CARPATHIAN J. MATH.
24 (2008), No. 2, 110 - 119

An algorithm for solving nonsmooth variational inequalities arising in frictional quasistatic contact problems

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ABSTRACT.
The aim of this paper is to give an algorithm for nonsmooth minimization using the generalized Jacobians with applications in contact problems. Some definitions, results and algorithms from nonsmooth analysis are presented which lead to some sufficient conditions such that the nonlinear and nondifferentiable system obtained in modeling the contact problems with friction, have one solution which is obtained by bundle methods.

1. INTRODUCTION

The contact problems are characterized by being highly nonlinear and nondifferentiable and are amongst the most difficult problems to handle. Many studies have been carried out on the frictional contact problems for several decades. The smoothing procedure is complex and the algorithms for solving nonsmooth equations have been developed by Pang (1990), Qi (1993). Cristiensen et all. (1998) presented the employment of Newton's method to formulate B-differentiable equations involving projection for the contact problems. The numerical treatment of the unilateral contact with dry friction is certainly one of the non smooth mechanics topics for which many efforts have been done in the last decade, see for example the fixed-point iteration approach [1], or the fixed-point convergence methods [2], [4] and [14]-[15].

In this paper a new model based on nonsmooth equations is proposed for the three-dimensional frictional quasistatic contact problems. Contact conditions for the slip direction of the contact nodes are modeled by formulating the Coulomb friction law as a nonsmooth equation.

2. ANALYSIS OF LOCALLY LIPSCHTZ CONTINUOUS FUNCTIONS

We consider useful to summarize the main features of the nonsmooth analysis, for introducing the procedure to solve the nonsmooth equations modeling the quasistatic functional contact problem from [3].

Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitzian but not necessary differentiable and consider the nonsmooth equation

$$F(x) = 0. \tag{2.1}$$

Received: 25.07.2008; In revised form: 03.09.2008; Accepted: 30.09.2008
2000 *Mathematics Subject Classification.* 49J52, 49J49, 74M10,74M15, 49M37, 90C46.
Key words and phrases. *Frictional quasistatic problems, nonsmooth analysis, generalized Jacobian, bundle methods.*

The most popular method for solving (2.1) is the Newton method based on the Clarke generalized Jacobian,

$$x_{k+1} = x_k + JF^{-1}(x_k)F(x_k) \tag{2.2}$$

where $JF(x_k) \in \partial F(x_k)$ and $\partial F(x_k)$ is the Clarke generalized Jacobian of the F at x_k [3]. The generalized Jacobian of F is defined as a convex hull of all $n \times n$ matrices obtained as the limit of a sequence $(JF(x_k))_k$ when $x_k \to x$, $k \to \infty$ and x_k is a point at which F is differentiable.

In the case when F from (2.1) is smooth (i.e. Frechet differentiable), we can use Newton's method for solving equation (2.1).

In the case when F is nonsmooth, $JF(x_k)$ may not exist, and therefore, it should be replaced by a generalized derivative. In this situation assume that the vector valued function $F : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitzian function and it is also nonsmooth. Then, by using Rodemacher's Theorem, F is differentiable almost everywhere.

Denote by D_F the set of points where F is differentiable, and by $\nabla F(x)$ a $n \times n$ Jacobian matrix of partial derivatives whenever x_k is a point at which the partial derivatives exist. By $\partial F(x)$ we denote the generalized derivative:

$$\partial F(x) = \left\{ V(x) = \lim_{k\to\infty} \nabla F(x_k) : \lim_{k\to\infty} x_k \to x,\ x_k \in D_F \right\}. \tag{2.3}$$

Definition 2.1. ([3]) The function F is *BD-regular* at x if all matrices $V \in \partial F(x)$ are nonsingular.

Definition 2.2. ([3]) If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitzian and nonsmooth function and for $\mathbf{h} \in \mathbb{R}^n - \{0\}$, $F'(\mathbf{x};\mathbf{h}) = \lim_{t\downarrow 0} \dfrac{F(\mathbf{x}+t\mathbf{h}) - F(\mathbf{x})}{t}$ exist, then we say that F is *directionally differentiable at* $\mathbf{x}$ or *B-differentiable function at* $\mathbf{x}$.

The fundamental distinction between the smooth (F-differentiable) and nonsmooth (B-differentiable) functions is the absence of linearity in the directional derivative for nonsmooth functions.

Definition 2.3. ([3]) If $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitzian continuous function on the open domain D and the limit

$$\lim_{\substack{y\downarrow\bar{y}\ t\downarrow 0 \\ V\in\partial F(x+t\bar{y})}} V\bar{y}$$

exist for every direction $y \in \mathbb{R}^n$, then the function F is said to be *semismooth* at $x \in D$.

Semismooth functions lie between locally Lipschitz continuous and continuously differentiable functions. It is known [20] that, if F is semismooth at x, then the usual directional derivative $F'(x;h)$ exists for every direction $h \in \mathbb{R}^n\backslash\{0\}$ and, moreover, the directional derivative is even a *Bouligand* derivative. This fact is crucial for the successful application of semismooth functions, since it allows to perform analytical investigations using computable derivatives (see [16] and [11]).

For an algorithm for nonsmooth minimization, it is of interest to find conditions, which identify a stationary point or a local minimum. The following theorem, adopted from [8], provides necessary conditions for a local minimum.

Theorem 2.1. ([8]) *For a locally Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}$ the following properties are equivalent:*

a) F has a stationary point $\boldsymbol{x}_0$;

b) $0 \in \partial F(\boldsymbol{x}_0)$;

c) $F'(\boldsymbol{x};\boldsymbol{h}) \geq 0$ for all $\boldsymbol{h} \in \mathbb{R}^n$ and $F(\boldsymbol{x}_0)$ is the global minimum of F.

Note that the third condition is suitable for practical computations.

3. Nonsmooth nonlinear equations set formulation for quasi-static contact conditions

Modeling the frictional contact between an elastic body and a rigid one or between two elastic bodies, leads to quasistatic variational inequalities of the form:

Find $u \in H^1(0,T;V)$ with

$$a(u(t)), v - \dot{u}(t)) + j(u(t), v) - j(u(t), \dot{u}(t)) \geq \langle f(t), v - \dot{u}(t)\rangle \,, \; \forall\, v \in V, \tag{3.4}$$

where V is a Banach space, $\dot{u}$ denotes the time derivative of the quantity $u(t,x)$, $j(u,v)$ is a nonsmooth functional defined on $V \times V$, $a(u,v)$ is a bilinear for virtual work produced by interval forces, f is a linear functional (virtual work produced by external forces) and $H^1(0,T;V)$ is a Hilbert space (Sobolev space). After finite element discretization, in spaces, and finite difference discretization in time, of this inequation, we obtain a big non-symmetric, nonlinear system with a non-differential term.

The contact condition and the friction law can be expressed as non-smooth equations in which the variables are related to the candidate contact nodes.

3.1. **Contact conditions.** Assume that the contact surface is smooth and both the deformation and strain are small such that the point-to-point contact model can be employed. There are two bodies Ω^1 and Ω^2 in contact.

On the potential contact boundary Γ_C, one may use the two contact boundaries in a local coordinate system (n,t,τ) defined on Γ_C, where the vector n, which points from the contact body Ω^1 to Ω^2 denotes the unit normal vector to Γ_C, and the vectors t, τ denote the two orthogonal unit tangential vectors on Γ_C.

Assume that there are NC pairs of possible candidate contact nodes on the potential contact boundary, and consider the contact conditions of the i^{th} pair of candidate contact nodes. The superscripts 1 and 2 denote the contact bodies Ω^1 and Ω^2, respectively and the subscripts n, t, τ denote the components in the directions of n, t, τ, respectively.

In the local coordinate system, the relative displacement between the two contact bodies used in the formulation, are defined as follows:

$$\begin{aligned} \Delta u_n^i &= u_n^{1i} - u_n^{2i} + \Delta u_n^0, & \Delta du_n^i &= du_n^{1i} - du_n^{2i} \\ \Delta du_t^i &= du_t^i - du_t^i, & \Delta du_\tau^i &= du_\tau^i - du_\tau^i, \end{aligned} \tag{3.5}$$

where Δu_n^i, Δu_n^0 denote the current and initial normal gap between the ith pair candidate contact nodes, respectively; Δdu_n^i, Δdu_t^i, Δdu_τ^i denote the incremental relative displacement between the ith pairs candidate nodes in the direction of η, t, τ, respectively.

3.1.1. *The law of action and reaction of the contact boundary* Γ_C. For the ith pair of candidate contact nodes, the conditions are

$$s_n^{1i} = -s_n^{2i} = s_n^i,\ s_t^{1i} = -s_t^{2i} = s_t^i,\ s_\tau^{1i} = -s_\tau^{2i} = s_\tau^i \tag{3.6}$$

where, for simplicity s_n^i, s_t^i, s_τ^i are used to denote the contact stresses on contact boundary Γ_C in the local coordinate system for the ith pairs of candidate contact nodes.

3.1.2. *Normal contact conditions for the contact state: separate, stick and slide state.* For separate state, the normal contact conditions are

$$s_n^i = 0,\ \Delta u_n^i \geq 0. \tag{3.7}$$

For stick and slide state, the normal contact conditions are

$$s_n^i \geq 0,\ \Delta u_n^i = 0. \tag{3.8}$$

The normal contact conditions can be written in a complementary form

$$\Delta u_n^i \geq 0,\ s_n^i \geq 0 \Leftrightarrow \Delta u_n^i s_n^i = 0.$$

3.1.3. *The Coulomb friction law in the three dimensional case.* For the stick state, it is given by

$$\sqrt{(s_t^i)^2 + (s_\tau^i)^2} \leq \mu s_n^i;\ \sqrt{(\Delta du_t^i)^2 + (\Delta du_\tau^i)^2} = 0, \tag{3.9}$$

while, for the slide state it is given by

$$\sqrt{(s_t^i)^2 + (s_\tau^i)^2} = \mu s_n^i;\ \sqrt{(\Delta du_t^i)^2 + (\Delta du_\tau^i)^2} \geq 0 \tag{3.10}$$

$$\theta_\tau = \theta_t + \pi, \tag{3.11}$$

where μ is the friction coefficient, θ_τ is the angle between the tangential contact stress and the local t-axis, and θ_t is the angle between the direction of slide and local t-axis on the contact surface.

The condition (3.11) means that the direction of the tangential force is on the same line with the slide force between contact surfaces, but opposite to each other.

3.2. **Incremental formulation of the quasistatic contact problem and its discretization.** This problem is obtained by using the backward finite difference approximation of the time derivative of (3.4).

If we set $u^k = u(x, t^k)$, $\Delta u^k = u^{k-1} - u^k$, $\Delta t^k = t^{k+1} - t^k$, $f^k = f(u^k)$, $\Delta f^k = f^{k+1} - f^k$, $k = 0, 1, \ldots, n-1$ and take $\dot{u}(t^{k+1}) = \Delta u^k / \Delta t^k$, we obtain for each moment t^k, the following quasi-variational problem from (3.4): Find Δu^k such that:

$$\begin{aligned} &a(\Delta u^k, v - \Delta u^k/\Delta t^k) + j(\Delta u^k, v) - j(u^k, \Delta u^k/\Delta t^k) \\ &\geq \Delta f^k(v - \Delta u^k/\Delta t^k) - h^k(\Delta u^k, v - \Delta u^k/\Delta t^k),\ \forall\, v \in V, \end{aligned} \tag{3.12}$$

where

$$h^k(\Delta u^k, v - \Delta u^k/\Delta t^k) = a(u^k, v - \Delta u^k/\Delta t^k) - f^k(v - \Delta u^k).$$

For each step t^k, after discretization with finite element method, it is used a Newton-Raphson method for the liniarization of the nonlinear equation:

$$K_N(du^k) = B(t^k), \tag{3.13}$$

where K_N is nonlinear stiffness matrix, $u^k = u(t^k)$ and B is the load vector. By introducing the concept of residual vector $R(u^k) = B(t^k) - K_N(du^k)$, the solution of (3.13) is the root of the equation

$$R(du^k) = 0$$

which is equivalent to

$$B^k - K_N(du^k) = 0.$$

By using Newton-Raphson method we have

$$B^k - \left[\frac{\partial K_N}{\partial du^{k+1}}\right] \delta du^{k+1} = 0, \tag{3.14}$$

and by denoting $\dfrac{\partial K_N}{\partial du^{k+1}} = K_T^k$, we obtain

$$\delta du^{k+1} = \left[K_N^k\right]^{-1} \cdot B^k \tag{3.15}$$

$$du^{k+1} = du^k + \delta du^{k+1} \tag{3.16}$$

$$u^{k+1} = u^k + du^{k+1}. \tag{3.17}$$

After discretization by the finite element method, the equilibrium of the contact system is obtained. The changes are made to some variables, for example, the contact stresses s_n, s_t, s_τ are changed into nodal contact forces P_n, P_t, P_τ.

In many cases, the potential contact region is relatively small comparing with the contact bodies. The condensed equilibrium equations, to the candidate contact nodes, involving the flexibility matrix $[F]$ in the local coordinate system can be expressed in the form:

$$\left\{\begin{array}{c} \Delta du_n \\ \Delta du_t \\ \Delta du_\tau \end{array}\right\} = [F] \left\{\begin{array}{c} dP_n \\ dP_t \\ dP_\tau \end{array}\right\} + \left\{\begin{array}{c} dq_n \\ dq_t \\ dq_\tau \end{array}\right\} \tag{3.18}$$

where dq_j $(j = n, t, \tau)$ are incremental relative displacements generated by the incremental load. The relationship between the total and incremental contact forces (displacements) is a follows:

$$\begin{array}{ll} P_j & = P_j^0 + dP_j, \\ \Delta u_j & = \Delta u_j + \Delta du_j, \ (j = n, t, \tau) \end{array} \tag{3.19}$$

where $P_j(\Delta u_j)$, $dP_j(\Delta du_j)$ and $P_j^0(\Delta u_j^0)$ $(j = n, t, \tau)$ denotes the components on the local j-axis of the total, incremental and initial contact forces (relative displacements), respectively, at the current load step.

On the contact surface, the norm of total tangential contact forces and the norm of incremental tangential relative displacements are denoted by P_{TAN} and Δdu_{TAN}, respectively, which are given by

$$P_{TAN} = \sqrt{(P_t)^2 + (P_\tau)^2} \qquad (3.20)$$
$$\Delta du_{TAN} = \sqrt{(\Delta du_t)^2 + (\Delta du_\tau)^2}.$$

Assume that the angle between the total tangential forces and local t-axis is θ, the relationships among P_{TAN}, P_t, P_τ and among Δdu_{TAN}, Δdu_t, Δdu_τ is:

$$\begin{aligned} P_t &= P_{TAN}\cos\theta,\ P_\tau = P_{TAN}\sin\theta, \\ \Delta du_t &= \Delta du_{TAN}\cos(\theta+\pi) \\ \Delta du_\theta &= \Delta du_{TAN}\sin(\theta+\pi). \end{aligned} \qquad (3.21)$$

The contact constraints can be expressed as non-smooth equations set in which the variables are related to the candidate contact nodes, as follows

$$\begin{aligned} f_1^i &= \min\left\{P_n^i, \Delta u_n^i\right\} = 0, \\ f_2^i &= \min\left\{\sqrt{(\Delta du_t^i)^2 + (\Delta du_\tau^i)^2}\,,\ \mu s_n^i - \sqrt{(s_t^i)^2 + (s_\tau^i)^2}\right\} = 0, \\ f_3^i &= \left|\Delta_t^i s_\tau^i - \Delta u_\tau^i s_t^i\right| + \max\{0, \Delta u_t^i s_t^i\} = 0. \end{aligned} \qquad (3.22)$$

4. Bundle methods for solution of the quasi-static contact problem

For a given load history the quasi-static problem is approximated by a sequence of incremental problems (3.12); although every problem from this sequence is a static one, it requires appropriate updating of the displacements and the loads after each increment.

Assuming that the solution before $(k+1)$th load step is known, the quasistatic frictional contact problem at the $(k+1)$th load step can then be described by incremental equilibrium equation.

In this algorithm are used two kinds of iterative procedures: Newton-Raphson method for the liniarization of the quasistatic contact problem and the bundle methods for solving of the nonsmooth equations set which describe the contact conditions.

4.1. **Bundle methods.** The contact constraints can be expressed as non-smooth equations set in which the variables are related to the candidate contact nodes:

$$F(P_n, P_t, P_\tau) \equiv \left\{f_1^1, f_1^2, \ldots, f_1^{NC}, f_2^1, f_2^2, \ldots, f_2^{NC}, f_3^1, f_3^2, \ldots, f_3^{NC}\right\}^T. \qquad (4.23)$$

Our aim is to give an algorithm for nonsmooth minimization, based on generalized derivatives defined by Clarke, consisting in solving the minimum problem

$$\min\{F(P_n, P_t, P_\tau)\}. \qquad (4.24)$$

To this end we use the bundle methods and property c) from Theorem 2.1, by previously computing $F(P_n, P_t, P_\tau)$ from (4.23) and $F'(P_n, P_t, P_\tau; \mathbf{h}) \in \partial F(P_n, P_t, P_\tau)$. It is known that the gradient $\nabla F(x)$ is a ascent direction, consequently $-\nabla F(x)$ is a descent direction and, moreover, this is the direction of the steepest descent. A steepest descent direction $\mathbf{h}$ has to satisfy

$$\min\{F'(P_n, P_t, P_\tau; \mathbf{h})\} \text{ subject to } \frac{1}{2}\|\mathbf{h}\| = 1, \qquad (4.25)$$

where the second condition only serves the purpose of normalization. By changing this normalization constrains to an inequality and applying duality theory, we find that this problem is equivalent to

$$\mathbf{h} = -\arg \min_{V \in \partial F(P_n, P_t, P_\tau)} \frac{1}{2} \|V\|_2^2, \tag{4.26}$$

where $V \equiv F'(P_n, P_t, P_\tau; \mathbf{h})$.

To solve either (4.25) or (4.26) one has to obtain the whole subdifferential or, at least a good approximation of (4.26). These are called 'bundle methods'.

The basic idea is: assume we have already obtained a set of subgradients $\{V_1, V_2, \ldots, V_k\}$ at x, which satisfy $V_j \in \partial F(x), j = 1, 2, \ldots, k$, where $V_j \equiv F'(P_n^j, P_t^j, P_\tau^j; \mathbf{h})$ and $x \equiv \{P_n, P_t, P_\tau\}$. From this set we can obtain an approximation of the subdifferential by building the convex polyhedron $W_k = co\big\{V_1, V_2, \ldots, V_k\big\}$. A partial direction of descent h_k has to satisfy

$$V_j^T \mathbf{h} < 0 \quad \text{for} \quad j = 1, 2, \ldots, k \tag{4.27}$$

or equivalenty

$$V^T \mathbf{h} < 0 \;\; \text{for} \;\; V \in W_k \tag{4.28}$$

since V is a convex combination of $V_1, V_2, \cdots, V_k$. To make sure that h_k is a descent direction, we have to find out whether $F'(P_n^j, P_t^j, P_\tau^j; \mathbf{h}) < 0$. For this, check whether $F(x + th_k) < F(x)$ for $t \to 0$. If h_k is a descent direction, then we are done. If not, we have to obtain a better approximation of the subdifferential, by adding a new subgradient to W_k. If we define $V^* = \lim JF(x + th_k)$, $t \to 0$, where $JF(x + th_k) \in \partial F(x + th_k)$, then the inequality $V^{*T} h_k \geq 0$ holds. But h_k has been selected to satisfy (4.27) and, because (4.27) and (4.28) are equivalent, we may deduce that $V^* \notin W_k$. This new subgradient V^* may be added to the bundle to obtain a better approximation of the subdifferential. The sequence of polyhedra fulfils $W_1 \subset W_2 \subset \cdots \subset W_k \subset W_{k+1} \subset \partial F(x)$. To implement a bundle algorithm, we must be able to build an *ε-subdifferential*, $\partial F_\varepsilon(x) = \cup \partial F(x'),\ x' \in N_\varepsilon$, where N_ε is an ε-neighborhood of x whose size decreases as the algorithm proceeds closer to minimum.

4.2. **Solution procedure.** The tangential stiffness K_T, at the beginning of current load step R_T^0 is taken as that at end of the previous load step, and assume that the solution before the $(m+1)$th load step is known. Iterations will be carried out until equilibrium is achieved. The solution procedure at the $(m+1)$th load step is described below.

Step 1. Solve equations (3.14) for δdu^k, and obtain increment and total displacement. For the initial iteration on equilibrium take $du^0 = 0$ and u^0 as the convergence value from the end of previous load step.

Step 2. Solve the contact of a ith iteration subproblem by using the non-smooth nonlinear equation method to obtain the contact forces.

The contact flexibility matrix $F(d)$ from the condensed equilibrium equations involving the flexibility matrix in the local coordinated system can be computed using the current tangential stiffness matrix K_T^i. The solution procedure of system (3.22) for contact of i-th iteration subproblem, can be described as follows:

set the counter $m = 1$, and perform iterations by incrementing m at the end of last equilibrium iteration in the current load step for a tolerance $\delta > 0$.

Step 2.1. For the case when all candidate contact pairs are in stick state we obtain dP_n^1, dP_t^1, dP_τ^1 from (3.19) and (3.18).

Step 2.2. The initial position $\mathbf{x}_0$ is given. Compute $F(\mathbf{x}_0)$ and $V_0 \in \partial F(\mathbf{x}_0)$. Set $W_0 = \{V_0\}$ and $k = 0$.

Step 2.2.1. Set $l = 0$.

Step 2.2.2. Solve $h_l = -\arg min \|V\|_2^2$ to obtain a search direction h_l. If $\|h_l\| \leq \delta$ then go to Step 2.4.

Step 2.2.3. Perform a line - search along h_l, and try to find a t such that $F(\mathbf{x}_k + th_l) < F(\mathbf{x}_k) - \varepsilon$. If t is valid go to Step 2.2.5.

Step 2.2.4. Get a new subgradient $V_{l+1} \in \partial_\varepsilon F(\mathbf{x}_k)$. Add V_{l+1} to bundle $W_{l+1} = \{V_1, V_2, \cdots, V_l, -h, V_{l+1}\}$. If necessary remove to subgradients from the set $\{V_1, V_2, \cdots, V_l\}$, increase l and go to Step 2.2.2.

Step 2.2.5. Set $\mathbf{x}_{k+1} = \mathbf{x}_k + th_l$. Clear the bundle and set $W_0 = \{V_0\}$ with $V_0 = \partial F(\mathbf{x}_{k+1})$. Increase k and go to Step 2.2.1.

Step 2.3. Compute $V(\mathbf{x}_k) \equiv V(P_n^k, P_t^k, P_\tau^k)$ and $\mathbf{x}_{k+1} \equiv (P_n^{k+1}, P_t^{k+1}, P_\tau^{k+1})$. Set $k = k + 1$ and proceed to step 2.2.2.

Step 2.4. Convert the incremental contact forces in the local coordinate system to those in the global coordinate system denoted as dP^m, and the residual forces $B^m = B^{m-1} + dP^m$. Solve equation (3.14) for δdu^{m+1} and then obtain du^{m+1} and u^{m+1}. On the each increment load, we compute a static contact problem and thus the residual force can be updated.

If the norm of the vector of residual forces B^m is less than a tolerance δ, equilibrium is achieved and the procedure is terminated. Let $m = m + 1$, compute tangential stiffness K_T^m and return to Step 1.

In order to prove the global convergence of this algorithm we must assume that the function F is locally Lipschitz continuous and the level set $\{\mathbf{x} \in \mathbb{R}^n | F(\mathbf{x}) \leq F(\mathbf{x}_1)\}$ is bounded for every starting point $(\mathbf{x}_1) \in \mathbb{R}^n$. Furthermore, we assume that each execution of the line search procedure is finite (the function F is assumed to be semi-smooth). Because the function F is not supposed to be convex, this algorithm either terminates at a stationary point or generates an infinite sequence $(\mathbf{x}_k)$ for which accumulation points are stationary for F, as shown by the following theorem

Theorem 4.2. ([18]) *If the level set $\{\boldsymbol{x} \in \mathbb{R}^n | F(\boldsymbol{x}) \leq F(\boldsymbol{x}_1)\}$ is bounded, then every accumulation point of the sequence $(\boldsymbol{x}_k)$ is stationary for F.*

Numerical example and concluding remarks. This example [19] has the advantage of being very elementary and that of giving different contact areas for given loading and coefficient of friction. Thus, we have open (or non-contact) area AB, sliding area BC and sticking area CD.

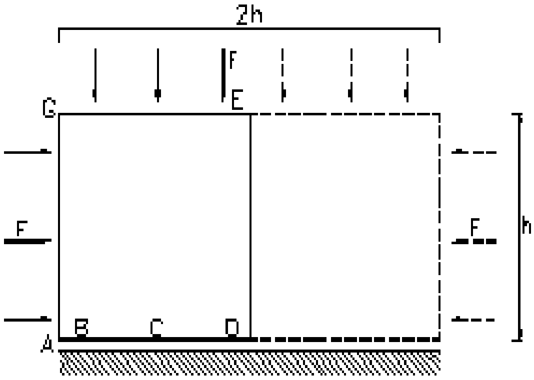

Figure 1. The geometry ($h = 40$ mm) and the loading

μ	F daN/mm^2	f daN/mm^2	Non-contact area AB mm	Sliding area BC mm	Stick area CD mm
1	10	-5	3.75	20	16.25
1	15	-5	5	20.75	7.5
0.2	10	-5	0	40	0
0.2	10	-15	0	22.5	17.5
0.2	10	-25	0	5	35

Table 1. Contact states for different loading cases

- The non-smooth bundle methods are based on generalized derivatives and is used to solve the minimum problem (4.24).
- The non-smooth bundle methods have the advantage of the small number of iterative steps and of the fast rate of convergence.

Acknowledgments. The research was supported by the CEEX Grant No. 2-CEEX-06-11-96/19.09.2006 of the Romanian Ministry of Education and Research.

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