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Fixed points for weakly compatible mappings satisfying an implicit relation in partially ordered metric spaces

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ABSTRACT.
Let $(X, d, \preceq)$ be a partially ordered metric space. Let F, G be two set valued mappings and f, g two single valued mappings on X. We obtained sufficient conditions for existence of common fixed point of F, G, f and g satisfying an implicit relation in X.

1. INTRODUCTION AND PRELIMINARIES

Let (X, d) be a metric space and $B(X)$ be the class of all non-empty bounded subsets of X. For $A, B \in B(X)$, let

$$\delta(A, B) := \sup\{d(a, b) : \ a \in A, \ b \in B\},$$

and

$$dist(a, B) := \inf\{d(a, b) : \ a \in A, \ b \in B\}.$$

If $A = \{a\}$, then we write $\delta(A, B) = \delta(a, B)$ and $\delta(A, B) = 0$ if and only if $A = B = \{x\}$. Also note that $dist(A, B) \leq \delta(A, B)$.

Let $F : X \to X$ be a set valued mapping i.e., $X \ni x \mapsto F(x)$ is a subset of X.
A point $x \in X$ is said to be a *fixed point* of the set valued mapping F if $x \in F(x)$.

Jungck [15] introduced the concept of compatible mappings which is a generalization of commuting mappings. Afterward Jungck and Rhoades [16] defined weakly compatible mappings and showed that compatible mappings are weakly compatible but converse need not be true. Many fixed point results have been obtained for compatible and weakly compatible mappings, see for instance [5, 9, 15, 7, 16] and reference cited therein.

Definition 1.1. [16] Two mappings $f : X \to X$ and $F : X \to B(X)$ are *weakly compatible* if they commute at their coincident points, that is if $f(x) \in F(x)$ then we have $Ff(x) = fF(x)$.

Definition 1.2. A *partial ordered set* consists of a set X and a binary relation $\preceq$ on X which satisfies the following conditions:

i) $x \preceq x$ (reflexivity);
ii) if $x \preceq y$ and $y \preceq x$ then $x = y$ (antisymmetry);
iii) if $x \preceq y$ and $y \preceq z$ then $x \preceq z$ (transitivity);

for all $x, \ y$ and z in X.

A set with a partial order $\preceq$ is called a *partially ordered set*.

Let $(X, \preceq)$ be a partially ordered set and $x, y \in X$. Elements x and y are said to be *comparable elements* of X if either $x \preceq y$ or $x \preceq y$.

Implicit relations in metric spaces have been considered by several authors in connection with solving nonlinear functional equations (see for instance [2, 3, 4, 28] and reference cited therein).

Let R_+ be the set of nonnegative real numbers and $\mathcal{T}$ be the set of continuous real valued functions $T : R_+^5 \to R$ satisfying the following conditions:

$\mathcal{T}_1 : T(t_1, t_2, ..., t_5)$ is non-decreasing in t_1 and non-increasing in $t_2, ..., t_5$.

$\mathcal{T}_2$: there exists $h \in (0, 1)$ such that

$$T(u, v, v, u, v + u) \leq 0,$$

or

$$T(u, v, u, v, u + v) \leq 0,$$

implies

$$u \leq hv.$$

$\mathcal{T}_3 : T(u, 0, 0, u, u) > 0$, $T(u, 0, u, 0, u) > 0$ and $T(u, u, 0, 0, 2u) > 0$, for all $u > 0$.

Sedghi and Altun in [28], obtained the following useful fixed point theorem for mappings satisfying an implicit relation.

Theorem 1.1. [28] *Let (X, d) be a complete metric space. Let $f, g : X \to X$ and $F, G : X \to B(X)$ be such that the following conditions are satisfied:*

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(1) $F(x) \subseteq g(X)$, $G(x) \subseteq f(X)$, *for each* $x \in X$.
(2) The pair (F, f) *and* (G, g) *are weakly compatible.*
(3) $g(X)$ *or* $f(X)$ *is closed.*
(4) $T(\delta(F(x), G(y)), d(f(x), g(y)), dist(f(x), F(x)), dist(g(y), G(y)),$
$$dist(f(x), G(y)) + dist(g(y), F(x))) \leq 0,$$
for all x, y *in* X *and for some* $T \in \mathcal{T}$.

Then there exists a unique $p \in X$ *with* $\{p\} = F(p) = \{f(p)\} = \{g(p)\} = G(p)$.

Existence of fixed point in partially ordered metric spaces has been recently considered in [22, 23, 20, 27, 24, 10, 25, 26, 1, 8, 19, 21, 18, 14, 6]. It is of interest to determine the existence of a fixed point in such a setting. This trend was initiated by Ran and Reurings in [27] where they extended the Banach contraction principle [17], in partially ordered sets with some application to linear and nonlinear matrix equations. Ran and Reurings [27] proved the following result.

Theorem 1.2. [27] *Let* $(X, \preceq)$ *be a partially ordered set such that for every pair* $x, y \in X$ *has an upper and lower bound. Let* d *be a metric on* X *such that* (X, d) *is a complete metric space. Let* $f : X \to X$ *be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:*

(1) there exists $\kappa \in (0, 1)$ *with*
$$d(f(x), f(y)) \leq \kappa d(x, y) \text{ for all } x \preceq y.$$
(2) there exists $x_0 \in X$ *with* $x_0 \preceq f(x_0)$ *or* $f(x_0) \preceq x_0$.

Then f *is a Picard Operator (PO), that is* f *has a unique fixed point* $x^* \in X$ *and for each* $x \in X$,
$$\lim_{n\to\infty} f^n(x) = x^*.$$

Theorem 1.2 was further extended and refined in [22, 23, 20, 24, 25, 19, 21, 14, 6, 18, 19, 21, 8, 18]. These results are hybrid of the two fundamental classical theorems; Banach's fixed point theorem [17] and Tarski's fixed point theorem [29, 11, 13]. Our aim in this paper is to obtain sufficient conditions for existence of common fixed point in a partially ordered metric space for two pairs of weakly compatible mapping satisfying an implicit relation. Our result generalized several known results.

2. Main Results

Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space.

We begin this section with the following theorem that gives the existence of a fixed point (not necessarily unique) in partially ordered metric space X for the set valued mappings and single valued mapping satisfying an implicit relation.

Theorem 2.3. *Let* $f, g : X \to X$ *and* $F, G : X \to B(X)$ *be such that the following conditions are satisfied:*

(1) $F(x) \subseteq g(X)$, $G(x) \subseteq f(X)$, *for each* $x \in X$.
(2) If $g(u) \in F(x)$ *then* $u \preceq x$, *if* $f(w) \in G(x)$ *then* $w \preceq x$.
(3) The pair (F, f) *and* (G, g) *are weakly compatible.*
(4) $g(X)$ *is closed and if* $y_n \in F(x_n)$ *be such that* $y_n \to y = g(v) \in g(X)$ *then* $x_n \preceq v$ *also* $x_n \preceq y$ *for all* n.
or
$f(X)$ *is closed and if* $y_n \in G(x_n)$ *be such that* $y_n \to y = f(v) \in f(X)$ *then* $x_n \preceq v$ *also* $x_n \preceq y$ *for all* n.
(5) $T(\delta(F(x), G(y)), d(f(x), g(y)), dist(f(x), F(x)), dist(g(y), G(y)),$
$$dist(f(x), G(y)) + dist(g(y), F(x))) \leq 0,$$
for all comparable elements x, y *of* X *and for some* $T \in \mathcal{T}$.

Then there exists $p \in X$ *(not necessarily unique) with* $\{p\} = F(p) = \{f(p)\} = \{g(p)\} = G(p)$.

Proof. Let $x_0 \in X$, then from assumptions 1 and 2, there exists $x_1 \in X$ such that
$$y_0 = g(x_1) \in F(x_0) \text{ with } x_0 \preceq x_1.$$

Again from assumptions 1 and 2, for this x_1 there exists $x_2 \in X$ such that
$$y_1 = f(x_2) \in G(x_1) \text{ with } x_1 \preceq x_2.$$

Since $x_0 \preceq x_1$, therefore by using assumption 5, we have,
$$T(\delta(F(x_0), G(x_1)), d(f(x_0), g(x_1)), dist(f(x_0), F(x_0)), dist(g(x_1), G(x_1)),$$
$$dist(f(x_0), G(x_1)) + dist(g(x_1), F(x_0))) \leq 0.$$

Using the facts, $d(y_0, y_1) \leq \delta(F(x_0), G(x_1))$, $dist(f(x_0), F(x_0)) \leq d(f(x_0), y_0)$, $dist(g(x_1), G(x_1)) \leq d(y_0, y_1)$, $dist(f(x_0), G(x_1)) + dist(g(x_1), F(x_0)) \leq d(f(x_0), y_1) + d(y_0, y_0)$ and by $\mathcal{T}_1$ we have,
$$T(d(y_0, y_1), d(fx_0, y_0), d(fx_0, y_0), d(y_0, y_1), d(fx_0, y_0) + d(y_0, y_1)) \leq 0,$$
that is,
$$T(u, v, v, u, u + v) \leq 0,$$

where $u = d(y_0, y_1)$, $v = d(f(x_0), y_0)$.

Next by using $\mathcal{T}_2$, we have ($u \leq hv$),

$$d(y_0, y_1) \leq hd(f(x_0), y_0). \tag{2.1}$$

Now for this x_2, we have the existence of x_3 in X such that

$$y_2 = g(x_3) \in F(x_2) \text{ with } x_2 \preceq x_3.$$

Again since $x_1 \preceq x_2$, therefore by assumption 5, we have,

$$T(\delta(F(x_2), G(x_1)), d(f(x_2), g(x_1)), dist(f(x_2), F(x_2)), dist(g(x_1), G(x_1)),$$
$$dist(f(x_2), G(x_1)) + dist(g(x_1), F(x_2))) \leq 0.$$

By using $\mathcal{T}_1$ we have,

$$T(d(y_2, y_1), d(y_1, y_0), d(y_1, y_2), d(y_0, y_1), d(y_0, y_1) + d(y_1, y_2)) \leq 0,$$

that is

$$T(u, v, u, v, u + v) \leq 0,$$

where $u = d(y_1, y_2)$, $v = d(y_0, y_1)$.

Next, by using $\mathcal{T}_2$ we have

$$d(y_1, y_2) \leq hd(y_0, y_1). \tag{2.2}$$

Continuing in this manner we can define a sequence $\{x_n\}$ with $x_n \preceq x_{n+1}$ such that

$$y_{2n} = g(x_{2n+1}) \in F(x_{2n}),$$

$$y_{2n+1} = f(x_{2n+2}) \in G(x_{2n+1}),$$

for $n = 0, 1, 2 \ldots$.

From assumption 5, we have

$$T(\delta(F(x_{2n}), G(x_{2n+1})), d(f(x_{2n}), g(x_{2n+1})), dist(f(x_{2n}), F(x_{2n})),$$
$$dist(g(x_{2n+1}), G(x_{2n+1})), dist(f(x_{2n}), G(x_{2n+1})) + dist(g(x_{2n+1}), F(x_{2n}))) \leq 0,$$

and by $\mathcal{T}_1$,

$$T(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}),$$
$$d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})) \leq 0.$$

That is

$$T(u, v, v, u, u + v) \leq 0,$$

where $u = d(y_{2n}, y_{2n+1})$, $v = d(y_{2n-1}, y_{2n})$.
Next, by using $\mathcal{T}_2$, there exists $h \in (0, 1)$ such that

$$d(y_{2n}, y_{2n+1}) \leq hd(y_{2n-1}, y_{2n}). \tag{2.3}$$

Therefore, we have

$$d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n) \leq h^2 d(y_{n-2}, y_{n-1}) \leq \ldots \leq h^n d(y_0, y_1).$$

Next we will show that (y_n) is a Cauchy sequence in X. Let $m > n$. Then

$$\begin{aligned}
d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \ldots. + d(y_{m-1}, y_m) \\
&\leq [h^n + h^{n+1} + h^{n+2} + \ldots. + h^{m-1}] d(y_0, y_1) \\
&= h^n [1 + h + h^2 \ldots. + h^{m-n-1}] d(y_0, y_1) \\
&= h^n \, \frac{1 - h^{m-n}}{1 - h} \, d(y_0, y_1) \\
&< \frac{h^n}{1 - h} \, d(y_0, y_1),
\end{aligned}$$

because $h \in (0, 1), 1 - h^{m-n} < 1$.

Therefore $d(y_n, y_m) \to 0$ as $n \to \infty$ implies that $\{y_n\}$ is a Cauchy sequence and hence there exists some point (say) p in the complete metric space X such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} g(x_{2n+1}) = p \in \lim_{n\to\infty} F(x_{2n}),$$

and

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} f(x_{2n+2}) = p \in \lim_{n\to\infty} G(x_{2n+1}).$$

Suppose that 4 holds, then there exists $v \in X$ such that $p = g(v) \in g(X)$ with $x_{2n} \preceq p$ and $x_{2n} \preceq v$.

Now by assumption 5, we have

$$T(\delta(F(x_{2n}),G(v)),d(f(x_{2n}),g(v)),dist(f(x_{2n}),F(x_{2n})),dist(g(v),G(v)),\\ dist(f(x_{2n}),G(v))+dist(g(v),F(x_{2n})))\leq 0,$$

letting $n\to\infty$ we get

$$T(\delta(p,G(v)),d(p,p),d(p,p),dist(p,G(v)),dist(p,G(v))+d(p,p))\leq 0.$$

From $\mathcal{T}_1$,

$$T(\delta(p,G(v)),0,0,\delta(p,G(v)),\delta(p,G(v)))\leq 0,$$

that is

$$T(u,0,0,u,u)\leq 0,$$

and from $\mathcal{T}_3$, we have $u=\delta(p,G(v))=0$, which gives $G(v)=\{p\}=\{g(v)\}$.
From weak compatibility of (G,g), we have

$$G(p)=Gg(v)=gG(v)=\{g(p)\}.$$

Next, since $x_{2n}\preceq p$, by using assumption 5, we have

$$T(\delta(F(x_{2n}),G(p)),d(f(x_{2n}),g(p)),dist(f(x_{2n}),F(x_{2n})),dist(g(p),G(p)),\\ dist(f(x_{2n}),G(p))+dist(g(p),F(x_{2n})))\leq 0.$$

It further gives

$$T(d(y_{2n},g(p)),d(y_{2n-1},g(p)),d(y_{2n-1},y_{2n}),d(g(p),g(p)),\\ d(y_{2n-1},g(p))+d(g(p),y_{2n}))\leq 0.$$

Letting $n\to\infty$, we have

$$T(d(p,g(p)),d(p,g(p)),d(p,p),d(g(p),g(p)),d(p,g(p))+d(g(p),p)\leq 0.$$

Thus,

$$T(u,u,0,0,2u)\leq 0.$$

From $\mathcal{T}_3$, we have $u=d(u,g(p))=0$, it gives $g(p)=p$.
Hence

$$Gp=\{gp\}=\{p\}.$$

Since (by assumption 1) $Gp\subseteq f(X)$, there exists $w\in X$ such that

$$f(w)\in G(p)=\{p\}$$

with $w\preceq p$ (by assumption 2).

Now from assumption 5, we have

$$T(\delta(F(w),G(p)),d(f(w),g(p)),dist(f(w),F(w)),dist(g(p),G(p)),\\ dist(f(w),G(p))+dist(g(p),F(w)))\leq 0.$$

It implies,

$$T(\delta(F(w),p),d(p,p),dist(p,F(w)),d(p,p),d(p,p)+dist(p,F(w)))\leq 0,$$

and by $\mathcal{T}_1$, we have

$$T(\delta(F(w),p),0,\delta(F(w),p),0,\delta(F(w),p))\leq 0.$$

That is,

$$T(u,0,u,0,u)\leq 0,$$

and by using $\mathcal{T}_3$, we have $u=\delta(F(w),p)=0$, which gives $F(w)=\{p\}=\{f(w)\}$.
From weak compatibility of (F,f), we have

$$F(p)=Ff(w)=fF(w)=\{f(p)\}.$$

Next, since $p\preceq p$, from assumption 5,

$$T(\delta(F(p),G(p)),d(f(p),g(p)),dist(f(p),F(p)),dist(g(p),G(p)),\\ dist(f(p),G(p))+dist(g(p),F(p)))\leq 0,$$

and

$$T(d(f(p),p),d(f(p),p),d(f(p),f(p)),d(p,p),d(f(p),p)+d(p,f(p)))\leq 0.$$

That is,

$$T(u,u,0,0,2u)\leq 0.$$

and by using $\mathcal{T}_3$, we have $u=d(f(p),p)=0$, which gives $f(p)=p$.

Hence $F(p)=\{f(p)\}=\{p\}=\{g(p)\}=G(p)$. □

Remark 2.1. In the Theorem 2.3 assumptions 2 and 4, we need only comparability of the elements, there is no need of monotonicity in the terms of the sequence. That is if we replace assumption 2 in Theorem 2.3 by the condition:

if $g(u) \in F(x)$ then $u \succeq x$, if $f(w) \in G(x)$ then $w \succeq x$,

or

if $g(u) \in F(x)$ then $u \preceq x$, if $f(w) \in G(x)$ then $w \succeq x$,

or

if $g(u) \in F(x)$ then $u \succeq x$, if $f(w) \in G(x)$ then $w \preceq x$,

then the conclusion remains true. Similarly for the assumption 4.

Note that in contrast with Theorem 1.3., we require the assumption 5, only for the comparable elements of the partially ordered metric space.

Corollary 2.1. *Let $F, G : X \to B(X)$ be such that the following conditions are satisfied:*

(1) *if $u \in F(x)$ then $u \preceq x$, if $w \in G(x)$, then $w \preceq x$;*
(2) *if $\{x_n\}$ is any sequence in X whose consecutive terms are comparable such that $x_n \to x$ then $x_n \preceq x$, for all n;*
(3) $$T(\delta(F(x),G(y)), d(f(x),g(y)), dist(f(x),F(x)), dist(g(y),G(y)), dist(f(x),G(y)) + dist(g(y),F(x))) \leq 0,$$
for all comparable elements x, y of X and for some $T \in \mathcal{T}$.

Then there exists $p \in X$ (not necessarily unique) with $\{p\} = F(p) = G(p)$.

Proof. This can be proved by taking f and g as identity mappings in Theorem 2.3. □

Corollary 2.1 can also be further extended as:

Corollary 2.2. *Let $F, G : X \to B(X)$ be such that the following conditions are satisfied:*

(1) *for any $x \in X$ there exists $u \in F(x)$ with $u \preceq x$, and $w \in G(x)$ with $w \preceq x$;*
(2) *if $\{x_n\}$ is any sequence in X whose consecutive terms are comparable such that $x_n \to x$ then $x_n \preceq x$, for all n;*
(3) $$T(\delta(F(x),G(y)), d(f(x),g(y)), dist(f(x),F(x)), dist(g(y),G(y)), dist(f(x),G(y)) + dist(g(y),F(x))) \leq 0,$$
for all comparable elements x, y of X and for some $T \in \mathcal{T}$.

Then there exists $p \in X$ (not necessarily unique) with $\{p\} = F(p) = G(p)$.

Proof. Let $x_0 \in X$, then from assumptions 1, there exists $x_1 \in F(x_0)$ such that $x_0 \preceq x_1$. For this x_1 choose $x_2 \in G(x_1)$ such that $x_1 \preceq x_2$.

Since $x_0 \preceq x_1$, therefore by using assumption 3, we have,

$$T(\delta(F(x_0),G(x_1)), d(x_0,x_1), dist(x_0,F(x_0)), dist(x_1,G(x_1)), dist(x_0,G(x_1)) + dist(x_1,F(x_0))) \leq 0.$$

By $\mathcal{T}_1$, we have

$$T(d(x_1,x_2), d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), d(x_0,x_1) + d(x_1,x_2)) \leq 0,$$

that is,

$$T(u,v,v,u,u+v) \leq 0,$$

where $u = d(x_1,x_2)$, $v = d(x_0,x_1)$.

Next by using $\mathcal{T}_2$, there exists $h \in (0,1)$ such that

$$d(x_1,x_2) \leq hd(x_0,x_1). \tag{2.4}$$

Now for this x_2, we have the existence of $x_3 \in F(x_2)$ such that $x_2 \preceq x_3$.
Again since $x_1 \preceq x_2$, therefore by assumption 3, we have

$$T(\delta(F(x_2),G(x_1)), d(x_2,x_1), dist(x_2,F(x_2)), dist(x_1,G(x_1)), dist(x_2,G(x_1)) + dist(x_1,F(x_2))) \leq 0.$$

By using $\mathcal{T}_1$, we have

$$T(d(x_3,x_2), d(x_1,x_2), d(x_2,x_3), d(x_1,x_2), d(x_1,x_2) + d(x_2,x_3)) \leq 0,$$

that is,

$$T(u,v,u,v,u+v) \leq 0,$$

where $u = d(x_2,x_3)$, $v = d(x_1,x_2)$.
Next, by using $\mathcal{T}_2$ and inequality 4, we have,

$$d(x_2,x_3) \leq hd(x_1,x_2) \leq h^2 d(x_0,x_1). \tag{2.5}$$

Therefore, we have

$$d(x_n,x_{n+1}) \leq hd(x_{n-1},x_n) \leq h^2 d(x_{n-2},x_{n-1}) ... \leq h^n d(x_0,x_1).$$

Continuing in this manner we can define a sequence $\{x_n\}$ with $x_n \preceq x_{n+1}$ such that $x_{2n+1} \in F(x_{2n})$ and $x_{2n+2} \in G(x_{2n+1})$, for $n = 0, 1, 2...$.

Next we will show that (x_n) is a Cauchy sequence in X. Let $m > n$. Then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + ... + d(x_{m-1}, x_m) \\ &\leq [h^n + h^{n+1} + h^{n+2} + ... + h^{m-1}]d(x_0, x_1) \\ &= h^n[1 + h + h^2.... + h^{m-n-1}]d(x_0, x_1) \\ &= h^n \frac{1-h^{m-n}}{1-h} d(x_0, x_1) \\ &< \frac{h^n}{1-h} d(x_0, x_1), \end{aligned}$$

because $h \in (0,1), 1 - h^{m-n} < 1$.

Therefore $d(x_n, x_m) \to 0$ as $n \to \infty$ implies that (x_n) is a Cauchy sequence, and hence there exists some point (say) x in the complete metric space X, such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = x \in \lim_{n\to\infty} F(x_{2n}),$$
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+2} = x \in \lim_{n\to\infty} G(x_{2n+1}),$$

and by assumption 2, $x_n \preceq x$ for all n.

Next

$$\begin{aligned} T(\delta(F(x_{2n}), G(x)), d(x_{2n}, x), dist(x_{2n}, F(x_{2n}), dist(x, G(x)), \\ dist(x_{2n}, G(x)) + dist(x, F(x_{2n}))) \leq 0, \end{aligned}$$

which gives,

$$\begin{aligned} T(\delta(x_{2n+1}, G(x)), d(x_{2n}, x), d(x_{2n}, x_{2n+1}), dist(x, G(x)), \\ dist(x, G(x)) + d(x, x_{2n+1})) \leq 0. \end{aligned}$$

Letting $n \to \infty$ and using $\mathcal{T}_1$, we get

$$T(\delta(x, G(x)), 0, 0, \delta(x, G(x)), \delta(x, G(x))) \leq 0,$$

that is

$$T(u, 0, 0, u, u) \leq 0,$$

and from $\mathcal{T}_3$, we have $u = \delta(x, Gx) = 0$, which gives $G(x) = \{x\}$.

Similarly

$$\begin{aligned} T(\delta(F(x), G(x_{2n+1})), d(x, x_{2n+1}), dist(x, F(x)), dist(x_{2n+1}, G(x_{2n+1})), \\ dist(x, G(x_{2n+1})) + dist(x_{2n+1}, F(x))) \leq 0, \end{aligned}$$

which implies

$$\begin{aligned} T(\delta(F(x), x_{2n+2}), d(x, x_{2n+1}), dist(x, F(x)), d(x_{2n+1}, x_{2n+2}), \\ d(x, x_{2n+2}) + dist(x_{2n+1}, F(x))) \leq 0. \end{aligned}$$

Letting $n \to \infty$ and using $\mathcal{T}_1$, we get

$$T(\delta(F(x), x), 0, \delta(F(x), x), 0, \delta(F(x), x)) \leq 0,$$

that is

$$T(u, 0, u, 0, u) \leq 0,$$

and from $\mathcal{T}_3$, we have $u = \delta(F(x), x) = 0$, which gives $F(x) = \{x\}$. □

Example 2.1. Let $X = \left\{(0,0), \left(0, \frac{-1}{2}\right), \left(\frac{1}{8}, 0\right), \left(\frac{-1}{8}, \frac{1}{8}\right)\right\}$ be a subset of R^2 with usual order defined as: for $(u, v),\ (x, y) \in X$, $(u, v) \leq (x, y)$ if and only if $u \leq x, v \leq y$. Let d be a metric on X defined as:

$$d((x_1, x_2), (y_1, y_2)) := \max\{| x_1 - y_1 |, | x_2 - y_2 |\},$$

so that (X, d) is a complete metric space.

Define $F, G : X \to B(X)$ and $f, g : X \to X$ as

$$F(x, y) = \left\{\left(\frac{-1}{8}, \frac{1}{8}\right)\right\},$$

$$G(x, y) = \begin{cases} \left\{\left(-\frac{1}{8}, \frac{1}{8}\right)\right\} & \text{if } x < y \\ \left\{(0,0), \left(\frac{1}{8}, 0\right)\right\} & \text{if } x \geq y \end{cases}$$

$$f(x,y)=\begin{cases}\left(-\frac{1}{8},\frac{1}{8}\right) & \text{if } x<y\\ (0,0) \text{ if } x=y & \\ \left(\frac{1}{8},0\right) & \text{if } x>y\end{cases}$$

$$G(x,y)=\begin{cases}\left(-\frac{1}{8},\frac{1}{8}\right) & \text{if } x<y\\ \left(0,-\frac{1}{2}\right) & \text{if } x\geq y\end{cases}$$

It is clear that

$$F(x,y)=\left(-\frac{1}{8},\frac{1}{8}\right)\subseteq g(X)=\left\{\left(-\frac{1}{8},\frac{1}{8}\right),\left(0,-\frac{1}{2}\right)\right\},$$
$$G(x,y)=\left\{\left(-\frac{1}{8},\frac{1}{8}\right),(0,0),\left(\frac{1}{8},0\right)\right\}=f(X).$$

For $\left(0,-\frac{1}{2}\right)\leq(0,0)\leq\left(\frac{1}{8},0\right)$ and

$$(0,0)\leq(0,0),\left(0,-\frac{1}{2}\right)\leq\left(0,-\frac{1}{2}\right),\left(\frac{1}{8},0\right)\leq\left(\frac{1}{8},0\right);$$

$$\begin{aligned}\delta\left(F\left(0,-\frac{1}{2}\right),G(0,0)\right)&=\delta\left(F(0,0),G\left(\frac{1}{8},0\right)\right)=\delta\left(F\left(0,-\frac{1}{2}\right),G\left(\frac{1}{8},0\right)\right)\\&=\delta\left(F(0,0),G(0,0)\right)=\delta\left(F\left(0,-\frac{1}{2}\right),G\left(0,-\frac{1}{2}\right)\right)\\&=\delta\left(F\left(\frac{1}{8},0\right),G\left(\frac{1}{8},0\right)\right)=\frac{1}{4},\end{aligned}$$

and

$$\begin{aligned}d\left(f\left(0,-\frac{1}{2}\right),g(0,0)\right)&=d\left(f(0,0),g\left(\frac{1}{8},0\right)\right)=d\left(f\left(0,-\frac{1}{2}\right),g\left(\frac{1}{8},0\right)\right)\\&=d\left(f(0,0),g(0,0)\right)=d\left(f\left(0,-\frac{1}{2}\right),g\left(0,-\frac{1}{2}\right)\right)\\&=d\left(f\left(\frac{1}{8},0\right),g\left(\frac{1}{8},0\right)\right)=\frac{1}{2}.\end{aligned}$$

For $\left(-\frac{1}{8},\frac{1}{8}\right)\leq\left(-\frac{1}{8},\frac{1}{8}\right)$;

$$\delta\left(F\left(-\frac{1}{8},\frac{1}{8}\right),G\left(-\frac{1}{8},\frac{1}{8}\right)\right)=0=d\left(\left(-\frac{1}{8},\frac{1}{8}\right),\left(-\frac{1}{8},\frac{1}{8}\right)\right).$$

Thus for all comparable elements of X we have

$$\begin{aligned}\delta(F(x),G(y))&=\frac{1}{4}\leq\frac{1}{2}\frac{1}{2}=\frac{1}{2}d(f(x),g(y))\\&=\frac{1}{2}\max\Big\{d(f(x),g(y)),dist(f(x),F(x)),dist(g(y),G(y)),\\&\frac{dist(f(x),G(y))+dist(g(y),F(x))}{2}\Big\}.\end{aligned}$$

Thus assumption 5 of Theorem 2.3 is satisfied with

$$T(t_1,...,t_5)=t_1-\alpha\max\{t_2,t_3,t_4,t_5/2\},$$

where $0\leq\alpha<1$. Also (F,f) and (G,g) are weakly compatible and only $\left(-\frac{1}{8},\frac{1}{8}\right)$ is the coincidence point where these pair commute. Consequently all conditions of Theorem 2.3 are satisfied and

$$\left\{\left(-\frac{1}{8},\frac{1}{8}\right)\right\}=F\left(-\frac{1}{8},\frac{1}{8}\right)=G\left(-\frac{1}{8},\frac{1}{8}\right)=f\left(-\frac{1}{8},\frac{1}{8}\right)=g\left(-\frac{1}{8},\frac{1}{8}\right).$$

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