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Anti-periodic solutions of functional difference equations with p-Laplacian

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ABSTRACT.

The p-Laplacian functional difference equation

$$\Delta\Big[\,\sigma(k)\phi_p\Big(\,\Delta x(k)\Big)\Big]=f(k,x(k),x(k+1),x(\tau_1(k),\cdots,x(\tau_m(k)),\quad k\in Z$$

is studied. Three new existence results for anti-periodic solutions of the equation are established.

1. INTRODUCTION

Discrete BVPs are fascinating and challenging fields of study. Recently, there exist several papers concerned with the existence of solutions of boundary value problems of functional difference equations, for example:

• in paper [9], the authors studied the existence of solutions of boundary value problems of first order difference equations with maxima and with nonlinear functional boundary value conditions. Such boundary conditions include, among others, initial, periodic, antiperiodic and multi point boundary value conditions, as particular cases.

• in [14], the authors investigated and proved the existence of positive solutions of periodic boundary value problems for discrete nonlinear equations. The criteria which leads to nonexistence of positive solutions were also obtained by employing the index theory for the positive smooth function f and lower and upper solutions when the function f is allowed to have nonconstant sign and to be singular at 0.

Let T be an positive integer. If a real number sequence $\{x(n)\}$ satisfies $x(n+T)=-x(n)$ for all $n\in\mathbb{Z}$, we call $\{x(n)\}$ a anti-periodic sequence. It is easy to see that $\{x(n)\}$ is $2T$-periodic if $\{x(n)\}$ is T-anti-periodic. Hence we actually get a $2T$-periodic solution of a functional difference equation if one obtains a T-antiperiodic solution of the same equation. In applications, anti-periodic solutions of differential or difference equations are useful for studies. There exist many papers concerned with the existence of solutions of the anti-periodic boundary value problems of functional differential equations, see [5], [6], [3]-[12] and the references therein, but we note that

• there is no paper concerned with the existence of anti-periodic solutions of p-Laplacian functional difference equations.

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In [4], the authors presented results for the existence of solutions of n-th order difference equations with anti-periodic boundary conditions by using the lower and upper solution methods.

Motivated by the reasons mentioned above, we study the existence of anti-periodic solutions of the following higher order nonlinear p-Laplacian functional difference equation

$$\Delta\Big[\sigma(k)\phi_p\left(\Delta x(k)\right)\Big] = f(k, x(k), x(k+1), x(\tau_1(k), \ldots, x(\tau_m(k)), \quad k \in \mathbb{Z}, \tag{1.1}$$

where

- $\mathbb{Z}$ is the set of all integers, $[\alpha, \beta] = \{\alpha, \cdots, \beta\}$ if $\alpha \leq \beta$ and $[\alpha, \beta] = \emptyset$ if $\alpha > \beta$ for $\alpha, \beta \in Z$;
- $\Delta x(k) = x(k+1) - x(k)$;
- $\tau_i : Z \to Z(i \in [1, m])$.
- $\phi_p(x) = |x|^{p-2}x$ for $x \neq 0$ and $\phi_p(0) = 0$ with its inverse function ϕ_q defined by $\phi_q(x) = |x|^{q-2}x$ for $x \neq 0$ and $\phi_q(0) = 0$, where $1/p + 1/q = 1$, $p > 1, q > 1$;
- $f : \mathbb{Z} \times \mathbb{R}^{m+2} \to \mathbb{R}$ is continuous in $u = (u, v, x_1, \cdots, x_m)$ for each $n \in \mathbb{Z}$;
- $\sigma(n)$ is a positive sequence.

Suppose that $T \geq 1$ is an integer. An anti-periodic sequence $\{x(k)\}_{k=-\infty}^{+\infty}$ is called an anti-periodic solution of equation (1.1) if it satisfies (1.1) for all $k \in \mathbb{Z}$.

The purpose of this paper is to use continuation theorem of coincidence degree, see [5], [6], [13], to establish sufficient conditions for the existence of at least one anti-periodic solution of equation (1.1). It is interesting that we allow f to be sublinear, at most linear or superlinear. This paper may be the first one concerned with the existence of anti-periodic solutions of the p-Laplacian functional difference equations.

The remainder is organized as follows. In Section 2, we make preparations, and the main results are presented at the end of this section. Two examples to illustrate the main results will be given in Section 3.

2. Main Results

In this section, we first give preparations. Then the main results are given at the end.

Let X and Y be Banach spaces, and $L : D(L) \bigcap X \to Y$ and $N : \ X \to Y$ the maps.

Definition 2.1. [13] The linear operator $L : D(L) \bigcap X \to Y$ is called a Fredholm operator of index zero if ImL is closed in Y and dim Ker$L$ = codim Im$L < +\infty$.

Definition 2.2. [13] If Ω is an open bounded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N : \ X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1. [13] *Let X and Y be Banach spaces. Suppose $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero with Ker$L = \{0\}$, $N : X \to Y$ is $L-$compact on any open bounded subset of X. If $0 \in \Omega \subset X$ is an open bounded subset and $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in [0, 1]$, then there exist at least one $x \in \Omega$ such that $Lx = Nx$.*

Let S be the set of all $T-$anti-periodic sequences. Choose $X = S \times S$. Define the norm $\|(x,y)\| = \max\{\max\{|x_i| : i \in \mathbb{Z}\}, \max\{|x_i| : i \in Z\}\}$. Then X is a real Banach space.

Let $L : X \to X$ be defined by

$$L\left(\begin{array}{c} x(n) \\ y(n) \end{array}\right) = \left(\begin{array}{c} \Delta x(n) \\ \Delta y(n) \end{array}\right), \quad (x,y) \in X.$$

Define $N : X \to Y$ by

$$N\left(\begin{array}{c} x(n) \\ y(n) \end{array}\right) = \left(\begin{array}{c} \phi^{-1}\left(\frac{y(n)}{\sigma(n)}\right) \\ f(n, x(n+1), x(\tau_1(n)), \cdots, x(\tau_m(n))) \end{array}\right), \quad (x,y) \in X.$$

Lemma 2.2. *The following results hold.*

(i) *Ker* $L = \left\{(x,y) \in X : \begin{array}{ll} x(n) = 0, & n \in \mathbb{Z}, \\ y(n) = 0, & n \in \mathbb{Z}, \end{array}\right\}$;

(ii) *L is a Fredholm operator of index zero;*

(iii) *Let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \neq \emptyset$, then N is L-compact on $\overline{\Omega}$;*

(iv) *$(x,y) \in X$ is a solution of $L(x,y) = N(x,y)$ implies that x is a solution of equation* (1.1).

Suppose the followings:

(H) there exist numbers $\beta > 0$, $\theta \geq 0$, nonnegative anti-periodic sequences $q_0(n)$, $p_i(n)$, $r(n)(i = 0, \ldots, m)$, continuous functions $g(n, x, y, x_1, \ldots, x_m)$, $h(n, x, y, x_1, \ldots, x_m)$ such that

$$f(n, x, y, x_1, \ldots, x_m) = g(n, x, y, x_1, \ldots, x_m) + h(n, x, y, x_1, \ldots, x_m) \tag{2.2}$$

and

$$g(n, x, y, x_1, \ldots, x_m)y \geq \beta |y|^{\theta+1}, \tag{2.3}$$

and

$$|h(n, x, y, x_1, \ldots, x_m)| \leq q_0(n)|x|^\theta + \sum_{s=0}^{m} p_i(n)|x_i|^\theta + r(n), \tag{2.4}$$

for all $n \in [0, T-1]$, $(x, y, x_1, \ldots, x_m) \in \mathbb{R}^{m+2}$;

(K) there exist numbers $\beta > 0$, $\theta > 0$, nonnegative anti-periodic sequences $q_0(n)$, $p_i(n)$, $r(n)(i = 0, \ldots, m)$, continuous functions $g(n, x, y, x_1, \ldots, x_m)$, $h(n, x, y, x_1, \ldots, x_m)$ such that

$$f(n, x, y, x_1, \ldots, x_m) = g(n, x, y, x_1, \ldots, x_m) + h(n, x, y, x_1, \ldots, x_m) \tag{2.5}$$

and

$$g(n, x, y, x_1, \ldots, x_m) \geq \beta |y|^{\theta}, \tag{2.6}$$

and

$$|h(n, x, y, x_1, \ldots, x_m)| \leq q_0(n)|x|^\theta + \sum_{s=0}^{m} p_i(n)|x_i|^\theta + r(n), \tag{2.7}$$

for all $n \in [0, T-1]$, $(x, y, x_1, \ldots, x_m) \in \mathbb{R}^{m+2}$.

Lemma 2.3. *Suppose that* (H) *holds and* $\sigma(n) = \sigma(T+n)$ *for all* $n \in \mathbb{Z}$. *Let* $\Omega_1 = \{(x,y) : L(x,y) = \lambda N(x,y),\ ((x,y),\lambda) \in D(L) \bigcap X \times (0,1)\}$. *Then* Ω_1 *is bounded if*

$$||q_0|| + ||p_0|| + T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| < \beta. \tag{2.8}$$

Proof. For $x \in \Omega_1$, we have $L(x,y) = \lambda N(x,y)$, $\lambda \in (0,1)$, so

$$\begin{cases} \Delta x(n) = \lambda \phi_q\left(\frac{y(n)}{\sigma(n)}\right), & n \in \mathbb{Z}, \\ \Delta y(n) = \lambda f(n, x(n), x(n+1), x(\tau_1(n)), \dots, x(\tau_m(n))), & n \in \mathbb{Z}. \end{cases} \tag{2.9}$$

It follows that

$$\begin{aligned} &\Delta\left[\sigma(n)\phi_p\left(\Delta x(n)\right)\right] x(n+1) \\ &= \lambda\phi_p(\lambda) f(n, x(n), x(n+1), x(\tau_1(n)), \dots, x(\tau_m(n)) x(n+1). \end{aligned} \tag{2.10}$$

Since $\sigma(T) = \sigma(0)$, $x(T+1) = -x(1)$, $\Delta x(T) = -\Delta x(0)$ imply that

$$\begin{aligned} & \sum_{n=0}^{T-1} \Delta[\sigma(n)\phi_p(\Delta x(n))] x(n+1) \\ = & \sum_{n=0}^{T-1} [\sigma(n+1)\phi_p(\Delta x(n+1)) - \sigma(n)\phi_p(\Delta x(n))][x(n+2) - \Delta x(n+1)] \\ = & \sum_{n=0}^{T-1} [\sigma(n+1)\phi_p(\Delta x(n+1)) x(n+2) - \sigma(n)\phi_p(\Delta x(n)) x(n+1)] \\ & - \sum_{n=0}^{T-1} \sigma(n+1)\phi_p(\Delta x(n+1)) \Delta x(n+1) \\ = & \sigma(T)\phi_p(\Delta x(T)) x(T+1) - \sigma(0)\phi_p(\Delta x(0)) x(1) \\ & - \sum_{n=0}^{T-1} \sigma(n+1)\phi_p(\Delta x(n+1)) \Delta x(n+1) \\ = & - \sum_{n=0}^{T-1} \sigma(n+1)\phi_p\Big(\Delta x(n+1)\Big) \Delta x(n+1), \end{aligned}$$

and $\phi_p(x) = x|x|^p$, we get

$$\sum_{n=0}^{T-1} f(n, x(n), x(n+1), x(\tau_1(n)), \dots, x(\tau_m(n)) x(n+1) \le 0.$$

Together with (2.2), (2.3) and (2.4), one gets that

$$\begin{aligned}
& \beta \sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \\
\leq \ & \sum_{n=0}^{T-1} g(n, x(n), x(n+1), w(\tau_1(n)), \ldots, x(\tau_m(n)) x(n+1) \\
\leq \ & -\sum_{n=0}^{T-1} h(n, x(n), x(n+1), x(\tau_1(n)), \ldots, x(\tau_m(n)) x(n+1) \\
\leq \ & \sum_{n=0}^{T-1} |h(n, x(n), x(n+1), x(\tau_1(n)), \ldots, x(\tau_m(n))| \, |x(n+1)| \\
\leq \ & \sum_{n=0}^{T-1} q_0(n)|x(n)|^{\theta}|x(n+1)| + \sum_{n=0}^{T-1} p_0(n)|x(n+1)|^{\theta+1} \\
& + \sum_{i=1}^{m} \sum_{n=0}^{T-1} p_i(n)|x(\tau_i(n))|^{\theta}|x(n+1)| + \sum_{n=0}^{T-1} r(n)|x(n+1)| \\
\leq \ & ||q_0|| \sum_{n=0}^{T-1} |x(n)|^{\theta}|x(n+1)| + ||p_0|| \sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \\
& + \sum_{i=1}^{m} ||p_i|| \sum_{n=0}^{T-1} |x(\tau_i(n))|^{\theta}|x(n+1)| + ||r|| \sum_{n=0}^{T-1} |x(n+1)|.
\end{aligned}$$

For $x_i \geq 0$, $y_i \geq 0$, Holder's inequality implies

$$\sum_{i=1}^{s} x_i y_i \leq \left(\sum_{i=1}^{s} x_i^p \right)^{1/p} \left(\sum_{i=1}^{s} y_i^q \right)^{1/q}, \quad 1/p + 1/q = 1, \quad q > 0, \ p > 1.$$

It follows that

$$\begin{aligned}
& \beta \sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \\
\leq \ & ||q_0|| \left(\sum_{n=0}^{T-1} |x(n)|^{\theta+1} \right)^{\theta/(\theta+1)} \left(\sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \right)^{1/(\theta+1)} \\
& + ||p_0|| \sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} + ||r|| T^{\theta/(\theta+1)} \left(\sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \right)^{1/(\theta+1)} \\
& + \sum_{i=1}^{m} ||p_i|| \left(\sum_{n=0}^{T-1} |x(\tau_i(n))|^{\theta+1} \right)^{\theta/(\theta+1)} \left(\sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \right)^{1/(\theta+1)}
\end{aligned}$$

$$\begin{aligned}
&= ||q_0||\sum_{u=0}^{T-1}|x(u+1)|^{\theta+1}+||p_0||\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\\
&+ ||r||T^{\theta/(\theta+1)}\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{1/(\theta+1)}\\
&+\sum_{i=1}^{m}||p_i||\left(\sum_{u\in\{\tau_i(n)-1:\ n=0,\dots,T-1\}}|x(u+1)|^{\theta+1}\right)^{\theta/(\theta+1)}\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{1/(\theta+1)}\\
&\le ||q_0||\sum_{u=0}^{T-1}|x(u+1)|^{\theta+1}+||p_0||\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\\
&+ ||r\|T^{\theta/(\theta+1)}\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{1/(\theta+1)}+T^{\theta/(\theta+1)}\sum_{i=1}^{m}\|p_i\|\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left(\beta-||q_0||-||p_0||-T^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||p_i||\right)\sum_{u=0}^{T-1}|x(u+1)|^{\theta+1}\\
&\le ||r||T^{\frac{\theta}{\theta+1}}\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{-\frac{\theta}{\theta+1}}.
\end{aligned}$$

It follows from (2.8) that $\beta-||q_0||-||p_0||-T^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||p_i||>0$. Then there is $M_1>0$ such that $\sum_{u=0}^{T-1}|x(u+1)|^{\theta+1}\le M_1$. So $|x(n+1)|\le M_1^{1/(\theta+1)}$ for all $n\in Z$. We get $||x||\le M_1^{1/(\theta+1)}=:M$.

It follows from (2.9) that

$$\begin{aligned}
|\Delta y(n)| &= |\lambda f(n,x(n),x(n+1),x(\tau_1(n)),\dots,x(\tau_m(n)))|\\
&\le \max_{\substack{n\in[0,T-1]\\ |x|\le M,|y|\le M\\ |x_i|\le M,i\in[1,m]}}\left|f(n,x,y,x_1,\dots,x_m)\right| =: M'.
\end{aligned}$$

Since $y(T)=-y(0)$, we get that there exist $0\le\xi\le T$ such that

$$\frac{0-y(0)}{\xi-0}=\frac{y(T)-y(0)}{T-0}.$$

Then

$$|y(0)|=\left|\frac{\xi}{T}(y(T)-y(0))\right|\le\sum_{k=0}^{T-1}|\Delta y(k)|\le TM'.$$

Hence

$$|y(n)|=\left|y(0)+\sum_{k=0}^{n-1}\Delta y(k)\right|\le TM'+\sum_{k=0}^{n-1}|\Delta y(k)|\le 2TM'$$

for all $n\in[0,T]$. It follows that

$$\|y\|\le 2TM'.$$

So

$$\|(x,y)\| \le \max\{M,\ \ 2TM'\}.$$

So Ω_1 is bounded. The proof of Lemma 2.3 is complete. □

Lemma 2.4. *Suppose that* (K) *holds and* $\sigma(n+T) = -\sigma(n)$ *for all* $n \in \mathbb{Z}$*. Let* $\Omega_1 = \{(x,y) : L(x,y) = \lambda N(x,y),\ ((x,y),\lambda) \in D(L)\bigcap X \times (0,1)\}$*. Then* Ω_1 *is bounded if*

$$\|q_0\| + \|p_0\| + T\sum_{i=1}^{m} ||p_i|| < \beta. \tag{2.11}$$

Proof. For $x \in \Omega_1$, we have $L(x,y) = \lambda N(x,y)$, $\lambda \in (0,1)$, so we have (2.9) and (2.10). Then $\sigma(T) = -\sigma(0)$, $x(T+1) = -x(1)$, $\Delta x(T) = -\Delta x(0)$ imply that

$$\begin{aligned}\sum_{n=0}^{T-1}\Delta[\sigma(n)\phi_p(\Delta x(n))] &= \sum_{n=0}^{T-1}[\sigma(n+1)\phi_p(\Delta x(n+1)) - \sigma(n)\phi_p(\Delta x(n))]\\ &= \sigma(T)\phi_p(\Delta x(T)) - \sigma(0)\phi_p(\Delta x(0)) = 0.\end{aligned}$$

So, we get

$$\sum_{n=0}^{T-1} f(n, x(n), x(n+1), x(\tau_1(n)), \cdots, x(\tau_m(n))) = 0. \tag{2.12}$$

It follows from (2.5), (2.6) and (2.7) that

$$\begin{aligned}&\beta\sum_{n=0}^{T-1}|x(n+1)|^\theta\\ \le\ &\sum_{n=0}^{T-1} g(n, x(n), x(n+1), w(\tau_1(n)), \cdots, x(\tau_m(n)))\\ \le\ &-\sum_{n=0}^{T-1} h(n, x(n), x(n+1), x(\tau_1(n)), \cdots, x(\tau_m(n)))\\ \le\ &\sum_{n=0}^{T-1} |h(n, x(n), x(n+1), x(\tau_1(n)), \cdots, x(\tau_m(n))|\\ \le\ &\sum_{n=0}^{T-1} q_0(n)|x(n)|^\theta + \sum_{n=0}^{T-1} p_0(n)|x(n+1)|^\theta + \sum_{i=1}^{m}\sum_{n=0}^{T-1} p_i(n)|x(\tau_i(n))|^\theta + \sum_{n=0}^{T-1} r(n)\\ \le\ &||q_0||\sum_{n=0}^{T-1}|x(n)|^\theta + ||p_0||\sum_{n=0}^{T-1}|x(n+1)|^\theta + \sum_{i=1}^{m}||p_i||\sum_{n=0}^{T-1}|x(\tau_i(n))|^\theta + ||r||T.\end{aligned}$$

We get that

$$\begin{aligned}
&\beta \sum_{n=0}^{T-1} |x(n+1)|^\theta \\
\le\ & (\|q_0\|+\|p_0\|) \sum_{n=0}^{T-1} |x(n+1)|^\theta + \sum_{i=1}^{m} \|p_i\| \sum_{u=\tau_i(n)-1:n\in[0,T-1]} |x(u+1)|^\theta + \|r\|T \\
\le\ & (\|q_0\|+\|p_0\|) \sum_{u=0}^{T-1} |x(u+1)|^\theta + T\sum_{i=1}^{m} \|p_i\| \sum_{u=0}^{T-1} |x(u+1)|^\theta + \|r\|T.
\end{aligned}$$

Since $\|q_0\|+\|p_0\|+T\sum\limits_{i=1}^{m}\|p_i\| < \beta$, we get that there exists a constant $M_1 > 0$ such that

$$\sum_{u=0}^{T-1} |x(u+1)|^\theta \le M_1.$$

Hence $|x(n+1)| \le M_1^{1/(\theta)} =: M$ for all $n \in \mathbb{Z}$. It follows that $\|x\| \le M$.

It follows from (2.9) that

$$\begin{aligned}
|\Delta y(n)| &= \Big| \lambda f(n, x(n), x(n+1), x(\tau_1(n)), \cdots, x(\tau_m(n))) \Big| \\
&\le \max_{\substack{n\in[0,T-1] \\ |x|\le M, |y|\le M \\ |x_i|\le M, i\in[1,m]}} |f(n,x,y,x_1,\cdots,x_m)| =: M'.
\end{aligned}$$

Since $y(T) = -y(0)$, we get that there exist $0 \le \xi \le T$ such that

$$\frac{0-y(0)}{\xi-0} = \frac{y(T)-y(0)}{T-0}.$$

Then

$$|y(0)| = \left|\frac{\xi}{T}(y(T)-y(0))\right| \le \sum_{k=0}^{T-1} |\Delta y(k)| \le TM'.$$

Hence

$$|y(n)| = \left| y(0) + \sum_{k=0}^{n-1} \Delta y(k) \right| \le TM' + \sum_{k=0}^{n-1} |\Delta y(k)| \le 2TM'$$

for all $n \in [0,T]$. It follows that $\|y\| \le 2TM'$. So $\|(x,y)\| \le \max\{M,\ 2TM'\}$.

So Ω_1 is bounded. The proof of Lemma 2.4 is complete. □

(**L**). *There exist numbers $\beta > 0$, $\theta > 0$, nonnegative sequences $q_0(n)$, $p_i(n)$, $r(n)$ $(i = 0, \ldots, m)$, functions $g(n,x,y,x_1,\ldots,x_m)$, $h(n,x,y,x_1,\ldots,x_m)$ such that*

$$f(n,x,y,x_1,\cdots,x_m) = g(n,x,y,x_1,\ldots,x_m) + h(n,x,y,x_1,\ldots,x_m) \tag{2.13}$$

and

$$g(n,x,y,x_1,\ldots,x_m) \le -\beta|y|^\theta, \tag{2.14}$$

and

$$|h(n,x,y,x_1,\cdots,x_m)| \le q_0(n)|x|^\theta + \sum_{s=0}^{m} p_i(n)|x_i|^\theta + r(n), \tag{2.15}$$

for all $n \in [0, T-1]$, $(x,y,x_1,\cdots,x_m) \in \mathbb{R}^{m+2}$.

Lemma 2.5. *Suppose that* (L) *holds and* $\sigma(n+T) = -\sigma(n)$ *for all* $n \in \mathbb{Z}$. *Let* $\Omega_1 = \{(x,y) : L(x,y) = \lambda N(x,y),\ ((x,y),\lambda) \in D(L)\bigcap X \times (0,1)\}$. *Then* Ω_1 *is bounded if* (2.11) *holds.*

Proof. For $x \in \Omega_1$, we have $L(x,y) = \lambda N(x,y)$, $\lambda \in (0,1)$, we get (2.9) and (2.10). Then one has (2.11). Since (2.13), (2.14) and (2.15) imply that

$$\begin{aligned} & \beta \sum_{n=0}^{T-1} |x(n+1)|^\theta \\ \le\ & -\sum_{n=0}^{T-1} g(n,x(n),x(n+1),w(\tau_1(n)),\cdots,x(\tau_m(n))) \\ \le\ & \sum_{n=0}^{T-1} h(n,x(n),x(n+1),x(\tau_1(n)),\cdots,x(\tau_m(n))) \\ \le\ & \sum_{n=0}^{T-1} |h(n,x(n),x(n+1),x(\tau_1(n)),\cdots,x(\tau_m(n))|. \end{aligned}$$

The remainder of the proof is similar to that of Lemma 2.4 and is omitted. □

The main results of this paper are as follows:

Theorem L1. *Suppose that* $\sigma(T) = \sigma(0)$, (H) *holds. Then equation* (1.1) *has at least one solution if* (2.8) *holds.*

Proof. It follows from Lemma 2.2 that L is a Fredholm operator of index zero and N is L-compact on each nonempty open bounded subset Ω of X. To apply Lemma 2.1, we should define an open bounded subset Ω of X centered at zero such that assumptions of Lemma 2.1 hold.

Since $\sigma(T) = \sigma(0)$, (H) and (2.8) hold, Lemma 2.3 implies that Ω_1, defined in Lemma 2.3, is bounded. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \Omega_1$ centered at zero. It follows from Lemma 2.3 that Then $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in [0,1]$. Thus Lemma 2.1 implies that $L(x,y) = N(x,y)$ has at least one solution in $D(L) \cap \overline{\Omega}$. So x is a solution of equation (1.1). The proof is complete. □

Theorem L2. *Suppose that* $\sigma(T) = -\sigma(0)$, (K) *holds. Then equation* (1.1) *has at least one solution if* (2.11) *holds.*

Proof. Use Lemma 2.1, Lemma 2.2 and Lemma 2.4. The proof is similar to that of Theorem L1 and is omitted. □

Theorem L3. *Suppose that* $\sigma(T) = -\sigma(0)$, (L) *holds. Then equation* (1.1) *has at least one solution if* (2.11) *holds.*

Proof. Use Lemma 2.1, Lemma 2.2 and Lemma 2.5. The proof is similar to that of Theorem L1 and is omitted. □

3. Two Examples

In this section, we present two examples to illustrate the main results in section 2.

Example 3.1. Consider the following equation

$$\Delta\left[\left(5+\sin\frac{n\pi}{T}\right)(\Delta x(n))^3\right] = \beta[x(n+1)]^{2k+1}+\sum_{i=1}^{m}p_i(n)[x(n-i)]^{2k+1}+r(n),\quad n\in\mathbb{Z}, \tag{3.16}$$

where $T\geq 1, k\geq 0$ are integers, $\sigma(n)=5+\sin\frac{n\pi}{T}$, $\beta>0$, $p_i(n), r(n)$ are $T-$anti-periodic sequences.

Corresponding to equation (1.1), we get $\phi_p(x)=|x|^2x$, $\theta=2k+1$ and

$$\begin{aligned}
&\sigma(T)=\sigma(0),\\
g(n,x,y,x_1,\dots,x_m) &= \beta y^{2k+1}+\sum_{i=1}^{m}p_i(n)x_i^{2k+1}+r(n),\\
g(n,x,y,x_1,\dots,x_m) &= \beta y^{2k+1},\\
h(n,x,y,x_1,\dots,x_m) &= \sum_{i=1}^{m}p_i(n)x_i^{2k+1}+r(n).
\end{aligned}$$

It is easy to see that (H) holds. It follows from Theorem L1 that (3.16) has at least one solution if

$$T^{\frac{2k+1}{2k+2}}\sum_{i=1}^{m}||p_i||<\beta.$$

Example 3.2. Consider the following equation

$$\Delta\left[\left(5-10\sin\frac{n\pi}{22}\right)(\Delta x(n))^3\right] = \beta[x(n+1)]^{2k}+\sum_{i=1}^{m}p_i(n)[x(n-i)]^{2k}+r(n),\quad n\in\mathbb{Z}, \tag{3.17}$$

where $\beta\neq 0, m>1, k\geq 1$ are integers, $p_i(n), r(n)$ are 11-anti-periodic sequences.

Corresponding to equation (1.1), we get $\phi_p(x)=|x|^2x$, $T=11$, $\sigma(n)=5-10\sin\frac{n\pi}{22}$, $\theta=2k$ and

$$\begin{aligned}
&\sigma(11)=-\sigma(0),\\
g(n,x,y,x_1,\cdots,x_m) &= \beta y^{2k}+\sum_{i=1}^{m}p_i(n)x_i^{2k}+r(n),\\
g(n,x,y,x_1,\cdots,x_m) &= \beta y^{2k},\\
h(n,x,y,x_1,\cdots,x_m) &= \sum_{i=1}^{m}p_i(n)x_i^{2k}+r(n).
\end{aligned}$$

It is easy to see that (K) holds if $\beta > 0$. It follows from Theorem L2 that (3.17) has at least one solution if

$$11\sum_{i=1}^{m}||p_i|| < \beta. \tag{3.18}$$

It is easy to see that (L) holds if $\beta < 0$. It follows from Theorem L3 that (3.17) has at least one solution if

$$11\sum_{i=1}^{m}||p_i|| < -\beta. \tag{3.19}$$

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References

[1] Aftabizadeh, A. R., Pavel, N. H. and Huang,Y. K., *Anti-periodic oscillations of some second-order differential equations and optimal control problems*, J. Comput. Appl. Math. **52** (1994), 3-21
[2] Aftabizadeh, A. R., Aizicovici, S. and Pavel, N. H., *On a class of second-order anti-periodic boundary value problems*, J. Math. Anal. Appl. **171** (1992), 301-320
[3] Aftabizadeh, A. R., Aizicovici, S. and Pavel, N. H., *Anti-periodic boundary value problems for higher order differential equations in Hilbert spaces*, Nonl. Anal. **8** (1992), 253-267
[4] Agarwal, R. P., Cabada, A., Otero-Espinar, V. and Dontha, S., *Existence and uniqueness of solutions for anti-periodic difference equations*, Arch. of Ineq. and Appl. **2** (2004), 397-412
[5] Agarwal, R. P., *Focal Boundary Value Problems for Differential and Difference Equations*, Kluwer, Dordrecht, 1998
[6] Agarwal, R. P., O'Regan, D. and Wong, P. J. Y., *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 1999
[7] Aizicovici, S. and Pavel, N. H., *Anti-periodic solutions to a class of nonlinear differential equations in Hilbert space*, J. Func. Anal. **99** (1991), 387-408
[8] Aizicovici, S., McKibben, M. and Reich, S., *Anti-periodic solutions to nonmonotone evolution equations with discontinuous nonlinearities*, Nonl. Anal. **43** (2001), 233-251
[9] Atici, F. M., Cabada, A. and Ferreiro, J., *First order difference equations with maxima and nonlinear functional boundary value conditions*, J. Diff. Equa. Appl. **12** (2006), 565-576
[10] Chen, Y., *Anti-periodic solutions for semilinear evolution equations*, J. Math. Anal. Appl. **315** (2006), 337-348
[11] Chen, Y., Wang, X. and Xu, H., *Anti-periodic solutions for semilinear evolution equations*, J. Math. Anal. Appl. **273** (2002), 627-636
[12] Chen, Y., Nieto, J. and O'Regan, D., *Anti-periodic solutions for fully nonlinear first-order differential equations*, Math. Comput. Modelling **46** (2007), 1183-1190
[13] Gaines, R. E. and Mawhin, J. L., *Coincidence Degree and Nonlinear Differential Equations*, Lecture Notes in Math. 568, Springer, Berlin, 1977
[14] Merdivenci Atici, F., Cabada, A. and Otero-Espinar, V., *Criteria for Existence and Nonexistence of Positive Solutions to a Discrete Periodic Boundary Value Problem*, J. Diff. Equa. Appl. **9** (2003), 765-775

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