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Simple paths of maximum length in star graphs

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ABSTRACT.
The star graph has been known as an attractive model for interconnection networks due to its topological properties, capacity to simulate other basic network topologies and possibility to design efficient communication algorithms. In this paper we study the possibility to embed an array between any two nodes of a star graph and then we show that between any two nodes of a n-star graph there is a simple path that contains at least $n! - 2$ nodes.

1. INTRODUCTION

An important aspect of designing a distributed system regards the design of the communication subsystem that means the design of its interconnection network. The design of the interconnection network suppose a compromise to achieve some objectives as: high transfer rate, small communication delay, simplicity, scalability, optimal rapport cost/performance.

An interconnection network can be modeled by a finite graph $G = (V, E)$, with V the set of vertices and E the set of edges. The vertices of the graph represent the nodes of the network, that is processing elements, and the edges correspond to the communication links. If the communication between processors is unidirectional then the graph is a directed graph, otherwise the graph is undirected. Two processors connected by a link in the network are called neighbours. The interconnection graph of the network is referred as the network topology.

In this paper we use the terms vertex and node, edge and link respectively array and path interchangeably.

A good model for interconnection networks must have some properties as: small degree (limit due to technical reasons), small diameter and average distance between nodes (small communication delay), maximum connectivity (optimal fault tolerance), embedding properties (efficient simulation of other networks) and modular structure (recursive scalability). A set of topologies that allow implementation of good communication algorithms and efficient simulation of other networks is the set of Cayley graphs. The properties of Cayley graphs are studied in [1], [2], [4]. The well known hypercubes, torus, butterfly, star and pancake graphs are members of the class of Cayley graphs and together with Fibonacci and extended Fibonacci cubes are called hypercube-like topologies.

The star graph topology was introduced by Akers in [2] together with the pancake graph, as interconnection topologies using as mathematical model the Cayley graph and possess the properties of Cayley graphs. In [2], [3], [4] topological

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properties of the star graph are studied, and optimal communication algorithms are given. Embeddings of paths, trees and hypercubes in star graphs are given in [2], [3], [5].

2. Preliminaries

Let $S_n = \left\{(s_1 \dots s_n) | s_i \in \{1, 2, \dots, n\},\, s_i \neq s_j \text{ for } i \neq j,\, i, j = \overline{1, n}\right\}$ be the set of permutations of $\{1, 2, \dots, n\}$ and $S = \{(i2 \dots (i-1)1(i+1) \dots n) | i = \overline{2, n}\} = \{g_i, i = \overline{2, n}\}$ be the set of $n-1$ transpositions of the first and any other element in the permutation, $g_i = \langle 1, i \rangle$, $i = \overline{2, n}$. The set S is a generating set on the permutations group $(S_n, \cdot)$ and its elements g_i, $i = \overline{2, n}$ are called generators.
The n-star graph ST_n is defined as the Cayley graph on $(S_n, \cdot)$ with the generating set S as:

Definition 2.1. The n-star graph $ST_n = (V, E)$, $n \geq 2$ is the graph with the vertices set $V = S_n$, the set of permutations of elements $\{1, 2, \dots, n\}$ and the edges set $E = \{(u, v) | u, v \in V,\ \exists\, i \in \{2, 3, \dots, n\} \text{ such that } v = u \cdot g_i\}$.

The n-star graph ST_n has $n!$ vertices labelled with the $n!$ permutations of elements $\{1, 2, \dots, n\}$ and there is an edge between two vertices u and $v \in S_n$ if their labels differ in only two positions i and j, where $i = 1$ and $j \in \{2, \dots, n\}$. In this case $u = v \cdot g_j$, $v = u \cdot g_j$ and we say that vertices u and v are connected along dimension j. The star graph is an undirected graph because if $v = u \cdot g_j$ then $u = v \cdot g_j$, $j = \overline{2, n}$.

The star graph of order 2, ST_2, 3, ST_3 and 4, ST_4 are represented in fig. 1.

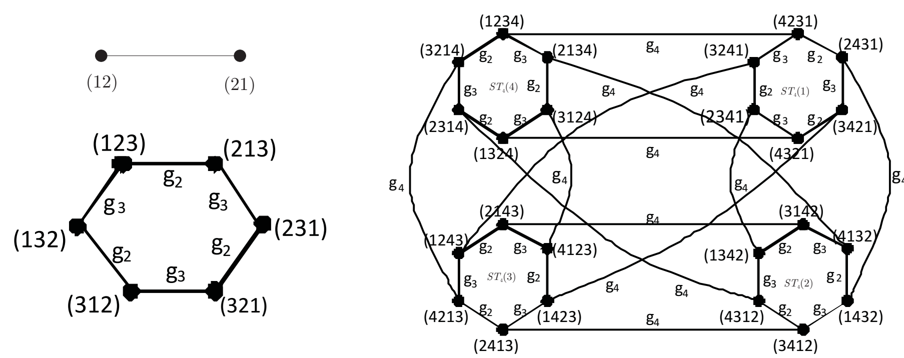

Fig. 1. Star graph of order 2, ST_2, 3, ST_3 and 4, ST_4

The n-star graph ST_n is symmetric, regular with degree $n-1$ and has $\dfrac{(n-1)n!}{2}$ edges. Its diameter is $\left\lfloor \dfrac{3(n-1)}{2} \right\rfloor$ subalgorithmic in the number of its vertices.
By fixing each different symbol $i \in \{1, 2, \dots, n\}$ in one particular position p, $p \in \{2, \dots, n\}$, n graphs denoted by $ST_{n,p}(j)$, $j = \overline{1, n}$ are obtained. Each of these graphs is isomorphic to ST_{n-1}, so we can say that one n-star graph ST_n can be recursively decomposed in n substars of order $(n-1)$, $ST_{n,p}(j)$, $j = \overline{1, n}$.

If the fixed position is the last of the permutation, $p = n$, then we denote the $(n-1)$-substars by $ST_n(j)$, $j = \overline{1,n}$.

In fig. 1, the decomposition of the 4-star ST_4 in 4 substars $ST_4(1)$, $ST_4(2)$, $ST_4(3)$ and $ST_4(4)$ is given.

3. Main results

One of the most important properties studied for the hypercube-like topologies is the property to contain hamiltonian cycles. According to Lovasz conjecture, all the Cayley graphs are hamiltonian. The extended Fibonacci cubes are hamiltonian [10] and the Fibonacci cubes with an even number of nodes are hamiltonian [9]. Methods and algorithms for construction of hamiltonian cycles in star graphs are given in [5]. We use this property of the star graph to show that any two nodes of a star graph can be connected using a path that contains at least $n! - 2$ nodes.

A basic result asserts that if there is a hamiltonian cycle in a star graph, this cycle is not unique.

Let $ST_n = (V_n, E_n)$ be a n-star graph and u and v two vertices of the graph.

Lemma 3.1. [Jwo] *If H is a hamiltonian cycle in ST_n and $(u, v) \in E_n$ is an edge of the star graph, then there is a hamiltonian cycle H_1 that contains the edge (u, v).*

Consequently, there is a hamiltonian path between any two neighbours in a star graph.

We give first a method to connect the n substars of a n-star graph. For simplicity we consider the n-star graph ST_n decomposed in n substars $ST_n(i)$, $i = \overline{1,n}$.

Theorem 3.1. *For any permutation $(k_1 k_2 \dots k_n) \in S_n$, the $(n-1)$-substars $ST_n(k_1)$, $ST_n(k_2), \dots, ST_n(k_n)$ of a star graph ST_n can be successively connected using the edges $(u^{k_i}, v^{k_{i+1}}) \in E_n$, $u^{k_i} \in ST_n(k_i)$, $v^{k_{i+1}} \in ST_n(k_{i+1})$, $1 \le i \le n-1$ such that v^{k_i} and u^{k_i} are connected through a hamiltonian path H_{k_i} in $ST_n(k_i)$, $2 \le i \le n-1$, and $u^{k_1} \in ST_n(k_1)$, $v^{k_n} \in ST_n(k_n)$.*

Proof. The substar $ST_n(k_1)$ contains $(n-2)!$ vertices with symbol k_2 on the first position of their label and we choose $u^1 = (k_2 u_2^1 \dots u_{n-1}^1 k_1) \in ST_n(k_1)$.
We consider then in $ST_n(k_2)$ the node

$$v^{k_2} = u^{k_1} \cdot g_n = (k_1 u_2^{k_1} \dots u_{n-1}^{k_1} k_2) = (v_1^{k_2} v_2^{k_2} \dots v_n^{k_2}) \in ST_n(k_2)$$

and the edge $(u^{k_1}, v^{k_2}) \in E_n$ connects the substars $ST_n(k_1)$ and $ST_n(k_2)$. From the $n-2$ neighbours of v^{k_2} in $ST_n(k_2)$ we choose u^{k_2} that has k_3 on its first position, $u^{k_2} = (k_3 v_2^{k_2} \dots v_{n-1}^{k_2} k_2)$. According to Lemma 3.1 there is a hamiltonian path H_{k_2} between v^{k_2} and u^{k_2} in $ST_n(k_2)$, $H_{k_2} = \{v^{k_2}, \dots, u^{k_2}\}$.

Using this method, we suppose the vertex $v^{k_i} \in ST_n(k_i)$, $i \le n-2$ choosen as the neighbour along dimension n of the node $u^{k_{i-1}} \in ST_n(k_{i-1})$. From the $n-2$ neighbours of $v^{k_i} \in ST_n(k_i)$ we choose u^{k_i} to be the neighbour with k_{i+1} on its first position, $u^{k_i} = (k_{i+1} v_2^{k_i} \dots v_{n-1}^{k_i} k_i)$. Between v^{k_i} and u^{k_i} there is an edge in $ST_n(k_i)$ and according to Lemma 3.1 there is a hamiltonian path $H_{k_i} =$

$\{v^{k_i}, \ldots, u^{k_i}\}$ in $ST_n(k_i)$. The node $v^{k_{i+1}}$ is the neighbour of u^{k_i} along dimension n, $v^{k_{i+1}} = u^{k_i} \cdot g_n \in ST_n(k_{i+1})$.

Repeating this method, we choose $v^{k_{n-1}} \in ST_n(k_{n-1}))$ the neighbour along dimension n of the node $u^{k_{n-2}} \in ST_n(k_{n-2})$. From the $n-2$ neighbours of $v^{k_{n-1}}$ in $ST_n(k_{n-1})$ we choose $u^{k_{n-1}}$ the neighbour with k_n on its first position, and there is a hamiltonian path $H_{k_{n-1}} = \{v^{k_{n-1}}, \ldots, u^{k_{n-1}}\}$ in $ST_n(k_{n-1})$ between nodes $v^{k_{n-1}}$ and $u^{k_{n-1}}$. The node v^{k_n} is the neighbour of $u^{k_{n-1}}$ along dimension n, $v^{k_n} = u^{k_{n-1}} \cdot g_n \in ST_n(k_n)$ and

$$L = \left\{u^{k_1}, v^{k_2}, \ldots, u^{k_2}, v^{k_3}, \ldots, u^{k_3}, \ldots, v^{k_{n-1}}, \ldots, u^{k_{n-1}}, v^{k_n}\right\}$$

connects the node $u^{k_1} \in ST_n(k_1)$ to $v^{k_n} \in ST_n(k_n)$ through a path that contains all nodes in $ST_n(k_i)$, $i = \overline{2, n-1}$. The path is represented in fig. 2. □

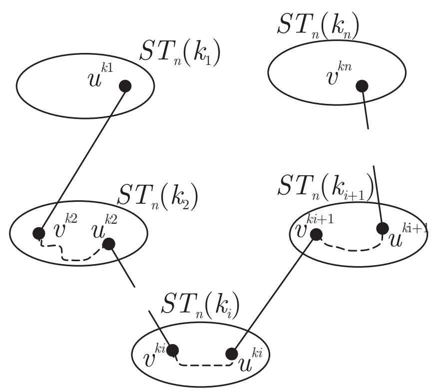

Fig. 2. Connection path between two substars in a star graph ST_n

According to Theorem 3.1, for two given $(n-1)$-substars $ST_n(i)$ and $ST_n(j)$, $i \neq j \in \{1, 2, \ldots, n\}$, there are several paths that connect them and contain all the nodes in the other $n-2$ substars. There are $(n-2)!$ ways to choose node u^1 in $ST_n(i)$ and there are $(n-2)!$ ways to choose the order of the order $n-2$ substars, so there are $[(n-2)!]^2$ different paths with the property in Theorem 3.1 that connect the $(n-1)$ substars $ST_n(i)$ and $ST_n(j)$.

Using Theorem 3.1 we give the main result of this paper.

Theorem 3.2. *For any two nodes of a n-star graph ST_n there is a simple path with at least $n! - 2$ nodes that connects them.*

Proof. We use the induction to prove this lemma.

For $n = 2$, $u = (12)$, $v = (21)$ and the path is $L = \{(12), (21)\}$ and contains $2! = 2$ nodes.

For $n = 3$, the paths between the identity node and any other nodes are are

$$(123) \xrightarrow{g_2} (213) \xrightarrow{g_3} (312) \xrightarrow{g_2} (132) \xrightarrow{g_3} (231) \xrightarrow{g_2} (321)$$
$$(123) \xrightarrow{g_2} (213) \xrightarrow{g_3} (312) \xrightarrow{g_2} (132) \xrightarrow{g_3} (231)$$
$$(123) \xrightarrow{g_3} (321) \xrightarrow{g_2} (231) \xrightarrow{g_3} (132) \xrightarrow{g_2} (312) \xrightarrow{g_3} (213)$$
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$$(123) \xrightarrow{g_3} (321) \xrightarrow{g_2} (231) \xrightarrow{g_3} (132)$$

Due to the symmetry of the star graph the other paths can be written in the same way.

We suppose that any two nodes in a $(n-1)$-star can be connected through a simple path that contains at least $(n-1)! - 2$ nodes.

Let u, v be two nodes of the n-star graph. Due to the symmetry of the n-star graph we can consider $u \in ST_n(i)$ and $v \in ST_n(j)$, $i \neq j \in \{1, 2, \ldots, n\}$, $u = (u_1 u_2 \ldots u_{n-1} u_n) = (u_1 u_2 \ldots u_{n-1} i)$, $v = (v_1 v_2 \ldots v_{n-1} v_n) = (v_1 \ldots v_{n-1} j)$.

We construct a path that will connect the nodes u and v using all nodes in $ST_n(k_2), ST_n(k_3), \ldots, ST_n(k_{n-1})$, $k_2, \ldots, k_{n-1} \in \{1, 2, \ldots, n\} \setminus \{i, j\}$ and $k_{n-1} \neq v_1$, $k_2 \neq u_1$. We choose u^{k_1} as the neighbour of u in $ST_n(i)$ with k_2 on its first position. There is a hamiltonian path between u and u^{k_1}, $H_{k_1} = \{u, \ldots, u^{k_1}\}$. Starting from u^{k_1}, according to Lemma 3.1 there is a simple path that contains all nodes in $ST_n(k_2), \ldots, ST_n(k_{n-1})$,

$$L = \left\{u^{k_1}, v^{k_2}, \ldots, u^{k_2}, v^{k_3}, \ldots, u^{k_3}, \ldots, v^{k_{n-1}}, \ldots, u^{k_{n-1}}\right\},$$
$$u^{k_{n-1}} = \left(k_n u_2^{k_{n-1}} \ldots u_{n-1}^{k_{n-1}} k_{n-1}\right) \in ST_n(k_{n-1}).$$

The neighbour along dimension n of $u^{k_{n-1}}$ is $v^{k_n} = u^{k_{n-1}} \cdot g_n \in ST_n(k_n)$ and $v^{k_n} \neq v$. According to induction hypothesis, between v^{k_n} and v there is a simple path L_{k_n} that contains at least $(n-1)! - 2$ nodes in $ST_n(k_n)$. The path obtained by concatenation of H_{k_1}, L and L_{k_n} contains $(n-1)! + (n-2) \cdot (n-1)! + (n-1)! - 2 = n! - 2$ nodes and connects u and v in ST_n. The path is illustrated in fig. 3. □

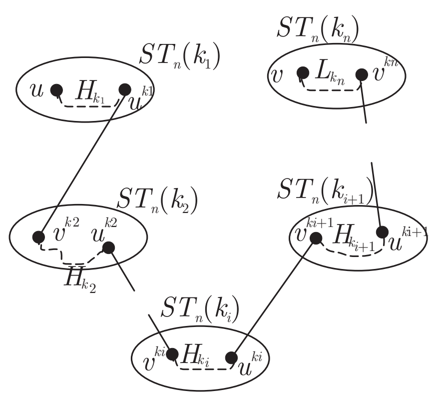

Fig. 3. Construction of a simple path with at least $n! - 2$ nodes between 2 nodes of a ST_n

There are several ways to choose the order of the substars $ST_n(k_2)$, ...,$ST_n(k_{n-1})$, so there are several simple paths with at least $n! - 2$ nodes that connect any two given nodes in the n-star graph ST_n. This property shows that in case of the existence of faulty links in the interconnection network, there is still the possibility to connect almost all nodes using a simple path.

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