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Sequences of almost contractions and fixed points

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1. PRELIMINARIES

We begin by recalling some notions and results we shall be using throughout the paper. If not mentioned, (X, d) is assumed to be a metric space.

Definition 1.1. A mapping $f : X \to X$ is a **Picard operator** if:
$i)$ $F_f = \{x^*\}$;
$ii)$ for any $x_0 \in X$ the sequence $(f^n(x_0))_{n\geq 0}$ converges to x^*,
where F_f represents the set of fixed points of the mapping f.

Definition 1.2. A mapping $f : X \to X$ is a **weakly Picard operator** if:
$iii)$ $F_f \neq \emptyset$;
$iv)$ for any $x_0 \in X$ the sequence $(f^n(x_0))_{n\geq 0}$ converges to a fixed point of f.

We also have to mention the types of convergence we shall be using.
If $g_n : X \to X$, $n \in X$ and $g : X \to X$, then:

Definition 1.3. The sequence $(g_n)_{n\geq 0}$ **converges** to g, denoted $g_n \xrightarrow{p} g$, as $n \to \infty$, if for any $\varepsilon > 0$ and any $x \in X$ there exists $N(\varepsilon, x) > 0$ such that for any $n \geq N(\varepsilon, x)$ the following holds:

$$d(g_n(x), g(x)) \leq \varepsilon.$$

Definition 1.4. The sequence $(g_n)_{n\geq 0}$ **converges uniformly** to g, denoted $g_n \xrightarrow{u} g$, as $n \to \infty$, if for any $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that for any $n \geq N(\varepsilon)$ and any $x \in X$ the following holds:

$$d(g_n(x), g(x)) \leq \varepsilon.$$

Definition 1.5. If g_n, $n \in \mathbb{N}$ and g are weakly Picard operators, then the sequence $(g_n)_{n\geq 0}$ **converges asymptotically** to g, denoted $g_n \xrightarrow{a} g$, as $n \to \infty$, if g_n^m converges (in a certain sense) to g^∞ as $n, m \to \infty$, where:

$$g^\infty(x) = \lim_{n\to\infty} g^n(x), x \in X.$$

The types of contractive operators that will be referred in this paper are mentioned in the following with their definitions. Some of them can also be found for example in [2], [5], [17].

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Definition 1.6 ([11]). A mapping $f : X \to X$ is called **Kannan mapping** if there exists $k \in \left[0, \frac{1}{2}\right)$ such that

$$d(f(x), f(y)) \leq k[d(x, f(x)) + d(y, f(y))], \text{ for any } x, y \in X. \tag{1.1}$$

In the framework of a complete metric space, any Kannan mapping is a Picard operator, as shown in [11].

Definition 1.7 ([8]). A mapping $f : X \to X$ is called **Chatterjea mapping** if there exists $c \in \left[0, \frac{1}{2}\right)$ such that

$$d(f(x), f(y)) \leq c[d(x, f(y)) + d(y, f(x))], \text{ for any } x, y \in X. \tag{1.2}$$

In the framework of a complete metric space, any Chatterjea mapping is a Picard operator, as shown in [8].

Definition 1.8 ([21]). A mapping $f : X \to X$ is called **Zamfirescu mapping** if there exist $\alpha, k, c \in \mathbb{R}$, $\alpha \in [0, 1)$, $k, c \in \left[0, \frac{1}{2}\right)$, such that for any $x, y \in X$ at least one of the following holds:

$i)$ $d(f(x), f(y)) \leq \alpha d(x, y)$;
$ii)$ $d(f(x), f(y)) \leq k[d(x, f(x)) + d(y, f(y))]$;
$iii)$ $d(f(x), f(y)) \leq c[d(x, f(y)) + d(y, f(x))]$.

In the framework of a complete metric space, any Zamfirescu mapping is a Picard operator, as shown in [21].

Definition 1.9 ([9]). A mapping $f : X \to X$ is called **quasi-contraction** if there exists $h \in (0, 1)$ such that

$$d(f(x), f(y)) \leq h \max\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}, \tag{1.3}$$

for any $x, y \in X$.

In the framework of a complete metric space, any quasi-contraction is a Picard operator, as shown in [9].

Definition 1.10 ([1]). A mapping $f : X \to X$ is said to satisfy **condition (B)** if there exist $\delta \in (0, 1)$ and $L \geq 0$ such that

$$d(f(x), f(y)) \leq \delta d(x, y) + L \min\{d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}, \tag{1.4}$$

for any $x, y \in X$.

In the framework of a complete metric space, any mapping satisfying condition (B) is a Picard operator, as shown in the very recent paper [1].

In the following we shall define the concept of *almost contraction*, but it has to be mentioned that in the original paper [6] this was termed as *weak contraction*. This "'old"' name has been used in various papers until recently, when the author [7] decided to replace it by a more suggestive one.

Definition 1.11 ([6]). A mapping $f : X \to X$ is called $(\delta, \mathbf{L})-$ **almost contraction** or simply **almost contraction** if there exist $\delta \in (0,1)$ and $L \geq 0$ such that

$$d(f(x), f(y)) \leq \delta d(x, y) + Ld(y, f(x)), \tag{1.5}$$

for any $x, y \in X$.

We note that in (1.3) we can have $\delta = 0$, provided that, in this case, $L = 0$.

Regarding this last type of operators we present partially Theorems 1 and 2 in [6] in the following.

Theorem 1.1 ([6]). *Let (X, d) be a complete metric space and $f : X \to X$ a $(\delta, L)-$ almost contraction, with $\delta \in (0, 1)$ and $L \geq 0$. Then f is a weakly Picard operator.*

Theorem 1.2 ([6]). *Let (X, d) be a complete metric space and $f : X \to X$ a $(\delta, L)-$ almost contraction, with $\delta \in (0, 1)$ and $L \geq 0$. If in addition there exist $\theta \in (0, 1)$ and $L_1 \geq 0$ such that:*

$$d(f(x), f(y)) \leq \theta d(x, y) + L_1 d(x, f(x)), \tag{1.6}$$

for any $x, y \in X$, then f is a Picard operator.

Remark 1.1. In section 2 of the paper [6] see also it is proved that all $\alpha-$contractions, Kannan mappings, Chatterjea mappings, Zamfirescu mappings and quasi-contractions with $h \in \left(0, \frac{1}{2}\right)$ are almost contractions with unique fixed point in complete metric spaces. Besides, in the recent paper [1] it is shown that all mappings satisfying condition (B) are almost contractions with unique fixed point. Thus the above Theorem 1.2 generalizes all the existence and uniqueness results concerning the types of operators previously mentioned.

The main results of this paper have as starting point two theorems published by S. B. Nadler in 1968 [12]. He considers here sequences of operators converging to an $\alpha-$contraction. The theorems are stated below without their proofs, which can be found in the original paper [12].

Theorem 1.3 ([12]). *Let (X, d) be a complete metric space and $g_n, g : X \to X$, $n \in \mathbb{N}$ such that:*

$i)$ each g_n has at least one fixed point, say $x_n \in F_{g_n}$, $n \in \mathbb{N}$;
$ii)$ g is an $\alpha-$ contraction, with $F_g = \{x^*\}$;
$iii)$ $g_n \xrightarrow{u} g$, $n \to \infty$.
Then $x_n \to x^*$, $n \to \infty$.

Theorem 1.4 ([12]). *Let (X, d) be a locally compact metric space and $g_n, g : X \to X$, $n \in \mathbb{N}$ such that:*

$i)$ g_n are $\beta-$contractions, with $F_{g_n} = \{x_n^*\}$, $n \in \mathbb{N}$;
$ii)$ g is $\alpha-$contraction, with $F_g = \{x^*\}$;
$iii)$ $g_n \xrightarrow{p} g$, $n \to \infty$.
Then $x_n \to x^*$, $n \to \infty$.

2. Sequences of operators converging to an almost contraction. Convergence theorems

One possible direction in which the previous results can be extended is to consider more general contraction conditions instead of the $\alpha-$ contractions used by Nadler. For example, in the paper [17] one can find such results concerning $\varphi-$contractions and contractive operators.

We will follow this direction, considering almost-contractions instead of the α-contractions used by Nadler.

The following theorem is a generalization of Theorem 1.3.

Theorem 2.5. *Let (X, d) be a complete metric space and $g_n, g : X \to X$, $n \in \mathbb{N}$ such that:*

$i)$ each g_n has at least a fixed point, $x_n \in F_{g_n}$, $n \in \mathbb{N}$;
$ii)$ g is $(\delta, L)-$ almost contraction;
$ii')$ there exist $\theta \in (0, 1)$ and $L_1 \geq 0$ such that:

$$d(g(x), g(y)) \leq \theta d(x, y) + L_1 d(x, g(x)),\ \text{for any } x, y \in X;$$

$iii)$ $g_n \xrightarrow{u} g$, $n \to \infty$.
Then $x_n \to x^*$, $n \to \infty$, where $F_g = \{x^*\}$.

Proof. Conditions $ii)$ and $ii')$ guarantee the existence and uniqueness of the fixed point x^* for g, according to Theorem 1.2. Thus we can write:

$$\begin{aligned} d(x_n, x^*) &= d(g_n(x_n), g(x^*)) \leq d(g_n(x_n), g(x_n)) + d(g(x_n), g(x^*)) = \\ &= d(g_n(x_n), g(x_n)) + d(g(x^*), g(x_n)) \leq \\ &\leq d(g_n(x_n), g(x_n)) + \theta d(x^*, x_n) + L_1 d(x^*, g(x^*)) = \\ &= d(g_n(x_n), g(x_n)) + \theta d(x^*, x_n), \end{aligned}$$

so

$$d(x_n, x^*) \leq \frac{1}{1-\theta} d(g_n(x_n), g(x_n)),$$

for any $n \in \mathbb{N}$.

As the sequence g_n converges uniformly to g when $n \to \infty$, it follows that $d(x_n, x^*) \to 0$, so

$$x_n \to x^*.$$ □

Considering Remark 1.1 above, Theorem 1.3 due to Nadler is a corollary of Theorem 2.5. Also the following corollaries can be stated:

Corollary 2.1. *Let (X, d) be a complete metric space and $g_n, g : X \to X$, $n \in \mathbb{N}$ such that:*

$i)$ each g_n has at least a fixed point, $x_n \in F_{g_n}$, $n \in \mathbb{N}$;
$ii)$ g is a Kannan mapping, with $F_g = \{x^*\}$;
$iii)$ $g_n \xrightarrow{u} g$, $n \to \infty$.
Then $x_n \to x^*$, $n \to \infty$.

Corollary 2.2. *Let (X, d) be a complete metric space and $g_n, g : X \to X$, $n \in \mathbb{N}$ such that:*

$i)$ each g_n has at least a fixed point, $x_n \in F_{g_n}$, $n \in \mathbb{N}$;

ii) *g is a Chatterjea mapping, with* $F_g = \{x^*\}$;
iii) $g_n \xrightarrow{u} g, n \to \infty$.
Then $x_n \to x^*, n \to \infty$.

Corollary 2.3. *Let* (X, d) *be a complete metric space and* $g_n, g : X \to X, n \in \mathbb{N}$ *such that:*
i) *each* g_n *has at least a fixed point,* $x_n \in F_{g_n}, n \in \mathbb{N}$;
ii) *g is a Zamfirescu mapping, with* $F_g = \{x^*\}$;
iii) $g_n \xrightarrow{u} g, n \to \infty$.
Then $x_n \to x^*, n \to \infty$.

Corollary 2.4. *Let* (X, d) *be a complete metric space and* $g_n, g : X \to X, n \in \mathbb{N}$ *such that:*
i) *each* g_n *has at least a fixed point,* $x_n \in F_{g_n}, n \in \mathbb{N}$;
ii) *g is a quasi-contraction with constant* $h \in (0, \frac{1}{2})$ *and* $F_g = \{x^*\}$;
iii) $g_n \xrightarrow{u} g, n \to \infty$.
Then $x_n \to x^*, n \to \infty$.

Corollary 2.5. *Let* (X, d) *be a complete metric space and* $g_n, g : X \to X, n \in \mathbb{N}$ *such that:*
i) *each* g_n *has at least a fixed point,* $x_n \in F_{g_n}, n \in \mathbb{N}$;
ii) *g satisfies condition (B), with* $F_g = \{x^*\}$;
iii) $g_n \xrightarrow{u} g, n \to \infty$.
Then $x_n \to x^*, n \to \infty$.

Similar to Theorem 1.4 we can state the following result, but in complete metric spaces, *not* in locally compact metric spaces: as $(\delta, L)-$almost contractions are not generally continuous, the pointwise convergence on a compact set does not imply the uniform one anymore.

Theorem 2.6. *Let* (X, d) *be a complete metric space and* $g_n, g : X \to X, n \in \mathbb{N}$ *such that:*
i) g_n *are* $(a, K)-$*almost contractions, with* $a \in (0, 1),\ K \geq 0,$ *for all* $n \in \mathbb{N}$;
i′) *there exist* $a_1 \in (0, 1)$ *and* $K_1 \geq 0$ *such that*

$$d(g_n(x), g_n(y)) \leq a_1 d(x, y) + K_1 d(x, g_n(x)),\ \text{for all } n \in \mathbb{N},\ x, y \in X;$$

ii) *g is* $(\delta, L)-$ *almost contraction;*
ii′) *there exist* $\theta \in (0, 1)$ *and* $L_1 \geq 0$ *such that:*

$$d(g(x), g(y)) \leq \theta d(x, y) + L_1 d(x, g(x)), \text{ for all } x, y \in X;$$

iii) $g_n \xrightarrow{p} g, n \to \infty$.
Then $x_n \to x^*, n \to \infty$, *where* $F_{g_n} = \{x_n^*\}, n \in \mathbb{N}$ *and* $F_g = \{x^*\}$.

Proof. Conditions $i')$ and $ii')$ make sure that $g_n, n \in \mathbb{N}$, and respectively g have each a unique fixed point. Thus we may denote by $F_{g_n} = \{x_n^*\}$, $n \in \mathbb{N}$ and $F_g = \{x^*\}$ the sets of fixed points for these operators.

Then for any $n \in \mathbb{N}$ we have:

$$\begin{aligned} d(x_n^*, x^*) &= d(g_n(x_n^*), g(x^*)) \leq d(g_n(x_n^*), g_n(x^*)) + d(g_n(x^*), g(x^*)) \leq \\ &\leq a_1 d(x_n^*, x^*) + K_1 d(x_n^*, g_n(x_n^*)) + d(g_n(x^*), g(x^*)) = \\ &= a_1 d(x_n^*, x^*) + d(g_n(x^*), g(x^*)), \end{aligned}$$

so

$$d(x_n^*, x^*) \leq \frac{1}{1-a_1} d(g_n(x^*), g(x^*)).$$

Now considering $iii)$, when $n \to \infty$ we get that:

$$d(x_n^*, x^*) \to 0, \text{ so } x_n \to x^*. \qquad \square$$

Having in view the *Remark* 1.1, we can formulate corollaries of *Theorem* 2.5 as well. For $\alpha-$ contractions such a corollary would be contained in *Theorem* 1.4, so we shall skip it.

Remark 2.2. As in Theorem 1.4, the sequence of almost contractions $(g_n)_{n\geq 0}$ has the same constants a, K, respectively a_1, K_1, for any $n \in \mathbb{N}$, a necessary condition in order to obtain the conclusion.

Corollary 2.6. *Let (X, d) be a complete metric space and $g_n, g : X \to X$, $n \in \mathbb{N}$ such that:*

$i)$ g_n are Kannan operators with the same constant $k \in \left(0, \frac{1}{2}\right)$, for all $n \in \mathbb{N}$;
$ii)$ g is Kannan operator;
$iii)$ $g_n \xrightarrow{p} g, n \to \infty$.
Then $x_n \to x^*, n \to \infty$, where $F_{g_n} = \{x_n^*\}, n \in \mathbb{N}$ and $F_g = \{x^*\}$.

Corollary 2.7. *Let (X, d) be a complete metric space and $g_n, g : X \to X$, $n \in \mathbb{N}$ such that:*

$i)$ g_n are Chatterjea operators with the same constant $c \in \left(0, \frac{1}{2}\right)$, for all $n \in \mathbb{N}$;
$ii)$ g is Chatterjea operator;
$iii)$ $g_n \xrightarrow{p} g, n \to \infty$.
Then $x_n \to x^*, n \to \infty$, where $F_{g_n} = \{x_n^*\}, n \in \mathbb{N}$ and $F_g = \{x^*\}$.

Corollary 2.8. *Let (X, d) be a complete metric space and $g_n, g : X \to X$, $n \in \mathbb{N}$ such that:*

$i)$ g_n are Zamfirescu operators with the same constant, for all $n \in \mathbb{N}$;
$ii)$ g is Zamfirescu operator;
$iii)$ $g_n \xrightarrow{p} g, n \to \infty$.
Then $x_n \to x^*, n \to \infty$, where $F_{g_n} = \{x_n^*\}, n \in \mathbb{N}$ and $F_g = \{x^*\}$.

Corollary 2.9. *Let (X, d) be a complete metric space and $g_n, g : X \to X$, $n \in \mathbb{N}$ such that:*

$i)$ g_n are quasi-contractions with the same constant $h \in \left(0, \frac{1}{2}\right)$, for all $n \in \mathbb{N}$;
$ii)$ g is quasi-contraction;
$iii)$ $g_n \xrightarrow{p} g, n \to \infty$.
Then $x_n \to x^*, n \to \infty$, where $F_{g_n} = \{x_n^*\}, n \in \mathbb{N}$ and $F_g = \{x^*\}$.

Corollary 2.10. *Let (X, d) be a complete metric space and g_n, $g : X \to X$, $n \in \mathbb{N}$ such that:*

$i)$ g_n satisfy condition (B) with the same constants, for all $n \in \mathbb{N}$;
$ii)$ g satisfies condition (B);
$iii)$ $g_n \xrightarrow{p} g, n \to \infty$.
Then $x_n \to x^*, n \to \infty$, where $F_{g_n} = \{x_n^*\}, n \in \mathbb{N}$ and $F_g = \{x^*\}$.

3. Comparison to another generalization of Nadler's result

In the previous section we showed how Theorem 2.5 extends Nadler's convergence theorem mentioned in this paper as Theorem 1.3. In paper [15] of I.A. Rus a similar generalization of the same theorem is proved, being included without a proof in paper [20], too. The question whether the two generalizations do coincide or not appears naturally. In this section we are going to answer this question.

Theorem 3.7 ([15]). *Let (X, d) be a complete metric space and $g_n, g : X \to X$, $n \in \mathbb{N}$ such that:*

$i)$ *each g_n has at least one fixed point, $x_n \in F_{g_n}$, $n \in \mathbb{N}$;*

$ii)$ *g is a Picard operator, with $F_g = \{x^*\}$;*

$iii)$ *$g_n \overset{a}{\to} g$, $n \to \infty$ (with respect to the uniform convergence).*

Then $x_n \to x^*$, $n \to \infty$.

Theorem 1.3 is a corollary of this theorem, as for $\alpha-$ contractions the following statement included in [20] as *Exemple 2.5* holds.

Proposition 3.1 ([20]). *Let (X, d) be a complete metric space and g_n, $g : X \to X$, $n \in \mathbb{N}$ such that:*

$i)$ *g is $\alpha-$contraction;*

$ii)$ *$g_n \overset{u}{\to} g$, $n \to \infty$.*

Then $g_n \overset{a}{\to} g$, $n \to \infty$, with respect to the uniform convergence.

The question now is whether this statement remains true for almost contractions, as this would mean that Theorem 2.5 is included in Theorem 3.7 above. The following example allows us to answer this question in the negative.

Example 3.1. Let

$$g_n : \mathbb{R} \to \mathbb{R},\ g_n(x) = \frac{1}{2^n} x, n \in \mathbb{N} \tag{3.7}$$

and

$$g : \mathbb{R} \to \mathbb{R},\ g(x) = \begin{cases} 0, & x \in (-\infty, 2] \\ -\dfrac{1}{2}, & x \in (2, \infty). \end{cases} \tag{3.8}$$

Then:

a) g is $(\delta, L)-$almost contraction with unique fixed point $x^* = 0$, so $g^\infty \equiv 0$;

b) $g_n \overset{u}{\not\to} g, n \to \infty$;

c) $g_n \overset{a}{\to} g, n \to \infty$.

Indeed, the mapping g in (3.8) is given in [4] as an example of non-continuous Kannan mapping with constant $k = \dfrac{1}{5}$ and unique fixed point $x^* = 0$. In [6], Proposition 1 states that any Kannan operator with constant k is also a $(\delta, L)-$almost contraction with $\delta = \frac{k}{1-k}$ and $L = \frac{2k}{1-k}$. It follows that our g is a $(\delta, L)-$almost contraction with

$$\delta = \frac{1}{4} \text{ and } L = \frac{1}{2}.$$

Further on, considering (3.8) and (3.7), it is obvious that $g_n \not\to g, n \to \infty$, and so $g_n \overset{u}{\not\to} g, n \to \infty$. As for the asimptotic convergence, things change. We have that:

$$g_n^2(x) = \frac{1}{2^n}(\frac{1}{2^n}x) = \frac{1}{2^{2n}}x, \dots, g_n^m(x) = \frac{1}{2^{mn}}x.$$

On the other hand, $g^\infty \equiv 0$, so

$$g_n^m \xrightarrow{u} g^\infty, m \to \infty, n \to \infty,$$

wich according to Definition 1.5 means that

$$g_n \xrightarrow{a} g, n \to \infty.$$

Conclusion. The uniform convergence of a sequence of mappings to an almost contraction with unique fixed point (which consequently is a Picard operator) does not imply the asimptotic convergence of this sequence. So Theorem 2.5 is not a corollary of Theorem 3.7, as the two results are *different generalizations* of the Theorem 1.3 given by Nadler.

Remark 3.3. The above example can be formulated in a more general form, considering instead of g_n and g, respectively,

$$f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \frac{1}{a^n}x, n \in \mathbb{N},$$

with $a \in \mathbb{R}$, $a > 1$ and

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} 0, & x \in (-\infty, b] \\ -\beta, & x \in (b, \infty), \end{cases}$$

where $\beta > 0$, $b > 0$, $\beta < b$. This f is similarly a non-continuous $(\delta, L)-$almost contraction, with $\delta = \frac{\beta}{b}$ and $L = \frac{2\beta}{b}$.

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