega) \in T(\omega, \xi(\omega))($ resp. $f(\omega, \xi(\omega)) = \xi(\omega))$. A measurable mapping $\xi : \Omega \to X$ is a random coincidence point of random operators $T : \Omega \times X \to CB(X)$ and $f : \Omega \times X \to X$ if for every $\omega \in \Omega$, $f(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$. A random operator $T : \Omega \times M \to 2^X$(resp. $f : \Omega \times X \to X)$ is said to be continuous (weakly continuous, etc.) if for each $\omega \in \Omega$, $T(\omega, .)$ (resp. $f(\omega, .)$) is continuous (weakly continuous, etc.). The pair $\{f, T\}$ of random operators is called commuting (R-subcommuting, etc.) if for each $\omega \in \Omega$, the pair $\{f(\omega, .), T(\omega, .)\}$ is so.

Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. The interest in this subject enhanced after publication of the survey paper by Bharucha-Reid [3]. Since then various types of random fixed point theorems have been obtained by numerous mathematicians; see, for example, [5, 12, 16, 21, 23, 25, 27].

Recently, Shahzad and Hussain [25] and Khan and Hussain [17] established random versions of the results of Latif and Tweddle [19] in the setting of Banach spaces and Frêchet spaces, respectively. In [20], Rhoades has generalized some results of Latif and Tweddle [19] by replacing commutativity condition of maps with R-subcommuting condition whereas, Shahzad [24] has studied their results for the class of R-subweakly commuting maps. In this paper, we introduce the notion of subcompatible multivalued maps and prove some random coincidence point results for this general class of maps. As applications, common random fixed point and random invariant approximation results are derived. Our results unify and extend many known results existing in the literature including those of Dotson [6], Hussain and Khan [9], Hussain and Jungck [10], Itoh [12], Jungck and Sessa [14], Khan and Hussain [17], Latif and Bano [18], Latif and Tweddle [19], Rhoades [20], Shahzad [24], Shahzad and Hussain [25] and Xu [27].

We shall need the following known results.

Lemma 1.1. [19] *Let M be a nonempty weakly compact subset of a Banach space X satisfying Opial's condition. Let f:$M\to X$ be a weakly continuous map and T:$M\to K(X)$ an f-nonexpansive multivalued map. Then $f-T$ is demiclosed.*

Theorem 1.1. ([2], Theorem 5.1). *Let (X,d) be a separable complete metric space, $T:\Omega\times X\to CB(X)$ a multivalued random operator, and $f:\Omega\times X\to X$ a continuous random operator such that $T(\omega,X)\subseteq f(\omega,X)$ for each $\omega\in\Omega$. If f and T are compatible and for all $x,y\in X$ and all $\omega\in\Omega$, we have*

$$H(T(\omega,x),T(\omega,y))\leq kd(f(w,x),f(\omega,y)),$$

$k\in(0,1)$, *then T and f have a random coincidence point.*

2. Main Results

We begin with the definition of subcompatible multivalued mappings.

Definition 2.1. Let M be a q-starshaped subset of a normed space X.
Let $f:M\to M$ and $T:M\to CB(M)$ be maps with $q\in F(f)$. We define $S_q(f,T):=\cup\{S(f,T_k):0\leq k\leq 1\}$, where $T_kx=(1-k)q+kTx$ and $S(f,T_k)=\{\{x_n\}\subset M:\lim_n fx_n=t\in\lim_n T_kx_n=A\subset M\Rightarrow\lim_n H(fT_kx_n,T_kfx_n)=0\}$. The maps f and T are called *subcompatible* if $fTx\in CB(M)$ for all $x\in M$ and $\lim_n H(fTx_n,Tfx_n)=0$ for all sequences $\{x_n\}\in S_q(f,T)$.

For selfmaps T and f of M with $q\in F(f)$, we define $S_q(f,T):=\cup\{S(f,T_k):0\leq k\leq 1\}$ where $T_kx=(1-k)q+kTx$ and $S(f,T_k)=\{\{x_n\}\subset M:\lim_n fx_n=\lim_n T_kx_n=t\in M\Rightarrow\lim_n\|fT_kx_n-T_kfx_n\|=0\}$. Now f and T are subcompatible if $\lim_n\|fTx_n-Tfx_n\|=0$ for all sequences $\{x_n\}\in S_q(f,T)$.

Clearly, subcompatible maps are compatible but the converse does not hold, in general, as the following example shows.

Example 2.1. Let $X=R$ with usual norm and $M=[1,\infty)$. Let $f(x)=2x-1$ and $T(x)=x^2$, for all $x\in M$. Let $q=1$. Then M is q-starshaped with $fq=q$. Note that f and T are compatible. For any sequence $\{x_n\}$ in M with $\lim_n x_n=2$, we have $\lim_n fx_n=\lim_n T_{\frac{2}{3}}x_n=3\in M\Rightarrow\lim_n\|fT_{\frac{2}{3}}x_n-T_{\frac{2}{3}}fx_n\|=0$. However, $\lim_n\|fTx_n-Tfx_n\|\neq 0$ and thus f and T are not subcompatible maps.

Note that R-subweakly commuting and R-subcommuting maps are subcompatible. The following simple example reveals that the converse is not true, in general.

Example 2.2. Let $X = R$ with usual norm and $M = [0, \infty)$. Let $f(x) = \frac{x}{2}$ if $0 \leq x < 1$ and $fx = x$ if $x \geq 1$, and $T(x) = \frac{1}{2}$ if $0 \leq x < 1$ and $Tx = x^2$ if $x \geq 1$. Then M is 1-starshaped with $f1 = 1$ and $S_q(f, T) = \{\{x_n\} : 1 \leq x_n < \infty\}$. Note that f and T are subcompatible but not R-weakly commuting for all $R > 0$. Thus f and T are neither R-subweakly commuting nor R-subcommuting maps.

The following result is our main theorem.

Theorem 2.2. *Let M be a nonempty subset of a Banach space X which is starshaped with respect to $q \in M$, and let $f : \Omega \times M \to M$ be a continuous affine random operator with $f(\omega, q) = q$ for each $\omega \in \Omega$. Let $T : \Omega \times M \to K(M)$ be an f-nonexpansive random operator such that $T(\omega, M) \subseteq f(\omega, M)$ for each $\omega \in \Omega$. Suppose that the pair $\{f(\omega, .), T(\omega, .)\}$ is subcompatible and that one of the following conditions is satisfied:*
(a) M is compact;
(b) M is separable weakly compact, $(f - T)(\omega, .)$ is demiclosed at 0 for each $\omega \in \Omega$;
(c) M is separable weakly compact and X satisfies Opial's condition.
Then f and T have a random coincidence point.

Proof. Choose a sequence $\{k_n\}$ of real numbers with $0 < k_n < 1$ and $k_n \to 1$ as $n \to \infty$. For each n, consider the random operator $T_n : \Omega \times M \to CB(M)$ defined by

$$T_n(\omega, x) = (1 - k_n)\, q + k_n T(\omega, x).$$

Then,

$$\begin{aligned} H\left(T_n(\omega, x), T_n(\omega, y)\right) &= k_n H(T(\omega, x), T(\omega, y)) \\ &\leq k_n d(f(\omega, x), f(\omega, y)) \end{aligned}$$

for each $x, y \in M$ and each $\omega \in \Omega$. Since $T(\omega, M) \subset f(\omega, M)$ and f is affine with $f(\omega, q) = q$ for each $\omega \in \Omega$, we have $T_n(\omega, M) \subset f(\omega, M)$ for each $\omega \in \Omega$. Further, since the pair $\{f, T\}$ is subcompatible and f is affine with $f(\omega, q) = q$, for any $\{x_m\} \subset M$ and $\omega \in \Omega$, with $\lim_m f(\omega, x_m) \in \lim_m T_n(\omega, x_m)$, we have

$$\begin{aligned} \lim_m H(T_n(\omega, f(\omega, x_m)), f(\omega, T_n(\omega, x_m)) &= k_n \lim_m H(T(\omega, f(\omega, x_m), f(\omega, T(\omega, x_m)) \\ &= 0. \end{aligned}$$

Thus each pair $\{f, T_n\}$ is compatible.

(a) By Theorem 1.1, there is a measurable map $\xi_n : \Omega \to M$ such that

$$f\left(\omega, \xi_n(\omega)\right) \in T_n\left(\omega, \xi_n(\omega)\right), \text{ for each } \omega \in \Omega.$$

For each n, define $G_n : \Omega \to CB(M)$ by $G_n(\omega) = cl\,\{\xi_i(\omega) : i \geq n\}$ and $G : \Omega \to CB(M)$ by $G(\omega) = \bigcap_{n=1}^{\infty} G_n(\omega)$. As in [12], G is measurable and has a measurable selector $\xi : \Omega \to M$. Fix $\omega \in \Omega$ arbitrarily. Then some subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ converges to $\xi(\omega)$. As f and T are continuous, so $T_m(\omega, \xi_m(\omega)) \to T(\omega, \xi(\omega))$ and $f(\omega, \xi_m(\omega)) \to f(\omega, \xi(\omega))$. Consequently,

$$d(f(\omega, \xi(\omega)), T(\omega, \xi(\omega))) = \lim_m d(f(\omega, \xi_m(\omega)), T_m(\omega, \xi_m(\omega)) = 0.$$

Hence $f(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$.

(b) Since M is separable and weakly compact, the weak topology on M is a metric topology (see e.g.[17], p. 161). It follows that M is a complete metric space. Thus, by Theorem 1.1, there is a measurable map $\xi_n : \Omega \to M$ such that

$$f\left(\omega, \xi_n(\omega)\right) \in T_n\left(\omega, \xi_n(\omega)\right), \quad \text{for each } \omega \in \Omega.$$

For each n, define $F_n : \Omega \to WK(M)$ by

$$F_n(\omega) = w - cl\left\{\xi_i(\omega) : i \geq n\right\},$$

where $WK(M)$ is the family of all nonempty weakly compact subsets of M and $w-cl$ denotes the weak closure. Define $F : \Omega \to WK(M)$ by $F(\omega) = \bigcap_{n=1}^{\infty} F_n(\omega)$. As before, the weak topology on M is a metric topology. Then as in ([12], proof of Theorem 2.5) (see also [17]), F is w-measurable and has a measurable selector ξ. This ξ is the desired random coincidence point of f and T. Indeed, fix $\omega \in \Omega$ arbitrarily. Then some subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ converges weakly to $\xi(\omega)$. Also, there is some $u_m \in T(\omega, \xi_m(\omega))$ such that

$$f(\omega, \xi_m(\omega)) - u_m = (1 - k_m)\left\{q - u_m\right\}.$$

The set M is bounded and $k_m \to 1$, it follows that $f(\omega, \xi_m(\omega)) - u_m \to 0$. Now $y_m = f(\omega, \xi_m(\omega)) - u_m \in (f - T)(\omega, \xi_m(\omega))$ and $y_m \to 0$. As $(f - T)(\omega, .)$ is demiclosed at 0, it follows that $f(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$.

(c) An affine and continuous map is weakly continuous [6], so by Lemma 1.1, $(f - T)(\omega, .)$ is demiclosed at 0. Hence, f and T have a random coincidence point ξ by part (b). □

R-subweakly commuting maps are subcompatible, so we obtain the following recent result as immediate corollary to our main theorem.

Corollary 2.1. ([24], Theorem 3.1). *Let M be a nonempty separable weakly compact subset of a Banach space X, which is q-starshaped, and let $f : \Omega \times M \to M$ be a continuous affine random operator such that $f(\omega, q) = q$ for each $\omega \in \Omega$.*

Let $T : \Omega \times M \to K(M)$ be an f-nonexpansive random operator such that $T(\omega, M) \subset f(\omega, M)$. Suppose that the pair $\{f, T\}$ is R-subweakly commuting and that one of the following two conditions is satisfied:

(a) $(f - T)(\omega, .)$ is demiclosed at 0 for each $\omega \in \Omega$;

(b) X satisfies Opial's condition.

Then f and T have a random coincidence point.

Recall that a Banach space X is almost smooth [15] if $SM(B)$ is dense in X^*, where $SM(B)$ is the set of all functionals of X^* which attain their norm at a smooth point of the unit ball $B = \{x \in X : \|x\| \leq 1\}$. A subset M of X is called Chebyshev if for each point $x \in X$, $P_M(x)$ is a singleton.

Every weakly compact Chebyshev subset of an almost smooth Banach space is convex [15], so we obtain the following result which properly contains Corollary 3.2 [24].

Corollary 2.2. *Let M be a nonempty Chebyshev subset of an almost smooth Banach space X, and let $f : \Omega \times M \to M$ be a continuous affine random operator.*

Let $T : \Omega \times M \to K(M)$ be an f-nonexpansive random operator such that $T(\omega, M) \subset f(\omega, M)$ for each $\omega \in \Omega$. Suppose that the pair $\{f(\omega, .), T(\omega, .)\}$ is subcompatible and that one of the following conditions is satisfied:

(a) M is compact;
(b) M is separable weakly compact, $(f-T)(\omega,.)$ is demiclosed at 0 for each $\omega\in\Omega$;
(c) M is separable weakly compact and X satisfies Opial's condition.
Then f and T have a random coincidence point.

From Theorem 2.2 we obtain the following common fixed point result.

Theorem 2.3. *Suppose that $M, f, T,$ and q satisfy all the hypotheses of Theorem 2.2. If for any $x\in M$ and $\omega\in\Omega$, $f(\omega,x)\in T(\omega,x)$ implies the existence of $\lim_n f^n(\omega,x)$, then f and T have a common random fixed point.*

Proof. By Theorem 2.2, T and f have a random coincidence point $\xi_0:\Omega\to M$ i.e. for each $\omega\in\Omega$, $f(\omega,\xi_0(\omega))\in T(\omega,\xi_0(\omega))$. Fix $\omega\in\Omega$. Since the pair $\{f,T\}$ is compatible so they commute at their coincidence points. Thus, we have $f^n(\omega,\xi_0(\omega))=f^{n-1}(\omega,f(\omega,\xi_0(\omega)))\in f^{n-1}(\omega,T(\omega,\xi_0(\omega)))=T(\omega,f^{n-1}(\omega,\xi_0(\omega)))$.

Let $\xi(\omega)=\lim_n f^n(\omega,\xi_0(\omega))$. The mapping ξ being the pointwise limit of measurable mappings is measurable. Taking limit as $n\to\infty$, we get $\xi(\omega)=f(\omega,\xi(\omega)\in T(\omega,\xi(\omega))$. Hence ξ is a common random fixed point of f and T. □

Theorem 2.4. *Let M be a nonempty separable weakly compact convex subset of a Banach space X, and let $f:\Omega\times M\to M$ be a continuous affine random operator.*
Let $T:\Omega\times M\to K(M)$ be an f-nonexpansive random operator such that $T(\omega,M)\subset f(\omega,M)$ for each $\omega\in\Omega$. Suppose that the pair $\{f(\omega,.),T(\omega,.)\}$ is subcompatible and that $(f-T)(\omega,.)$ is semiconvex for each $\omega\in\Omega$. Then f and T have a random coincidence point.

Proof. Choose a sequence $\{k_n\}$ of real numbers with $0<k_n<1$ and $k_n\to 1$ as $n\to\infty$. For each n, define the random operator $T_n:\Omega\times M\to CB(M)$ by

$$T_n(\omega,x)=(1-k_n)\,q+k_nT(\omega,x),$$

where $q=f(\omega,q)$ for each $\omega\in\Omega$. Then, as in the proof of Theorem 2.2, we have

$$f\left(\omega,\xi_n(\omega)\right)\in T_n\left(\omega,\xi_n(\omega)\right)$$

for each $\omega\in\Omega$. Fix $\omega\in\Omega$ arbitrarily. For each n, there is some $u_n\in T(\omega,\xi_n(\omega))$ such that

$$f(\omega,\xi_n(\omega))-u_n=(1-k_n)\left\{q-u_n\right\}.$$

The set M is bounded and $k_n\to 1$, it follows that $f(\omega,\xi_n(\omega))-u_n\to 0$ and hence $d(f(\omega,\xi_n(\omega)),T(\omega,\xi_n(\omega)))\to 0$ as $n\to\infty$. Thus $\inf\{d(f(\omega,x),T(\omega,x)):x\in M\}=0$. Define a mapping $h_n:\Omega\times M\to R$ as $h_n(\omega,x)=d(f(\omega,x),T(\omega,x))-\frac{1}{n}$, $n\geq 1$. Then by Rybinski ([21], Lemmas 1 and 2), each h_n is a Caratheodory function (that is, continuous in x and measurable in ω).
For each n, define $G_n(\omega)=\{x\in M:h_n(\omega,x)<0\}$ and $L_n(\omega)=cl(G_n(\omega))$. Then as in ([24], proof of Theorem 3.2), $L(\omega)=\bigcap\limits_{n=1}^{\infty}L_n(\omega)$ is measurable and has a measurable selector ξ. This ξ is the desired random coincidence point of f and T. □

As applications of Theorem 2.2 and Theorem 2.4, we obtain the following random approximation theorem which extends Theorem 7 of Jungck and Sessa [14], and the main results of Latif and Bano [18], Sahab et al. [22] and Singh [26] to the wider class of subcompatible maps.

Theorem 2.5. *Let X be a Banach space, $u \in X$, and $M \subseteq X$.*
$f : \Omega \times X \longrightarrow X$ and $T : \Omega \times X \longrightarrow K(X)$ be two random operators. Suppose that $P_M(u)$ is nonempty, $q-$ starshaped, f is affine and continuous mapping, $f(\omega, q) = q$ for each $\omega \in \Omega$, T is f-nonexpansive and for every $\omega \in \Omega$, $T(\omega, \partial M \cap M) \subseteq M$, $f(\omega, P_M(u)) = P_M(u)$, $\sup_{y\in T(\omega,x)} \|y - u\| \le \|f(\omega, x) - u\|$ for all $x \in P_M(u)$. Suppose that the pair $\{f(\omega, .), T(\omega, .)\}$ is subcompatible on $P_M(u)$ and one of the following conditions is satisfied:

(a) $P_M(u)$ is compact;

(b) $P_M(u)$ is separable weakly compact, and $(f - T)(\omega, .)$ is demiclosed at 0 for each $\omega \in \Omega$;

(c) $P_M(u)$is separable weakly compact, and X satisfies Opial's condition;

(d) $P_M(u)$ is separable weakly compact and convex instead of $q-$ starshaped and $(f - T)(\omega, .)$ is semiconvex for each $\omega \in \Omega$.

Then T and f have a random coincidence point $\psi : \Omega \to P_M(u)$. If, in addition, for any $v \in M$ and $\omega \in \Omega$, $f(\omega, f(\omega, v)) = f(\omega, v)$ whenever $f(\omega, v) \in T(\omega, v)$, then there exists a common random fixed point $\xi : \Omega \to P_M(u)$ of f and T.

Proof. Fix $\omega \in \Omega$. Let $x \in P_M(u)$. Then $\|(1 - \lambda)x + \lambda u - u\| < \|x - u\| = d(u, M)$ for all $\lambda \in (0, 1)$. Thus, $\{(1 - \lambda)x + \lambda u : \lambda \in (0, 1)\} \cap M = \emptyset$ and so $x \in M \cap \partial M$. Since, $T(\omega, M \cap \partial M) \subseteq M$, it follows that $T(\omega, x) \subseteq M$. Now let $z \in T(\omega, x)$. As $f(\omega, x) \in P_M(u)$,

$$\|z - u\| \le \sup_{y\in T(\omega,x)} \|y - u\| \le \|f(\omega, x) - u\| = d(u, M).$$

Thus $z \in P_M(u)$ and hence $T(\omega, x) \subseteq P_M(u)$. Moreover, $T(\omega, P_M(u)) \subseteq P_M(u) = f(\omega, P_M(u))$. Thus in each case, f and T have a random coincidence point $\psi : \Omega \to P_M(u)$, i.e., $f(\omega, \psi(\omega)) \in T(\omega, \psi(\omega))$ for each $\omega \in \Omega$ (for (a)-(c), we apply Theorem 2.2, and for (d), we use Theorem 2.4). Let $\xi(\omega) = f(\omega, \psi(\omega))$ for $\omega \in \Omega$. Then $\xi : \Omega \to P_M(u)$ is measurable. Fix $\omega \in \Omega$ arbitrarily. Since T and f are compatible, we have $\xi(\omega) = f(\omega, \psi(\omega)) = f(\omega, \xi(\omega)) = f(\omega, f(\omega, \psi(\omega)) \in f(\omega, T(\omega, \psi(\omega)) = T(\omega, f(\omega, \psi(\omega)) = T(\omega, \xi(\omega))$. Thus ξ is a common random fixed point of f and T. □

Corollary 2.3. *Let M be a subset of a Banach space X, $f : \Omega \times X \to X$ and $T : \Omega \times X \to K(X)$. Assume that $P_M(u)$ is nonempty q-starshaped, f is affine and continuous mapping, T is f-nonexpansive on $P_M(u)$, and $T(\omega, P_M(u)) \subset f(\omega, P_M(u)) \subset P_M(u)$ and $f(\omega, q) = q$ for each $\omega \in \Omega$. Suppose that the pair $\{f(\omega, .), T(\omega, .)\}$ is subcompatible on $P_M(u)$ and one of the conditions $((a) - (d))$ in Theorem 2.5 is satisfied. Then T and f have a random coincidence point $\psi : \Omega \to P_M(u)$. If, in addition, for any $v \in M$ and $\omega \in \Omega$, $f(\omega, f(\omega, v)) = f(\omega, v)$ whenever $f(\omega, v) \in T(\omega, v)$, then there exists a common random fixed point $\xi : \Omega \to P_M(u)$ of f and T.*

3. Concluding Remarks

(1) Theorem 2.2, Theorem 2.3 and Theorem 2.5 (a-c) remain valid if starshapedness of the set M is replaced by the following property (N) considered for single-valued case in [8, 11] and for multivalued case in [7, 25]:
A set M is said to have property (N) with respect to (w.r.t) T if,

(i) $T : M \to C(M)$,

(ii) $(1 - k_n)q + k_nTx \subset M$, for some $q \in M$ and a fixed sequence of real numbers $k_n(0 < k_n < 1)$ converging to 1 and for each $x \in M$.
Consequently, we obtain the stochastic generalizations of recent results due to Hussain [7] and Hussain and Berinde [8].

(2) Theorem 2.2-Corollary 2.1 and Theorems 2.4-Corollary 2.3 remain valid in the setup of a Frêchet space (X, d). Consequently, recent results due to Khan and Hussain [17] and ([24], Theorem 3.4 and Corollary 3.5) are extended to the class of subcompatible maps. Moreover, these results provide stochastic versions of the corresponding results in [9] and [11].

(3) Theorem 2.2 extends Theorem 3.4 in [12], Theorem 6 [14], Theorem 2.2 and Theorem 2.3 in [19] and Theorem 1 (ii) by Xu [27].

(4) Theorem 2.3 extends the common random fixed point result in [24].

(5) Theorem 2.4 contains properly Theorem 3.2 and Corollary 3.4 [24].

(6) Following the above arguments and those in [10], we may extend Theorem 2.2-Theorem 2.12 and Theorem 2.14-Corollary 2.16 of Hussain and Jungck [10] from R-subweakly commuting maps to subcompatible maps by using Theorem 3.1 of Jungck [13].

(7) In Theorem 2.1, Ćirić et al. [5] (see also [4]) employed a very general contractive condition given in (1.2). Utilizing this result of Ćirić et al. [5], we can similarly prove the coincidence point results without any commutativity condition on maps. These results in turn will generalize the corresponding results obtained in section 3 by Shahzad and Hussain [25]. We leave details to the reader.

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