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# A new type of approximating sequence for the solution of the Cauchy problem consisting of piecewise linear functions

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ABSTRACT.

This paper continues the study of a new successive approximation method for solving the Cauchy problem. The motivation for considering such a method consists in the fact that in practice real difficulties arise in the problem of computing integrals from the respective recurrence relations. We give below an original method to avoid these difficulties by defining a similar recurrence for solving the Cauchy problem, so that the obtaining approximating sequence consists in functions which are piecewise linear. We prove the (uniform) convergence and finally, a numerical example is given.

## 1. INTRODUCTION

The Picard theorem (e.g. [1], [5], [9]) is one of the most known result in the theory of existence and uniqueness of the solution of the Cauchy problem. Moreover, the Picard theorem gives us a method for finding approximations of the solution, also called the successive approximations method. Other two new type of successive approximations method were given in [7]. Theoretically, the solution of the Cauchy problem is the (uniform) limit of the approximating sequence, but in practice real difficulties arise in the problem of computing integrals from the recurrence relation. It is not suitable for finding the solution since computing the integrals in each iteration step is not possible in general. Even for numerical computations it is of no great help, since evaluating the integrals is too time consuming. Here we give an original method to avoid these difficulties and the idea is to use a similar recurrence to construct an approximating sequence consisting in functions which are linear on subintervals. Remark that this new type of recurrence and other ideas we introduce here can also be applied to a large class of ordinary differential equations. In this way, the problem of numerical calculations of the integrals is solved, because the integrated functions are piecewise linear. One of the approximating sequence given in [7] is

$$y_{n+1}(x) = f\left(x, y_0 + \int_{x_0}^{x} y_n(t)\,\mathrm{d}\,t\right)$$

and we will consider here the new form

$$w_{n+1}(z_k) = f\left(z_k, y_0 + \int_{x_0}^{z_k} w_n(t)\,\mathrm{d}\,t\right)$$

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where $w_{n+1}$ is linear on each interval $[z_k, z_{k+1}]$, $-m \leq k \leq m-1$. The advantage is that the integrated function is piecewise linear and the integral can be easily calculated in concrete cases. We can see here once again the powerful of the method developed in [7], because this linearization we present here does not work in case of the classical Picard iteration.

## 2. The results

In practise arise real difficulties in the problem of computing the integrals from the recurrence and we will give here an original method to avoid these difficulties. This method is to consider the approximating sequence $(w_n)_{n\in\mathbb{N}}$ consisting of functions which are linear on some subintervals. Let us consider again the Cauchy problem

$$\begin{cases} y' & = f(x,y) \\ y(x_0) & = y_0, \end{cases} \tag{2.1}$$

with global continuity and lipschizianity properties with respect to the second argument, $|f(x,y_1) - f(x,y_2)| \leq L|y_1 - y_2|$. Then we have local solvability for the Cauchy problem (2.1), where the solution is defined at least on the interval $y : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$ with $0 < \delta < \min\left\{a, \dfrac{b}{\|f\|}\right\}$ and $||f|| = \max\limits_{(x,y)\in D} |f(x,y)|$. Let $0 < r < \min\left\{\delta, L^{-1}\right\}$, $m \in \mathbb{N}$ be given and let

$$\Delta = (x_0 - r = z_{-m} < z_{-m+1} < ... < z_0 = x_0 < ... < z_m = x_0 + r)$$

be an arbitrary division of the interval $[x_0 - r, x_0 + r]$ with the norm denoted by $\|\Delta\|$. Let us define the functions sequence $w_n : [x_0 - r, x_0 + r] \to \mathbb{R}$,

$$w_{n+1}(z_k) = f\left(z_k, y_0 + \int_{x_0}^{z_k} w_n(t)\,\mathrm{d}\,t\right), \quad -m \leq k \leq m, \ (w_0 = 0) \tag{2.2}$$

where $w_{n+1}$ is linear on each interval $[z_k, z_{k+1}]$, $-m \leq k \leq m-1$,

$$w_{n+1}(x) = (1-\lambda)w_{n+1}(z_k) + \lambda w_{n+1}(z_{k+1}), \ \forall x = (1-\lambda)z_k + \lambda z_{k+1}, \ \lambda \in [0,1].$$

**Theorem 2.1.** *The functions sequence $(w_n)_{n\in\mathbb{N}}$ defined by the recurrence (2.2) converges uniformly on each interval $[x_0 - r, x_0 + r] \subseteq [x_0 - \delta, x_0 + \delta]$, with $0 < r < L^{-1}$.*

*Proof.* First, for $-m \leq k \leq m$, we have

$$\begin{aligned} &|w_{n+2}(z_k) - w_{n+1}(z_k)| \\ &= \left| f\left(z_k, y_0 + \int_{x_0}^{z_k} w_{n+1}(t)\,\mathrm{d}\,t\right) - f\left(z_k, y_0 + \int_{x_0}^{z_k} w_n(t)\,\mathrm{d}\,t\right)\right| \\ &\leq L\left|\left(y_0 + \int_{x_0}^{z_k} w_{n+1}(t)\,\mathrm{d}\,t\right) - \left(y_0 + \int_{x_0}^{z_k} w_n(t)\,\mathrm{d}\,t\right)\right| \\ &= L\left|\int_{x_0}^{z_k} (w_{n+1}(t) - w_n(t))\,\mathrm{d}\,t\right| \leq L|z_k - x_0| \cdot \|w_{n+1} - w_n\| \leq Lr\,\|w_{n+1} - w_n\|. \end{aligned}$$

Now let $x \in [x_0 - r, x_0 + r]$ be arbitrary, say $x \in [z_k, z_{k+1}]$, for some integer $-m \leq k \leq m-1$. Then there exists $\lambda \in [0,1]$ so that $x = (1-\lambda)z_k + \lambda z_{k+1}$ and using also the linearity of $w_{n+2}$ and $w_{n+1}$ on $[z_k, z_{k+1}]$, we derive

$$\begin{aligned}
|w_{n+2}(x) - w_{n+1}(x)| &= |w_{n+2}((1-\lambda)z_k + \lambda z_{k+1}) - w_{n+1}((1-\lambda)z_k + \lambda z_{k+1})| \\
&= |(1-\lambda)w_{n+2}(z_k) + \lambda w_{n+2}(z_{k+1}) - (1-\lambda)w_{n+1}(z_k) - \lambda w_{n+1}(z_{k+1})| \\
&\le (1-\lambda)\,|w_{n+2}(z_k) - w_{n+1}(z_k)| + \lambda\,|w_{n+2}(z_{k+1}) - w_{n+1}(z_{k+1})| \\
&\le (1-\lambda)Lr\,\|w_{n+1} - w_n\| + \lambda Lr\,\|w_{n+1} - w_n\| = Lr\,\|w_{n+1} - w_n\|\,.
\end{aligned}$$

By taking the supremum with respect to $x \in [x_0 - r, x_0 + r]$ in the inequality

$$|w_{n+2}(x) - w_{n+1}(x)| \le Lr\,\|w_{n+1} - w_n\| \tag{2.3}$$

we obtain

$$\|w_{n+2} - w_{n+1}\| \le Lr\,\|w_{n+1} - w_n\|\,, \tag{2.4}$$

for all integers $n \in \mathbb{N}$. By induction,

$$\|w_{n+s+1} - w_{n+s}\| \le (Lr)^s\,\|w_{n+1} - w_n\|\,. \tag{2.5}$$

Now, for $n, p \in \mathbb{N}$, we have

$$\begin{aligned}
\|w_{n+p} - w_n\| &= \|(w_{n+p} - w_{n+p-1}) + (w_{n+p-1} - w_{n+p-2}) + \ldots + (w_{n+1} - w_n)\| \\
&\le \|w_{n+p} - w_{n+p-1}\| + \|w_{n+p-1} - w_{n+p-2}\| + \ldots + \|w_{n+1} - w_n\| \\
&\le \big((Lr)^{p-1} + (Lr)^{p-2} + \ldots + Lr + 1\big) \cdot \|w_{n+1} - w_n\| \\
&= \frac{1-(Lr)^p}{1-Lr} \cdot \|w_{n+1} - w_n\| \le \frac{1-(Lr)^p}{1-Lr} \cdot (Lr)^n \cdot \|w_1 - w_0\| \le \frac{(Lr)^n}{1-Lr} \cdot \|w_1\|\,.
\end{aligned}$$

It results that $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the B-space $C([x_0 - r, x_0 + r])$ endowed with the supremum norm, thus it converges uniformly to a limit denoted $w \in C([x_0 - r, x_0 + r])$. □

By taking $p \to \infty$ in the inequality

$$\|w_{n+p} - w_n\| \le \frac{(Lr)^n}{1-Lr} \cdot \|w_1\|\,, \tag{2.6}$$

we obtain the error estimate

$$\|w - w_n\| \le \frac{(Lr)^n}{1-Lr} \cdot \|w_1\| = \frac{(Lr)^n}{1-Lr} \cdot ||f||\,. \tag{2.7}$$

**Definition 2.1.** For each real number $\varepsilon > 0$, we say that a derivable function $\omega : [x_0 - r, x_0 + r] \to \mathbb{R}$ with continuous derivative is $\varepsilon-$ fixed point for the Cauchy problem (2.1) if $\omega(x_0) = y_0$ and

$$|\omega'(x) - f(x, \omega(x))| \le \varepsilon,\ \forall\, x \in [x_0 - r, x_0 + r]\,.$$

From now assume that there exist $\alpha, L > 0$ so that

$$|f(x_1, y_1) - f(x_2, y_2)| \le \alpha\,|x_1 - x_2| + L\,|y_1 - y_2|\,, \tag{2.8}$$

condition which assures on $f$ the lipschitzianity in the second argument.
We establish some estimations we will use later.

**Lemma 2.1.** *If $f$ satisfies* (2.8)*, then the following relations hold true:*
*a)* $||w|| \le ||f||$; *b)* $|w_n(z_{k+1}) - w_n(z_k)| \le (\alpha + L\,||f||)\,||\Delta||$.

*Proof.* a) From (2.2), $|w_{n+1}(z_k)| \leq ||f||$ and $w_{n+1}$ is linear on subintervals, so for $x \in [x_0 - r, x_0 + r]$, we have $|w_{n+1}(x)| \leq \max_{-m \leq k \leq m} |w_{n+1}(z_k)| \leq ||f||$. By taking the supremum with respect to $x \in [x_0 - r, x_0 + r]$, it results that $||w_{n+1}|| \leq ||f||$, then by $n \to \infty$, $||w|| \leq ||f||$.

b) We have

$$\begin{aligned}
&|w_n(z_{k+1}) - w_n(z_k)| \\
&= \left| f\left(z_{k+1}, y_0 + \int_{x_0}^{z_{k+1}} w_{n-1}(t)dt\right) - f\left(z_k, y_0 + \int_{x_0}^{z_k} w_{n-1}(t)dt\right)\right| \\
&\leq \alpha\, |z_{k+1} - z_k| + L \left| y_0 + \int_{x_0}^{z_{k+1}} w_{n-1}(t)dt - y_0 - \int_{x_0}^{z_k} w_{n-1}(t)dt \right| \\
&= \alpha\, |z_{k+1} - z_k| + L \left| \int_{z_k}^{z_{k+1}} w_{n-1}(t)dt \right| \leq \alpha\, |z_{k+1} - z_k| + L\, |z_{k+1} - z_k| \cdot ||w_{n-1}|| \\
&\leq (\alpha + L\, ||f||)\, ||\Delta||\,.
\end{aligned}$$

□

**Theorem 2.2.** *Assume that $f$ satisfies (2.8) and the division $\Delta$ is chosen with $||\Delta|| \leq \dfrac{\varepsilon}{2(\alpha + L\, ||f||)}$. Then the function $\omega : [x_0 - r, x_0 + r] \to \mathbb{R}$ given by $\omega(x) = y_0 + \int_{x_0}^{x} w(t)dt$ is $\varepsilon-$fixed point for the Cauchy problem (2.1).*

*Proof.* We have $\omega'(x) = w(x)$. By taking the limit as $n \to \infty$ in the recurrence relation (2.2), we obtain

$$w(z_k) = f\left(z_k, y_0 + \int_{x_0}^{z_k} w(t)dt\right), \tag{2.9}$$

or $\omega'(z_k) = f(z_k, \omega(z_k))$, $-m \leq k \leq m$. Further, let $x \in [x_0 - r, x_0 + r]$, say that $x \in [z_k, z_{k+1}]$, for some integer $-m \leq k \leq m - 1$. Then

$$\begin{aligned}
|\omega'(x) - f(x, \omega(x))| &= |\omega'(x) - f(x, \omega(x)) - (\omega'(z_k) - f(z_k, \omega(z_k))| \\
&= |\omega'(x) - \omega'(z_k) + f(z_k, \omega(z_k)) - f(x, \omega(x))| \\
&\leq |f(z_k, \omega(z_k)) - f(x, \omega(x))| + |\omega'(x) - \omega'(z_k)|
\end{aligned} \tag{2.10}$$

and we intend to make these expressions as small as possible. First

$$\begin{aligned}
|f(z_k, \omega(z_k)) - f(x, \omega(x))| &\leq \alpha\, |z_k - x| + L\, |\omega(z_k) - \omega(x)| \\
&\leq \alpha\, ||\Delta|| + L \left| \int_{z_k}^{x} w(t)dt \right| \leq \alpha\, ||\Delta|| + L\, ||\Delta|| \cdot ||f|| \\
&= (\alpha + L\, ||f||)\, ||\Delta||\,,
\end{aligned} \tag{2.11}$$

where we used the inequality $||w|| \leq ||f||$. Then with Lemma 2.1, b), we have

$$\begin{aligned}
|\omega'(x) - \omega'(z_k)| &= |w(x) - w(z_k)| = \lim_{n \to \infty} |w_n(x) - w_n(z_k)| \\
&\leq \lim_{n \to \infty} |w_n(z_{k+1}) - w_n(z_k)| \leq (\alpha + L\, ||f||)\, ||\Delta||\,.
\end{aligned} \tag{2.12}$$

By adding (2.11) and (2.12), $|\omega'(x) - f(x, \omega(x))| \leq 2(\alpha + L\, ||f||)\, ||\Delta|| \leq \varepsilon$, so $\omega$ is $\varepsilon-$fixed point of the Cauchy problem (2.1). □

Now we are preoccupied about the error estimates. For sake of simplicity, we will work now only on the interval $[x_0, x_0 + r]$. We give the following

**Theorem 2.3.** *Assume that* $\omega : [x_0, x_0 + r] \to \mathbb{R}$ *is a* $\varepsilon-$ *fixed point of the Cauchy problem* (2.1) *and* $y : [x_0, x_0 + r] \to \mathbb{R}$ *is the (unique) solution of the Cauchy problem* (2.1)*. Then:*

*a)* $|\omega'(x) - y'(x)| \le \varepsilon + L \int_{x_0}^{x} |\omega'(t) - y'(t)|\, dt$*, for all* $x \in [x_0, x_0 + r]$*;*

*b)* $||\omega - y|| \le \dfrac{\exp(Lr) - 1}{L}\, \varepsilon$*, where this norm is considered on* $[x_0, x_0 + r]$*.*

*Proof.* a) We have

$$\begin{aligned} |\omega'(x) - y'(x)| &= |(\omega'(x) - f(x, \omega(x))) + (f(x, \omega(x)) - f(x, y(x)))| \\ &\le |\omega'(x) - f(x, \omega(x))| + |f(x, \omega(x)) - f(x, y(x))| \\ &\le \varepsilon + L\, |\omega(x) - y(x)| = \varepsilon + L \left| \int_{x_0}^{x} (\omega'(t) - y'(t))\, dt \right| \\ &\le \varepsilon + L \int_{x_0}^{x} |(\omega'(t) - y'(t))|\, dt. \end{aligned}$$

b) Follows from Gronwall's inequality. If $0 < A \in \mathbb{R}$, $u \ge 0$ and

$$u(x) \le A + \int_{x_0}^{x} B(t) u(t) dt,$$

for all $x \in [x_0, x_0 + r]$, then $u(x) \le A \exp\left( \int_{x_0}^{x} B(t) dt \right)$, for all $x \in [x_0, x_0 + r]$. In our case, with $u(x) = |\omega'(x) - y'(x)|$, from a), we have

$$|\omega'(x) - y'(x)| \le \varepsilon + L \int_{x_0}^{x} |\omega'(t) - y'(t)|\, dt,$$

so

$$|\omega'(x) - y'(x)| \le \varepsilon \exp\left( \int_{x_0}^{x} L' dt \right) = \varepsilon \exp\left( L(x - x_0) \right) \le \varepsilon \exp(Lr).$$

Thus for all $x \in [x_0, x_0 + r]$, we have

$$\begin{aligned} |\omega(x) - y(x)| &= \left| \int_{x_0}^{x} (\omega'(t) - y'(t)) dt \right| \le \int_{x_0}^{x} |(\omega'(t) - y'(t))|\, dt \\ \le \varepsilon \int_{x_0}^{x} \exp(L(t - x_0)) dt &= \varepsilon\, \frac{\exp(L(t - x_0))}{L} \Bigg|_{t = x_0}^{t = x} \le \frac{\exp(Lr) - 1}{L}\, \varepsilon \end{aligned}$$

and the conclusion follows by taking the supremum on $[x_0, x_0 + r]$. □

The classical way in iteration theory is that the approximating sequence converges (uniform) to the solution. We have here an interesting situation; the approximating sequence converges to a $\varepsilon-$fixed point, which is as close as we want from the exact solution. Hence it is necessary a result which give the error estimates when we approximate the exact solution by $\omega_n$, where

$$\omega_n(x) = y_0 + \int_{x_0}^{x} w_n(t) dt, \quad n \in \mathbb{N}.$$

Thus we give the following

**Theorem 2.4.** *Assume that the Cauchy problem (2.1) is defined by the continuous function* $f : \{(x, y) \| \ |x - x_0| \le a,\ |y - y_0| \le b\} \to \mathbb{R}$ *for which there exist* $\alpha, L > 0$ *such that* $|f(x_1, y_1) - f(x_2, y_2)| \le \alpha\,|x_1 - x_2| + L\,|y_1 - y_2|$ *and the sequence* $(w_n)_{n\in\mathbb{N}}$, *with* $w_0 = 0$ *is defined by the recurrence (2.2) on* $[x_0, x_0 + r]$, *with* $r < \min\left\{a, \dfrac{b}{||f||}, L^{-1}\right\}$ *and* $||\Delta|| \le \dfrac{\varepsilon L}{4(\exp(Lr) - 1)(\alpha + L\,||f||)}$.
*Then for every positive integer* $n \ge \log_{Lr} \dfrac{\varepsilon(1 - Lr)}{2\,||f||}$, *we have* $||\omega_n - y|| \le \varepsilon$, *where* $y : [x_0, x_0 + r] \to \mathbb{R}$ *be the unique solution of the Cauchy problem (2.1).*

*Proof.* We have $n \ge \log_{Lr} \dfrac{\varepsilon(1 - Lr)}{2r\,||f||} \Leftrightarrow \dfrac{r(Lr)^n}{1 - Lr}\ ||f|| \le \dfrac{\varepsilon}{2}$, so

$$||\omega_n - y|| = ||(\omega_n - \omega) + (\omega - y)|| \le ||\omega_n - \omega|| + ||\omega - y|| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where the inequalities $||\omega_n - \omega|| \le \dfrac{\varepsilon}{2}$ and $||\omega - y|| \le \dfrac{\varepsilon}{2}$ will be demonstrated next. First, using (2.7),

$$|\omega_n(x) - \omega(x)| = \left|\int_{x_0}^{x} (w_n(t) - w(t))dt\right| \le r\,||w_n - w|| \le \frac{r(Lr)^n}{1 - Lr}\ ||f|| \le \frac{\varepsilon}{2},$$

so $||\omega_n - \omega|| \le \dfrac{\varepsilon}{2}$. On the other hand, from $||\Delta|| \le \dfrac{\varepsilon L}{4(\exp(Lr) - 1)(\alpha + L\,||f||)}$, it results using Theorem 2.2, that $\omega$ is $\dfrac{\varepsilon L}{2(\exp(Lr) - 1)}$, fixed point for the equation (2.1). Further, with Theorem 2.3, b) it results that $|\omega - y| \le \dfrac{\varepsilon}{2}$. □

## 3. A numerical example

Let us consider the Cauchy problem for the second order differential equation

$$\frac{d^2u}{dx^2} - \frac{2x}{1 - x^2}\frac{du}{dx} + \frac{20}{1 - x^2}u = 0 \tag{3.13}$$

with the initial conditions

$$\begin{cases} u(0) = 3/2 \\ u'(0) = 0. \end{cases} \tag{3.14}$$

The exact solution of this problem is the Legendre polynomial of the fourth degree $L_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$. Usual techniques applied to equation (3.13) may reduce our problem to the following first order system of differential equations

$$\begin{cases} y_1' = y_2 \\ y_2' = \dfrac{2x}{1 - x^2}\, y_2 - \dfrac{20}{1 - x^2}\, y_1 \end{cases} \tag{3.15}$$

with the initial condition

$$\begin{cases} y_1(0) = 3/2 \\ y_2(0) = 0. \end{cases} \tag{3.16}$$

The relationships which transform equation (3.13) into the equivalent system (3.15) are $y_1 = u,\ y_2 = u'$. Using the vectorial notations $y = (y_1, y_2)^T$ and $f(x,y) = \left(y_2, \frac{2x}{1-x^2}\, y_2 - \frac{20}{1-x^2} + y_1\right)^T$ the problem (3.15), (3.16) may be written in the general form $y' = f(x,y)$. Following the theory presented in the previous section we construct the sequence $\{w_n\}_{n\in\mathbb{N}}$

$$w_{n+1}^k = \begin{cases} f\big(z_k, \sum_{j=0}^{k-1}(w_n^j + w_n^{j+1})\cdot(z_{j+1}-z_j)/2\big) & \text{for } k>0 \\ 0 & \text{for } k=0 \\ f\big(z_k, -\sum_{j=k}^{1}(w_n^j + w_n^{j+1})\cdot(z_{j+1}-z_j)/2\big) & \text{for } k<0 \end{cases} \tag{3.17}$$

where $w_n^k = w_n(z_k)\ n \geq 0,\ -m_1 \leq k \leq m_2$ and $w_0$ is the initial approximation (eventually $w_0 = 0$). Each $w_n$ is a vectorial function having each component a piecewise linear continuous function uniquely determined by its values in the points of the considered mesh. For our purpose we considered an uniform mesh ($z_k = x_0 + k\cdot h,\ -m_1 \leq k \leq m_2$) and stopped the iteration process (3.17) when the difference between two successive iterations is sufficiently small. To each $w_n$ we associate a vectorial function $y^{(n)}(x) = y_0 + \int_{x_0}^{x} w_n(t)dt$.

Figure 1 shows the the exact solution together with several iterations ($y^{(n)}$) when $m_1 = m_2 = 20$ and $h = 0.0475$.

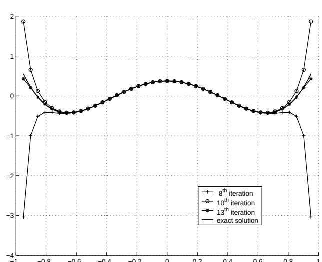


*Figure* 1

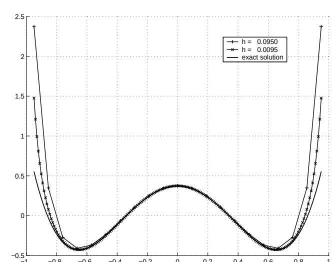


*Figure* 2

For $n = 22$ the difference between the last two iteration $y^{(22)},\ y^{(21)}$ is less than $10^{-3}$ while the error in the mesh points is $\max_k |u(z_k) - y_1^{(22)}(z_k)| = 0.00626$.
Figure 2 presents the exact solution and the 10-th iteration for two different choices of the mesh step size $h = 0.095$ ($m_1 = m_2 = 10$) and $h = 0.0095$ ($m_1 = m_2 = 100$). The second row in Table 1 gives the values of the iteration counts such that the difference between the last two iteration to be less than $10^{-3}$. The values in the third row represent the exact error in the mesh points.

| h | 0.0950 | 0.0475 | 0.0190 | 0.0136 | 0.0095 | 0.0063 |
|---|---|---|---|---|---|---|
| n | 33 | 22 | 21 | 21 | 21 | 21 |
| $\max_k \lvert u(z_k) - y_1^{(n)}(z_k)\rvert$ | 0.0243 | 0.0063 | 0.0010 | 0.0005 | 0.0003 | 0.0001 |

Table 1

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