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On the asymptotic behaviour of finite Markov chains

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ABSTRACT.
We give two sufficient conditions for weak and strong ergodicity of a nonhomogeneous finite Markov chain in terms of similar properties of a certain chain of smaller size.

1. PRELIMINARIES

Consider a finite Markov chain with state space $S = \{1, ..., r\}$ and transition matrices $(P_n)_{n\geq1}$. We shall refer to it as the finite Markov chain $(P_n)_{n\geq1}$. For all integers $m \geq 0$, $n > m$, define

$$P_{m,n} = P_{m+1}P_{m+2}...P_n = ((P_{m,n})_{ij})_{i,j\in S}.$$

Assume that the limit

$$\lim_{n\to\infty} P_n = P \tag{1.1}$$

exists and that the limit matrix P has $p \geq 1$ irreducible aperiodic closed classes and, perhaps transient states, so that it has the form

$$P = \begin{pmatrix} S_1 & 0 & ... & 0 & 0 \\ 0 & S_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & S_p & 0 \\ L_1 & L_2 & ... & L_p & T \end{pmatrix}, \tag{1.2}$$

where S_i, for $i = \overline{1,p}$, are $r_i \times r_i$ transition matrices associated with the p irreducible aperiodic closed classes, T concerns the transitions of the chain as long as it stays in the $r - \sum_{t=1}^{p} r_t$ transient states and the L_i concern transitions from the transient states into the ergodic sets corresponding to S_i, $i = \overline{1,p}$.

Markov chains of this type occur in simulated annealing, a stochastic algorithm for global optimization. We refer to van Laarhoven and Aarts [6] for a general exposition and historical background.

Definition 1.1. We say that a probability distribution $\mu = (\mu_1, ..., \mu_r)$ is *invariant* with respect to an $r \times r$ stochastic matrix P if we have $\mu P = \mu$.

We shall need the following result

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Theorem 1.1. *Consider a finite homogeneous Markov chain with state space S having the transition matrix P of the form* (1.2)*. Then*

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \Gamma_1 & 0 & ... & 0 & 0 \\ 0 & \Gamma_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \Gamma_p & 0 \\ \Omega_1 & \Omega_2 & ... & \Omega_p & 0 \end{pmatrix}, \tag{1.3}$$

where

$$\Gamma_i = \begin{pmatrix} \mu_1^{(i)} & ... & \mu_{r_i}^{(i)} \\ ... & ... & ... \\ \mu_1^{(i)} & ... & \mu_{r_i}^{(i)} \end{pmatrix}$$

is a strictly positive $r_i \times r_i$ matrix, $\forall\, i = \overline{1,p}$; each row of the matrix Γ_i is the invariant probability vector $\mu^{(i)} := (\mu_1^{(i)}, ..., \mu_{r_i}^{(i)})$ with respect to the matrix S_i, $\forall\, i = \overline{1,p}$, and

$$\Omega_i = \begin{pmatrix} \mu_1^{(i)} z_{r_1+r_2+...+r_p+1,i} & ... & \mu_{r_i}^{(i)} z_{r_1+r_2+...+r_p+1,i} \\ ... & ... & ... \\ \mu_1^{(i)} z_{r,i} & ... & \mu_{r_i}^{(i)} z_{r,i} \end{pmatrix}$$

is an $(r - \sum\limits_{t=1}^{p} r_t) \times r_i$ matrix, $\forall\, i = \overline{1,p}$, where z_{ji}= probability that the chain will enter and thus, will be absorbed in S_i given that the initial state is j, $\forall\, j = \overline{\sum\limits_{t=0}^{p} r_t, r}$, $\forall\, i = \overline{1,p}$ (with convention $r_0 = 1$).

Proof. For the form of Γ_i, $i = \overline{1,p}$, see, e.g., [3, p. 123] and for Ω_i, $i = \overline{1,p}$, see, e.g., [5, p. 91]. □

Remark 1.1. Clearly,

$$z_{ji} \geq 0,\ \forall\, j = \overline{\sum_{t=0}^{p} r_t, r},\ \forall i = \overline{1,p}, \tag{1.4}$$

and

$$\sum_{i=1}^{p} z_{ji} = 1, \forall\, j = \overline{\sum_{t=0}^{p} r_t, r}. \tag{1.5}$$

A vector $x \in \mathbb{C}^n$ will be understood as a row vector and x' is its transpose. Set $\mathbf{e} = \mathbf{e}(n) = (1, 1, ..., 1) \in \mathbb{R}^n$ and $\mathbf{0} = \mathbf{0}(n) = (0, 0, ..., 0) \in \mathbb{R}^n$. Let $(e_i)_{i=\overline{1,n}}$ be the canonical basis of the linear space $\mathbb{R}^n$.

Theorem 1.2. ([1] and [7]). *Let $A = -I_r + P$ with P of the form* (1.2)*. Then there exists a nonsingular $r \times r$ complex matrix Q such that*

$$A = QJQ^{-1}, \tag{1.6}$$

where J is an $r \times r$ Jordan matrix. Q reads as

$$Q = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 & \dots \\ 0 & 1 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \dots \\ z_{r_1+r_2+\dots+r_p+1,1} & z_{r_1+r_2+\dots+r_p+1,2} & \dots & z_{r_1+r_2+\dots+r_p+1,p} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ z_{r,1} & z_{r,2} & \dots & z_{rp} & \dots \end{pmatrix},$$

where the first column contains 1 in the $\overline{1, r_1}$ rows, the i-th column contains 1 in the $\overline{r_{i-1}+1, r_i}$ rows, $\forall\, i = \overline{2, p}$, and the last $r-p$ columns comprise complex numbers. For z_{ji}, $j = \overline{\sum_{t=0}^{p} r_t, r}$, $i = \overline{1, p}$, we have the meaning given in Theorem 1.1. The inverse Q^{-1} has the form

$$Q^{-1} = \begin{pmatrix} \mu_1^{(1)} & \dots & \mu_{r_1}^{(1)} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 & \mu_1^{(p)} & \dots & \mu_{r_p}^{(p)} & 0 & \dots & 0 \\ q_{p+1,1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & q_{p+1,r} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ q_{r,1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & q_{rr} \end{pmatrix},$$

where $\mu^{(i)}$ is the invariant probability vector with respect to S_i, $\forall\, i = \overline{1, p}$, and the last $r-p$ rows comprise complex numbers.

Proof. See [7]. □

Remark 1.2. (a) We shall need some spectral properties of A. We have

$$\lambda_1 = 0 \tag{1.7}$$

is an eigenvalue of A whose algebraic multiplicity is equal to its geometric multiplicity and equal to p. All other distinct eigenvalues $\lambda_2, \dots, \lambda_{l+s}$ of A satisfy

$$|\lambda_i + 1| < 1 \quad \text{and} \quad \mathrm{Re}(\lambda_i) < 0, \forall\, i = \overline{2, l+s}.$$

(b) From (1.7) it follows (see, e.g., [2, pp. 129-131])

$$J = \begin{pmatrix} J_1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & J_2 & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_l & 0 & \dots & \dots \\ 0 & 0 & \dots & 0 & J_{l+1} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & J_{l+s} \end{pmatrix},$$

where $J_1 = \mathbf{0}_{p\times p}$, J_k is a diagonal $m_k \times m_k$ matrix with entries the eigenvalue λ_k whose algebraic and geometric multiplicities are identical, $\forall\, k = \overline{2, l}$, and

$$J_{l+i} = \begin{pmatrix} \lambda_{l+i} & \varepsilon_1^{(i)} & 0 & \dots & \dots & 0 \\ 0 & \lambda_{l+i} & \varepsilon_2^{(i)} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \lambda_{l+i} & \varepsilon_{m_{l+i}-1}^{(i)} \\ 0 & 0 & \dots & \dots & 0 & \lambda_{l+i} \end{pmatrix},$$

for $i = \overline{1, s}$, are $m_{l+i} \times m_{l+i}$ matrices corresponding to eigenvalues whose geometric multiplicities are smaller than their algebraic multiplicities and $\varepsilon_t^{(i)} \in \{0, 1\}$, $\forall\, t = \overline{1, m_{l+i} - 1}$, $\forall\, i = \overline{1, s}$. Clearly, $p + m_2 + \dots + m_{l+s} = r$.

If $A = (A_{ij})$ is an $m \times n$ matrix, then for $M \subseteq \{1, \dots, m\}$, $N \subseteq \{1, \dots, n\}$, $M, N \neq \emptyset$, we define

$$A_{M\times N} = (A_{ij})_{(i,j)\in M\times N},$$

and

$$A_{i,N} = \sum_{j\in N} A_{ij}, \forall\, i \in \{1, \dots, m\}.$$

Definition 1.2. (see, e.g., [3]). A sequence of stochastic matrices $(P_n)_{n\geq 1}$ is said to be *weakly ergodic* if $\forall\, m \geq 0$, $\forall\, i, j, k \in S$

$$\lim_{n\to\infty} [(P_{m,n})_{ik} - (P_{m,n})_{jk}] = 0.$$

A stochastic matrix whose rows are identical is said to be stable.

Theorem 1.3. (see, e.g., [3]). *A sequence of stochastic matrices $(P_n)_{n\geq 1}$ is weakly ergodic if and only if there exist stable stochastic matrices $\Pi_{m,n}$, $0 \leq m < n$, such that*

$$\lim_{n\to\infty} (P_{m,n} - \Pi_{m,n}) = 0,\ \forall\, m \geq 0.$$

Proof. See, e.g., [3, p. 218]. □

Definition 1.3. (see, e.g., [3]). A sequence of stochastic matrices $(P_n)_{n\geq 1}$ is said to be *strongly ergodic* if $\forall\, m \geq 0$, $\forall\, i, j \in S$ the limit

$$\lim_{n\to\infty} (P_{m,n})_{ij} := (\pi_m)_j,$$

exists and does not depend on i.

Remark 1.3. (see, e.g., [3, p. 223]) It is easy to prove that if a Markov chain is strongly ergodic, then $(\pi_m)_j$ is also independent of m, $\forall\, m \geq 0$. Therefore, a sequence of stochastic matrices $(P_n)_{n\geq 1}$ is strongly ergodic if and only if there exists a stable stochastic matrix Π such that

$$\lim_{n\to\infty} P_{m,n} = \Pi,\ \forall\, m \geq 0.$$

Definition 1.4. (see, e.g., [4, p. 144]) Let P be an $r \times r$ stochastic matrix. The *ergodic coefficient* of P, denoted by $\delta(P)$, is defined by

$$\delta(P) = 1 - \min_{1\leq i,k\leq r} \sum_{j=1}^{r} \min(P_{ij}, P_{kj}).$$

Theorem 1.4. (see, e.g., [4]) *If P is a stochastic $n \times p$ matrix and $R = (R_{ij})$ is a real $m \times n$ matrix with $Re' = \mathbf{0}$, then $|||RP|||_\infty \leq |||R|||_\infty \delta(P)$.*

Proof. See, e.g., [4, p. 147]. □

Theorem 1.5. (see, e.g., [4]) *A sequence of stochastic matrices $(P_n)_{n\geq 1}$ is weakly ergodic if and only if $\lim_{n\to\infty} \delta(P_{m,n}) = 0$, $\forall\, m \geq 0$.*

Proof. See, e.g., [4, p. 149]. □

Proposition 1.1. (see, e.g., [4]) *Let $(a_{nk})_{n,k\geq 1}$ be a doubly indexed sequence of real numbers such that $\lim_{n\to\infty} a_{nk} = a_k$ exists, $\forall\, k \geq 1$. If there exists a sequence of non-negative numbers $(b_k)_{k\geq 1}$ such that $|a_{nk}| \leq b_k$, $\forall\, n \geq 1$, $\forall\, k \geq 1$, and $\sum_{k=1}^{\infty} b_k < \infty$, then*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty} a_k.$$

Proof. See, e.g., [4, p. 29]. □

2. A RESULT ON RECURRENCE RELATIONS

In this section we shall give a result related to sequences defined by recurrence relations.

Proposition 2.2. *Let $(X_n)_{n\geq 0} = ((X_n)_1, ..., (X_n)_p)_{n\geq 0}$ be a sequence of real vectors, each vector having $p > 1$ components, which satisfy the recurrence relation*

$$X_{n+1} = X_n C_{n+1} + R_n,\ \forall\, n \geq 0, \tag{2.8}$$

where $(R_n)_{n\geq 0} = ((R_n)_1, ..., (R_n)_p)_{n\geq 0}$ is a sequence of real vectors such that $R_n e' = 0$, $\forall\, n \geq 0$, and $\sum_{n=0}^{\infty} |||R_n|||_\infty < \infty$ and $(C_n)_{n\geq 1}$ is a sequence of $p \times p$ stochastic matrices. Then the following statements hold.

(i) If $X_n e' = 0$, $\forall\, n \geq 0$, and $(C_n)_{n\geq 1}$ is weakly ergodic, then

$$\lim_{n\to\infty} X_n = \mathbf{0}.$$

(ii) *If* $X_n\boldsymbol{e}' = 1$, $\forall\, n \geq 0$, *and* $(C_n)_{n\geq 1}$ *is strongly ergodic, with* $\lim\limits_{n\to\infty} C_{m,n} = \Pi = \boldsymbol{e}'\pi$, $\forall\, m \geq 0$, *then*

$$\lim_{n\to\infty} X_n = \pi.$$

Proof. Applying the recurrence relation (2.8) successively we obtain

$$X_{n+1} = X_0 C_{0,n+1} + \big[R_n + \sum_{k=0}^{n-1} R_k C_{k+1,n+1}\big],\ \forall\, n \geq 1. \tag{2.9}$$

First, assuming that $(C_n)_{n\geq 1}$ is weakly ergodic we shall prove that

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} R_k C_{k+1,n+1} = \mathbf{0}. \tag{2.10}$$

We have

$$\begin{aligned} |||\sum_{k=0}^{n-1} R_k C_{k+1,n+1}|||_\infty &\leq \sum_{k=0}^{n-1} |||R_k C_{k+1,n+1}|||_\infty \\ &\leq \sum_{k=0}^{n-1} |||R_k|||_\infty \delta(C_{k+1,n+1}) \end{aligned}$$

(using Theorem 1.4).

Next, we choose $a_{nk} = |||R_k|||_\infty \delta(C_{k+1,n+1})$, $\forall n, k \geq 0$ (take $a_{nk} = 0$ if $k > n$). Then, by Theorem 1.5, it follows that $\lim\limits_{n\to\infty} a_{nk} = 0 := a_k$, $\forall\, k \geq 0$. Further, $|a_{nk}| \leq b_k := |||R_k|||_\infty$, $\forall\, n \geq 0$, since $\delta(C_{k,n}) \leq 1$, $\forall\, n, k,\ n > k \geq 0$. The conditions of Proposition 1.1 are fulfilled, so $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} a_{nk} = \sum\limits_{k=0}^{\infty} a_k = 0$, which means (2.10).

Now, we shall prove (i). By Theorem 1.3, it follows that there exists a sequence of stable stochastic matrices $\Pi_{m,n}$, $0 \leq m < n$, such that

$$\lim_{n\to\infty} (C_{m,n} - \Pi_{m,n}) = \mathbf{0},\ \forall\, m \geq 0.$$

Next, letting $n \to \infty$ in (2.9), using (2.10), $\sum\limits_{n=0}^{\infty} |||R_n|||_\infty < \infty$, and the hypothesis of (i), it follows

$$\lim_{n\to\infty} X_n = X_0 \lim_{n\to\infty} (C_{0,n} - \Pi_{0,n}) = \mathbf{0}.$$

(because $X_k \Pi_{m,n} = \mathbf{0}$, $\forall\, m, n$, $0 \leq m < n$, $\forall\, k \geq 0$).

Now, we shall prove (ii). Letting $n \to \infty$ in (2.9), using (2.10), $\sum\limits_{n=0}^{\infty} |||R_n|||_\infty < \infty$, and the hypothesis of (ii), it follows

$$\lim_{n\to\infty} X_n = X_0 \lim_{n\to\infty} C_{0,n} = X_0 \Pi = \pi.$$

□

3. Weak and strong ergodicity results

In this section an earlier study of the author from [7] is continued. We shall give sufficient conditions for weak and strong ergodicity of a nonhomogeneous Markov chain in terms of similar behaviour of a certain nonhomogeneous Markov chain of smaller size. Our main result is given in Theorem 3.6.

In the sequel, we shall consider $(P_n)_{n\geq 1}$ be a nonhomogeneous Markov chain with state space $S = \{1, 2, ..., r\}$ such that $P_n \to P$ as $n \to \infty$. Suppose that P has exactly $p \geq 1$ irreducible aperiodic closed classes S_i, $i = \overline{1,p}$, and, possibly, transient states, i.e., P is of the form (1.2). Let $\mu^{(i)}$ be the invariant probability vector with respect to S_i, $\forall\, i = \overline{1,p}$, and z_{ji}, $j = \overline{\sum_{t=0}^{p} r_t, r}$, $i = \overline{1,p}$, as in Theorem 1.1. Let $V_n = P_n - P$, $\forall\, n \geq 1$, where $\lim_{n\to\infty} V_n = \mathbf{0}_{r\times r}$. Let Q and Q^{-1} as in Theorem 1.2. Set

$$\widetilde{V}_n = Q^{-1} V_n Q,\ \forall\, n \geq 1, \tag{3.11}$$

and

$$C_n = I_p + (\widetilde{V}_n)_{M\times M}, \forall n \geq 1, \text{ where } M = \{1, ..., p\}. \tag{3.12}$$

Proposition 3.3. *C_n is a stochastic matrix, $\forall\, n \geq 1$.*

Proof. Let q_i be the ith row of the matrix Q^{-1} and $\widetilde{q}_j$ the jth column of the matrix Q. We can write

$$(\widetilde{V}_n)_{ij} = q_i V_n \widetilde{q}_j,\ \forall\, i, j = \overline{1,p},\ \forall\, n \geq 1.$$

Then

$$\sum_{j=1}^{p} (\widetilde{V}_n)_{ij} = \sum_{j=1}^{p} q_i V_n \widetilde{q}_j = q_i V_n \sum_{j=1}^{p} \widetilde{q}_j = q_i V_n \mathbf{e}' =$$

(because $(V_n)_{i,S} = 0, \forall\, i \in S$)

$$= q_i \cdot \mathbf{0}' = 0,\ \forall\, i = \overline{1,p}.$$

Further

$$(\widetilde{V}_n)_{ij} = q_i V_n \widetilde{q}_j = (\mu_1^{(i)}, ..., \mu_{r_i}^{(i)})(V_n)_{S_i\times(S_j\cup T)} \begin{pmatrix} \mathbf{e}'(r_j) \\ z_j' \end{pmatrix},\ \forall\, i, j = \overline{1,p}, \tag{3.13}$$

(in this context by S_j and T we mean the set corresponding to the jth recurrent class and the set of transient states, respectively), where

$$z_j := (z_{r_1+r_2+\cdots+r_p+1,j}, \ldots, z_{r,j}),\quad \forall\, j = \overline{1,p}.$$

Since $(V_n)_{l,S} = 0$, $\forall l \in S$, and $(V_n)_{l,S\setminus S_i} = (P_n - P)_{l,S\setminus S_i} \in [0,1]$, $\forall\, l \in S_i$, $\forall\, i = \overline{1,p}$, $\forall\, n \geq 1$, it follows $(V_n)_{l,S_i\bigcup T} \in [-1, 0]$, $\forall\, l \in S_i$, $\forall\, i = \overline{1,p}$ and $(V_n)_{l,S_j\bigcup T} \in [0,1]$, $\forall\, l \in S_i, \forall\, i = \overline{1,p}, \forall\, j = \overline{1,p}$, $i \neq j$.

Using $0 \leq z_{ji} \leq 1$, $\forall\, j = \overline{\sum_{t=0}^{p} r_t, r}$, $\forall\, i = \overline{1,p}$ and the observation above, we recognize in (3.13) a convex combination, namely, for $i \neq j$, $i, j \in M$, $n \geq 1$,

$$(\widetilde{V}_n)_{ij} = \sum_{t=1}^{r_i} \mu_t^{(i)} a_t^{(j)} \in [0,1], \tag{3.14}$$

since $a_t^{(j)} \in [0,1]$, $\forall\, t = \overline{1, r_i}$, and for $i = j$, $i \in M$, $n \geq 1$,

$$(\widetilde{V}_n)_{ii} = \sum_{t=1}^{r_i} \mu_t^{(i)} b_t^{(i)} \in [-1, 0], \tag{3.15}$$

since $b_t^{(i)} \in [-1, 0]$, $\forall\, t = \overline{1, r_i}$.
Finally, from $(\widetilde{V}_n)_{i,M} = 0$, $\forall\, i = \overline{1, p}$, (3.14), and (3.15) the conclusion follows. □

The following theorem is the main result of this paper.

Theorem 3.6. *Suppose that*

$$\sum_{n=1}^{\infty} |||(\widetilde{V}_n)_{(S\setminus M)\times M}|||_{\infty} < \infty, \tag{3.16}$$

where $M = \{1, ..., p\}$. Then the following statements hold.
(i) If $(C_n)_{n\geq 1}$ is weakly ergodic, then $(P_n)_{n\geq 1}$ is weakly ergodic.
(ii) If $(C_n)_{n\geq 1}$ is strongly ergodic, then $(P_n)_{n\geq 1}$ is strongly ergodic.

Proof. Let $m \geq 0$. By the Chapman-Kolmogorov equation we have

$$P_{m,n+1} = P_{m,n} P_{n+1}, \ \forall\, n > m.$$

By subtracting $P_{m,n}$ from both sides, we obtain

$$P_{m,n+1} - P_{m,n} = P_{m,n}[-I_r + P_{n+1}], \ \forall\, n > m. \tag{3.17}$$

Set

$$t_{m,n}^{(i)} = ((P_{m,n})_{i,1}, ..., (P_{m,n})_{i,r}), \ \forall\, i \in S, \forall n > m. \tag{3.18}$$

Then equations (3.17) read as

$$t_{m,n+1}^{(i)} - t_{m,n}^{(i)} = t_{m,n}^{(i)}[-I_r + P_{n+1}], \ \forall\, i \in S, \ \forall\, n > m.$$

We remark that $t_{m,n}^{(i)}$ defined in (3.18) are solutions of equations of the type

$$x_{m,n+1} - x_{m,n} = x_{m,n}[-I_r + P_{n+1}], \ \forall\, n > m, \tag{3.19}$$

under the conditions

$$(x_{m,n})_i \in [0,1], \ \forall\, i \in S, \ \sum_{i=1}^{r} (x_{m,n})_i = 1, \ \forall\, n > m, \tag{3.20}$$

or

$$(x_{m,n})_i \in [-1,1], \ \forall\, i \in S, \ \sum_{i=1}^{r} (x_{m,n})_i = 0, \ \forall\, n > m. \tag{3.21}$$

We are interested in the asymptotic behaviour of the proposed solutions of (3.19) under conditions (3.20) or (3.21).

Setting $A = -I_r + P$, we can benefit of the result given in Theorem 1.2. Further, setting

$$y_{m,n} = x_{m,n} Q, \ \forall\, n > m, \tag{3.22}$$

equations (3.19) amount to

$$y_{m,n+1} - y_{m,n} = y_{m,n} J + y_{m,n} \widetilde{V}_{n+1}, \ \forall\, n > m. \tag{3.23}$$

Using the same arguments as in [7], we have

$$\lim_{n\to\infty}(y_{m,n})_i = 0,\ \forall\, i = \overline{p+1, r}. \tag{3.24}$$

From (3.23), following the same steps as in [7], we obtain

$$Y_{m,n+1} = Y_{m,n}C_{n+1} + R_{m,n},\ \forall\, n > m,$$

where $Y_{m,n} = ((y_{m,n})_1, ..., (y_{m,n})_p)$ and $R_{m,n} = ((R_{m,n})_1, ..., (R_{m,n})_p)$ with

$$(R_{m,n})_i = \sum_{j=p+1}^{r} (y_{m,n})_j (\widetilde{V}_{n+1})_{ji},\ \forall\, i = \overline{1,p},\ \forall\, n > m.$$

By (3.16), using the fact that $\exists \overline{M} \geq 0$ such that $|y_{m,n}| \leq \overline{M}$, $\forall\, m, n$, $0 \leq m < n$, it follows that $\sum_{n=m}^{\infty} |||R_{m,n}|||_\infty < \infty$, $\forall\, m \geq 0$.

In order to prove (i), let $i, j \in S$. Let

$$x_{m,n} = x_{m,n}(i,j) = t_{m,n}^{(i)} - t_{m,n}^{(j)},\ \forall\, n > m,$$

where $t_{m,n}^{(i)}$ was defined in (3.18), $\forall\, i \in S$. Then, using (1.5), we have (for $y_{m,n} := x_{m,n}Q$)

$$\sum_{k=1}^{p}(y_{m,n})_k = \sum_{k=1}^{p} x_{m,n}\widetilde{q}_k = x_{m,n}\sum_{k=1}^{p}\widetilde{q}_k = x_{m,n}\mathbf{e}' = 0,\ \forall\, n > m \tag{3.25}$$

($\widetilde{q}_k$ was defined in the proof of Proposition 3.3).

By Proposition 2.2 (i), it follows

$$\lim_{n\to\infty} Y_{m,n} = \mathbf{0}.$$

This, (3.24), and $x_{m,n} = y_{m,n}Q^{-1}$, give us

$$\lim_{n\to\infty}[(P_{m,n})_{ik} - (P_{m,n})_{jk}] = 0,\ \forall\, k \in S. \tag{3.26}$$

Therefore $(P_n)_{n\geq 1}$ is weakly ergodic.

Now, we shall prove (ii). Let $i \in S$. Let

$$x_{m,n} = x_{m,n}(i) = t_{m,n}^{(i)}, \forall n > m,$$

where $t_{m,n}^{(i)}$ was defined in (3.18), $\forall i \in S$.

Then, using (1.5), we have (for $y_{m,n} := x_{m,n}Q$)

$$\sum_{k=1}^{p}(y_{m,n})_k = \sum_{k=1}^{p} x_{m,n}\widetilde{q}_k = x_{m,n}\sum_{k=1}^{p}\widetilde{q}_k = x_{m,n}\mathbf{e}' = 1,\ \forall\, n > m. \tag{3.27}$$

By Proposition 2.2 (ii) it follows

$$\lim_{n\to\infty} Y_{m,n} = \pi.$$

This and $x_{m,n} = y_{m,n}Q^{-1}$ give

$$\lim_{n\to\infty}((P_{m,n})_{i,r_0+...+r_{t-1}}, ..., (P_{m,n})_{i,r_1+...+r_t}) = (\mu_1^{(t)}\pi_t, ..., \mu_{r_t}^{(t)}\pi_t),\ \forall\, t = \overline{1,p}.$$

Therefore, using (3.24), $(P_n)_{n\geq 1}$ is strongly ergodic. □

Example 3.1. Consider the chain $(P_n)_{n\geq 1}$ given by

$$P_n = \begin{pmatrix} 1-\frac{2}{n+1}+\frac{1}{2n} & \frac{1}{2n} & 0 & \frac{2}{n+1}-\frac{1}{n} \\ \frac{1}{2n} & 1-\frac{2}{n+1}+\frac{1}{2n} & 0 & \frac{2}{n+1}-\frac{1}{n} \\ 0 & 0 & 1-\frac{1}{n} & \frac{1}{n} \\ \frac{1}{2}-\frac{1}{2(n+1)} & \frac{1}{2}-\frac{1}{2(n+1)} & 0 & \frac{1}{n+1} \end{pmatrix}, \ \forall n \geq 1.$$

$(P_n)_{n\geq 1}$ is weakly (even strongly) ergodic because are fulfilled the conditions of Theorem 3.6. These are left to the reader.

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