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# Approximating common fixed points of noncommuting discontinuous weakly contractive mappings in metric spaces

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ABSTRACT.

In this paper we prove the existence of coincidence points and common fixed points for a large class of noncommuting discontinuous contractive type mappings in metric spaces. Moreover, a method for approximating the coincidence points and common fixed points is also constructed, for which both a priori and a posteriori error estimates are obtained.
These results generalize, extend and unify several well-known recent related results in literature.

## 1. INTRODUCTION

Having in view that many of the most important nonlinear problems of applied mathematics reduce to solving a given operator equation which in turn may be reduced to finding the fixed points of a certain mapping or the common fixed points of two mappings, the study of fixed and common fixed points of mapping satisfying certain contractive conditions attracted more research work in the last three decades, see for example [25] and the very recent monographs [10], [27].

Among these (common) fixed point theorems, only a few are important from a practical point of view, that is, they provide a *constructive* method for finding the fixed points or the common fixed points of the mappings involved, and only seldom offer information on the error estimate (or rate of convergence) of the iterative method used to approximate the (common) fixed point.

But, from a practical point of view it is important not only to know that the (common) fixed point exists (and, possibly, is unique), but also to be able to effectively construct that (common) fixed point.

Very recently M. Abbas and G. Jungck [3], obtained existence results of coincidence and common fixed points for noncommuting discontinuous contractive mappings in a cone metric space.

In this paper, inspired by the results in [3], we generalize, extend and unify several results in [3] and in some other related papers, and also provide an iterative method for approximating these points. A priori and a posteriori error estimates, expressed by a unique formula, as well as the rate of convergence for this method, are also obtained.

## 2. PRELIMINARIES

The classical Banach's contraction principle is one of the most useful results in nonlinear analysis. In a metric space setting its statement is given by the next theorem.

**Theorem B.** *Let $(X, d)$ be a complete metric space and $T : X \longrightarrow X$ a map satisfying*

$$d(Tx, Ty) \le a\, d(x, y)\,, \quad \text{for all } x, y \in X\,, \tag{2.1}$$

*where $0 \le a < 1$ is constant. Then:*

*$(p1)$ $T$ has a unique fixed point $x^*$ in $X$;*

*$(p2)$ The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \dots \tag{2.2}$$

*converges to $x^*$, for any $x_0 \in X$.*

*$(p3)$ The following estimate holds:*

$$d(x_{n+i-1}, x^*) \le \frac{a^i}{1-a}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \dots;\ i = 1, 2, \dots \tag{2.3}$$

*$(p4)$ The rate of convergence of Picard iteration is given by*

$$d(x_n, x^*) \le a\, d(x_{n-1}, x^*)\,, \quad n = 1, 2, \dots \tag{2.4}$$

**Remarks.**

A map satisfying $(p1)$ and $(p2)$ in Theorem B is said to be a *Picard operator*, see [25], [26], while a mapping satisfying (2.1) is usually called *strict contraction* or *$a$-contraction* or *Banach contraction*.

Theorem B shows therefore that any strict contraction is a Picard operator.

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Theorem B has many applications in solving nonlinear equations. Its merit is not only to state the existence and uniqueness of the fixed point of the strict contraction $T$ but also to show that the fixed point can be approximated by means of Picard iteration (2.2). Moreover, for this iterative method both *a priori*

$$d(x_n, x^*) \leq \frac{a^n}{1-a}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \dots$$

and *a posteriori*

$$d(x_n, x^*) \leq \frac{a}{1-a}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \dots;\ i = 1, 2, \dots$$

error estimates are available, which were both encapsulated in (2.3), following an idea from [29].

On the other hand, the inequality (2.4) shows that the rate of convergence of Picard iteration is linear in the class of strict contractions.

Despite these important features, Theorem B suffers from one drawback - the contractive condition (2.1) forces $T$ be continuous on $X$.

It was then natural to ask if there exist or not weaker contractive conditions which do not imply the continuity of $T$. This was answered in the affirmative by R. Kannan [16] in 1968, who proved a fixed point theorem which extends Theorem B to mappings that need not be continuous on $X$ (but are continuous at their fixed point), see [23], by considering instead of (2.1) the next condition: there exists $b \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big], \quad \text{for all } x, y \in X\,. \tag{2.5}$$

Following the Kannan's theorem, a lot of papers were devoted to obtaining fixed point and common fixed point theorems for various classes of contractive type conditions that do not require the continuity of $T$, see for example, [25], [10] and the references therein.

One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [13], is based on a condition similar to (2.5): there exists $c \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big], \quad \text{for all } x, y \in X. \tag{2.6}$$

For a presentation and comparison of such kind of fixed point theorems, see [21], [22], [17], [4], [12] and [10]. For other related results, see [1], [2], [18] and [19].

These fixed point results were then complemented by important results regarding the existence of common fixed points of such contractive type mappings. So, Jungck [14] proved in 1976 a common fixed point theorem for commuting maps, thus generalizing Theorem B. In the same spirit, very recently M. Abbas and G. Jungck [3], obtained coincidence and common fixed point theorems for the class of Banach contractions, Kannan contractions and Chatterjea contractions in cone metric spaces, without making use of the commutative property, but based on the so called concept of weakly compatible mappings, introduced by Jungck [15].

On the other hand, in 1972, Zamfirescu [28] obtained a very interesting fixed point theorem, by combining the contractive conditions (2.1) of Banach, (2.5) of Kannan and (2.6) of Chatterjea.

Note that, as shown by Rhoades [21], the contractive conditions (2.1) and (2.5), as well as (2.1) and (2.6), and (2.5) and (2.6), respectively, are independent.

We give here the complete statement of Zamfirescu's fixed point theorem, including also the error and rate of convergence estimates, similar to that given in the very recent paper [11], in view of its extension to coincidence and common fixed points. A complete proof of Theorem 2.1 can be found in [9] and [10].

**Theorem 2.1.** *Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a mapping for which there exist $a \in [0, 1)$, $b, c \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in X$, at least one of the following conditions is true:*

$(z_1)$ $d(Tx, Ty) \leq a\, d(x, y)$;

$(z_2)$ $d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big]$;

$(z_3)$ $d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big]$.

*Then the Picard iteration $\{x_n\}$ defined by (2.2) and starting from $x_0 \in X$ converges to the unique fixed point $x^*$ of $T$ with the following error estimate*

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \dots;\ i = 1, 2, \dots$$

*where* $\delta = \max\left\{a, \dfrac{b}{1-b}, \dfrac{c}{1-c}\right\}$.

*Moreover, the convergence rate of the Picard iteration is given by*

$$d(x_n, x^*) \leq \delta \cdot d(x_{n-1}, x^*)\,, \quad n = 1, 2, \dots \tag{2.7}$$

It is therefore the main aim of this paper to extend and unify all the results in [3], Theorem 2.1 and several other related results in literature, by proving a general result regarding the existence, the uniqueness and the approximation of (coincidence) common fixed points of two discontinuous weakly contractive mappings of Zamfirescu type. To this end we need some notions and results from [3] and [15].

**Definition 2.1.** ([3]) Let $S$ and $T$ be selfmaps of a nonempty set $X$. If there exists $x \in X$ such that $Sx = Tx$ then $x$ is called a *coincidence point* of $S$ and $T$, while $y = Sx = Tx$ is called a *point of coincidence* of $S$ and $T$. If $Sx = Tx = x$, then $x$ is a common fixed point of $S$ and $T$.

**Definition 2.2.** ([15]) Let $S$ and $T$ be selfmaps of a nonempty set $X$. The pair of mappings $S$ and $T$ is said to be *weakly compatible* if they commute at their coincidence points.

The next Proposition, which is given in [3] as Proposition 1.4, will be needed to prove the last part in our main result.

**Proposition 2.1.** *Let $S$ and $T$ be weakly compatible selfmaps of a nonempty set X. If $S$ and $T$ have a unique point of coincidence $y = Sx = Tx$, then $y$ is the unique common fixed point of $S$ and $T$.*

## 3. Main results

In [3], the authors obtained three coincidence and common fixed point theorems, corresponding to Banach contraction condition (Theorem 2.1), Kannan's contractive condition (Theorem 2.3) and Chatterjea's contractive condition (Theorem 2.4), respectively, in cone metric spaces. We state in the following the one corresponding to Kannan's contractive condition, in view of its extension and generalization.

**Theorem 3.2.** *Let $(X, d)$ be a cone metric space and $P$ a cone with normal constant $K$. Suppose that the mappings $f,\ g : X \to X$ satisfy the contractive condition*

$$d(fx, fy) \leq k\,[d(fx, gx) + d(fy, gy)]\,,\ \forall\, x, y \in X, \tag{3.8}$$

*where $k \in \left[0, \frac{1}{2}\right)$ is a constant. If the range of $g$ contains the range of $f$ and $g(X)$ is a complete subspace of $X$, then $f$ and $g$ have a unique coincidence point in $X$. Moreover, if $f$ and $g$ are weakly compatible, then then $f$ and $g$ have a unique common fixed point in $X$.*

In the present paper, we restrict ourselves to state an prove our main result in a usual metric space setting, since its formulation in the abstract case of a cone metric space with normal cone will be given in a future paper. Note that our technique of proof is significantly different from the one used in [3].

**Theorem 3.3.** *Let $(X, d)$ be a metric space and let $T,\ S : X \to X$ be two mappings for which there exist $a \in [0, 1)$, $b, c \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in X$, at least one of the following conditions is true:*

$(z_1)$ $d(Tx, Ty) \leq a\, d(Sx, Sy)$;

$(z_2)$ $d(Tx, Ty) \leq b\big[d(Sx, Tx) + d(Sy, Ty)\big]$;

$(z_3)$ $d(Tx, Ty) \leq c\big[d(Sx, Ty) + d(Sy, Tx)\big]$.

*If the range of $S$ contains the range of $T$ and $S(X)$ is a complete subspace of $X$, then $T$ and $S$ have a unique coincidence point in $X$. Moreover, if $T$ and $S$ are weakly compatible, then $T$ and $S$ have a unique common fixed point in $X$.*

*In both cases, the iteration $\{Sx_n\}$ defined by (3.15) converges to the unique (coincidence) common fixed point $x^*$ of $S$ and $T$, for any $x_0 \in X$, with the following error estimate*

$$d(Sx_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta}\, d(Sx_n, Sx_{n-1})\,, \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots \tag{3.9}$$

*where $\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$.*

*The convergence rate of the iteration $\{Sx_n\}$ is given by*

$$d(Sx_n, x^*) \leq \delta \cdot d(Sx_{n-1}, x^*)\,, \quad n = 1, 2, \ldots \tag{3.10}$$

*Proof.* We first fix $x, y \in X$. At least one of ($z_1$), ($z_2$) or ($z_3$) is true. If ($z_2$) holds, then

$$\begin{aligned} d(Tx, Ty) &\leq b[d(Sx, Tx) + d(Sy, Ty)] \leq \\ &\leq b\{d(Sx, Tx) + [d(Sy, Sx) + d(Sx, Tx) + d(Tx, Ty)]\}. \end{aligned}$$

So

$$(1-b)\ d(Tx, Ty) \leq 2b\, d(Sx, Tx) + b\, d(Sx, Sy),$$

which yields

$$d(Tx, Ty) \leq \frac{2b}{1-b}\ d(Sx, Tx) + \frac{b}{1-b}\ d(Sx, Sy). \tag{3.11}$$

If (z$_3$) holds, then similarly we get

$$d(Tx,Ty) \le \frac{2c}{1-c}\ d(Sx,Tx) + \frac{c}{1-c}\ d(Sx,Sy). \tag{3.12}$$

Therefore, by denoting

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\},$$

we have $0 \le \delta < 1$ and then, by $(z_1)$, (3.11) and (3.12), we find that, for all $x, y \in X$, the following inequality

$$d(Tx,Ty) \le 2\delta \cdot d(Sx,Tx) + \delta \cdot d(Sx,Sy) \tag{3.13}$$

holds. In a similar manner we obtain

$$d(Tx,Ty) \le 2\delta \cdot d(Sx,Ty) + \delta \cdot d(Sx,Sy), \tag{3.14}$$

valid for all $x, y \in X$.

Let now $x_0$ be an arbitrary point in $X$. Since $T(X) \subset S(X)$, we can choose a point $x_1$ in $X$ such that $Tx_0 = Sx_1$. Continuing in this way, for a $x_n$ in $X$, we can find $x_{n+1} \in X$ such that

$$Sx_{n+1} = Tx_n,\ n = 0, 1, \dots \tag{3.15}$$

If $x := x_n,\ y := x_{n-1}$ are two successive terms of the sequence defined by (3.15), then by (3.14) we have

$$d(Sx_{n+1}, Sx_n) = d(Tx_n, Tx_{n-1}) \le 2\delta \cdot d(Sx_n, Tx_{n-1}) + \delta \cdot d(Sx_n, Sx_{n-1}),$$

which in view of (3.15) yields

$$d(Sx_{n+1}, Sx_n) \le \delta \cdot d(Sx_n, Sx_{n-1}),\ n = 0, 1, 2 \dots. \tag{3.16}$$

Now by induction, from (3.16) we obtain

$$d(Sx_{n+k}, Sx_{n+k-1}) \le \delta^k \cdot d(Sx_n, Sx_{n-1}),\ n = 0, 1, 2 \dots;\ k = 1, 2, \dots, \tag{3.17}$$

and then, for $p > i$, we get after straightforward calculations

$$d(Sx_{n+p}, Sx_{n+i-1}) \le \frac{\delta^i(1-\delta^{p-i+1})}{1-\delta} \cdot d(Sx_n, Sx_{n-1}),\ n \ge 0;\ i \ge 1. \tag{3.18}$$

For $i = 1$ and then by an inductive process, (3.18) yields

$$d(Sx_{n+p}, Sx_n) \le \frac{\delta}{1-\delta} \cdot d(Sx_n, Sx_{n-1}) \le \frac{\delta^n}{1-\delta} \cdot d(Sx_1, Sx_0),\ n = 0, 1, 2 \dots,$$

which shows that $\{Sx_n\}$ is a Cauchy sequence.

Since $S(X)$ is complete, there exists a $x^*$ in $S(X)$ such that

$$\lim_{n\to\infty} Sx_{n+1} = x^*. \tag{3.19}$$

We can find $p \in X$ such that $Sp = x^*$. By (3.15) and (3.16) we further have

$$d(Sx_n, Tp) = d(Tx_{n-1}, Tp) \le \delta d(Sx_{n-1}, Sp) \le \delta^{n-1} d(Sx_1, Sp),$$

which shows that we also have

$$\lim_{n\to\infty} Sx_n = Tp. \tag{3.20}$$

By (3.19) and (3.20) it results now that $Tp = Sp$, that is, $p$ is a coincidence point of $T$ and $S$ (or $x^*$ is a point of coincidence of $T$ and $S$).

Now let us show that $T$ and $S$ have a unique point of coincidence. Assume there exists $q \in X$ such that $Tq = Sq$. Then, by (3.13) we get

$$d(Sq, Sp) = d(Tq, Tp) \le 2\delta d(Sq, Tq) + \delta d(Sq, Tp) = \delta d(Sq, Sp)$$

which shows that $Sq = Sp = x^*$, that is, $T$ and $S$ have a unique point of coincidence, $x^*$.

Now if $T$ and $S$ are weakly compatible, by Proposition 1 it follows that $x^*$ is their unique common fixed point.

The estimate (3.10) is obtained from (3.18) by letting $p \to \infty$, while (3.11) is obtained by (3.13) by taking $x = x_n$ and $y = x^*$. □

**Particular cases**

1) If in Theorem 3.3, condition $(z_1)$ holds for all $x, y \in X$, then by Theorem 3.3 we obtain a common fixed point result corresponding to Theorem 2.1 in [3].

2) If in Theorem 3.3, condition $(z_2)$ holds for all $x, y \in X$, then by Theorem 3.3 we obtain a common fixed point result corresponding to Theorem 2.3 in [3], that is, Theorem 2.1 in this paper.

3) If in Theorem 3.3, condition $(z_3)$ holds for all $x, y \in X$, then by Theorem 3.3 we obtain a result on the existence, uniqueness and approximation of the common fixed point, stated in Theorem 3.4, corresponding to Theorem 2.4 in [3].

But, what is extremely important, our result also provides both *a priori* and *a posteriori* error estimate, as well as information on the rate of convergence of the iterative process that approximates that common fixed point.

**Theorem 3.4.** *Let $(X,d)$ be a metric space and let $T,\ S: X \to X$ be two mappings for which there exist $c \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in X$,*

$d(Tx,Ty) \le c\big[d(Sx,Ty) + d(Sy,Tx)\big].$

*If the range of $S$ contains the range of $T$ and $S(X)$ is a complete subspace of $X$, then $T$ and $S$ have a unique coincidence point in $X$. Moreover, if $T$ and $S$ are weakly compatible, then $T$ and $S$ have a unique common fixed point in $X$.*

*In both cases, the iteration $\{Sx_n\}$ defined by (3.15) converges to the unique (coincidence) common fixed point $x^*$ of $S$ and $T$, for any $x_0 \in X$, with the following error estimate*

$$d(Sx_{n+i-1}, x^*) \le \frac{c^i}{1-c}\, d(Sx_n, Sx_{n-1})\,, \quad n = 0,1,2,\ldots;\ i = 1,2,\ldots$$

*The convergence rate of the iteration $\{Sx_n\}$ is given by*

$$d(Sx_n, x^*) \le c \cdot d(Sx_{n-1}, x^*)\,, \quad n = 1,2,\ldots$$

## 4. Conclusions and an example

By Theorem 3.3 we can obtain a common fixed point result for mappings that satisfy a single contractive condition, as in the next theorem.

**Theorem 4.5.** *Let $(X,d)$ be a metric space and let $T,\ S: X \to X$ be two mappings for which there exist $h \in [0,1)$ such that*

$$d(Tx,Ty) \le h \cdot \max\Big\{d(Sx,Sy), [d(Sx,Tx) + d(Sy,Ty)]/2,$$

$$[d(Sx,Ty) + d(Sy,Tx)]/2\Big\}\text{, for all } x, y \in X\,. \tag{4.21}$$

*If the range of $S$ contains the range of $T$ and $S(X)$ is a complete subspace of $X$, then $T$ and $S$ have a unique coincidence point in $X$. Moreover, if $T$ and $S$ are weakly compatible, then $T$ and $S$ have a unique common fixed point in $X$.*

*In both cases, the iteration $\{Sx_n\}$ defined by (3.15) converges to the unique (coincidence) common fixed point $x^*$ of $S$ and $T$, for any $x_0 \in X$, with the following error estimate*

$$d(Sx_{n+i-1}, x^*) \le \frac{h^i}{1-h}\, d(Sx_n, Sx_{n-1})\,, \quad n = 0,1,2,\ldots;\ i = 1,2,\ldots$$

*The convergence rate of the iteration $\{Sx_n\}$ is given by*

$$d(Sx_n, x^*) \le h \cdot d(Sx_{n-1}, x^*)\,, \quad n = 1,2,\ldots$$

*Proof.* The contractive condition (4.21) is equivalent to Zamfirescu's conditions, see [21], so the theorem follows by Theorem 3.3. ☐

**Remark 4.1.** We mention that, as noted in [20], the assumption "$S(X)$ is a complete metric space", used in all Theorems 3.2-4.5, appears to be too restrictive in applications.

Therefore, similarly to the paper [20], this assumption could be replaced by the more practical and slightly relaxed condition "there exists a complete metric subspace $Y \subseteq X$ such that $T(X) \subseteq Y \subseteq S(X)$", under which all results established in these paper remain valid.

The next example shows that the generalizations given by our results in this paper are effective.

**Example 4.1.** Let $X = [0,1]$ with the usual norm. Let $T, S : X \to X$ be defined by

$$Tx = \begin{cases} \frac{x}{4}, & 0 \le x < 1 \\ \frac{1}{3}, & x = 1 \end{cases} \quad \text{and} \quad Sx = \begin{cases} x\,, & 0 \le x < 1 \\ \frac{2}{3}, & x = 1, \end{cases} \quad \text{respectively.}$$

We have $T(X) = [0,1/4) \cup \{1/3\} \subset [0,1/3]$. So, see Remark 1, there exists the complete metric subspace $Y = [0,1/3]$ such that $T(X) \subseteq Y \subseteq S(X) = [0,1]$. Moreover, 0 is the unique coincidence point of $S$ and $T$ and, since, obviously, $T$ and $S$ commute at 0, $S$ and $T$ are weakly compatible.

In order to show that $S$ and $T$ do satisfy the contractive conditions in Theorem 3.3 (and also in Theorem 4.5), let us denote

$$M_1 = [0,1) \times [0,1);\ M_2 = [0,1) \times \{1\} \cup \{1\} \times [0,1);\ M_3 = \{1\} \times \{1\}.$$

Clearly, $[0,1] \times [0,1] = M_1 \cup M_2 \cup M_3$. For $(x,y) \in M_1$, $S$ and $T$ satisfy condition $(z_1)$, with constant $a = 1/4$ which immediately holds:

$$\left|\frac{x}{4} - \frac{y}{4}\right| \le a\,|x-y|\,.$$

For $(x, y) \in M_3$, $Tx = Ty$ and so $(z_2)$ is obviously satisfied. Consider now $(x, y) \in M_2$. Due to the symmetry of the contractive condition $(z_3)$, it suffices to show that $(z_3)$ is satisfied for all $x \in [0, 1)$ and $y = 1$. As $Tx = \dfrac{x}{4}$, $T1 = \dfrac{1}{3}$, $Sx = x$ and $S1 = \dfrac{2}{3}$, condition $(z_3)$ reduces to show that there exists a constant $c$, $0 \leq c < \dfrac{1}{2}$, such that

$$\left|\frac{x}{4} - \frac{1}{3}\right| \leq c\left(\left|x - \frac{1}{3}\right| + \left|\frac{2}{3} - \frac{x}{4}\right|\right), \forall\, x \in [0, 1). \tag{4.22}$$

We shall prove that (4.22) holds with $c = \dfrac{3}{7} < \dfrac{1}{2}$, that is,

$$\left|\frac{x}{4} - \frac{1}{3}\right| \leq \frac{3}{7}\left(\left|x - \frac{1}{3}\right| + \left|\frac{2}{3} - \frac{x}{4}\right|\right), \forall\, x \in [0, 1). \tag{4.23}$$

As, for $x \in [0, 1)$, $\dfrac{x}{4} < \dfrac{1}{4} < \dfrac{1}{3}$, we have $\left|\dfrac{x}{4} - \dfrac{1}{3}\right| = \dfrac{1}{3} - \dfrac{x}{4}$. Similarly, since $\dfrac{2}{3} > \dfrac{1}{4} > \dfrac{x}{4}$, we get $\left|\dfrac{2}{3} - \dfrac{x}{4}\right| = \dfrac{2}{3} - \dfrac{x}{4}$ and therefore, (4.23) becomes

$$\frac{1}{3} - \frac{x}{4} \leq \frac{3}{7}\left(\left|x - \frac{1}{3}\right| + \frac{2}{3} - \frac{x}{4}\right), \forall\, x \in [0, 1). \tag{4.24}$$

If $0 \leq x \leq \dfrac{1}{3}$, then $\left|x - \dfrac{1}{3}\right| = \dfrac{1}{3} - x$ and (4.24) reduces to $x \leq \dfrac{1}{3}$, which is true.

If $\dfrac{1}{3} \leq x < 1$, then $\left|x - \dfrac{1}{3}\right| = x - \dfrac{1}{3}$ and (4.24) reduces to $x \geq \dfrac{1}{3}$, which is also true. This proves that (4.22) is satisfied, that is, $S$ and $T$ satisfy condition $(z_3)$ on $M_3$.

Note that $T$ and $S$ do not satisfy condition $(z_1)$ on the whole space $X$. Indeed, this would imply that there exists a constant $a$, $0 \leq a < 1$, such that

$$|Tx - Ty| \leq a\,|x - y|\,, \forall\, x, y \in [0, 1]. \tag{4.25}$$

Take $0 \leq x < 1$ and $y = 1$ to get

$$\frac{1}{3} - \frac{x}{4} \leq a(1 - x),$$

which by letting $x \to 1$ yields the contradiction $\dfrac{1}{12} \leq 0$.

So, the metric space variant of Theorem 2.1 in [3] do not apply. Moreover, $S$ and $T$ do not satisfy $(z_2)$ on the whole $X$. Indeed, for $0 \leq x < 1$ and $y = 1$, condition $(z_2)$ reduces to

$$\left|\frac{x}{4} - \frac{1}{3}\right| \leq b\left(\left|x - \frac{x}{4}\right| + \left|1 - \frac{2}{3}\right|\right), \forall\, x \in [0, 1), \tag{4.26}$$

where $0 \leq b < \dfrac{1}{2}$. If we take $x = 0$ in (4.26), we get

$$\frac{1}{3} \leq b \cdot \frac{1}{3} < \frac{1}{6},$$

a contradiction.

This shows that $S$ and $T$ do not satisfy $(z_2)$ on the whole $X$ and hence, the metric space variant of Theorem 2.3 in [3] cannot be applied here.

In view of Remark 1 above, both Theorem 3.3 and Theorem 4.5 in our paper do apply and 0 is the unique common fixed point of $S$ and $T$.

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## References

[1] Altun, I., Hancer, H.A. and Turkoglu, D., *A common fixed point theorem on d-complete topological spaces*, Carpathian J. Math., **24** (2008), No. 1, 1–7
[2] Babu, G. V. R., Sandhya, M. L. and Kameswari, M. V. R., *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math. **24** (2008), No. 1, 8–12
[3] Abbas, M. and Jungck, G., *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341** (2008), 416–420
[4] Berinde, V., *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math., **19** (2003), No. 1, 7–22
[5] Berinde, V., *Picard iteration converges faster than the Mann iteration in the class of quasi-contractive operators*, Fixed Point Theory Appl. **2004** (2004), No. 2, 97–105
[6] Berinde, V., *On the convergence of Ishikawa iteration for a class of quasi contractive operators*, Acta Math. Univ. Comen., **73** (2004), No. 1, 119–126
[7] Berinde, V., *A convergence theorem for some mean value fixed point iterations in the class of quasi contractive operators*, Demonstratio Math., **38** (2005), No. 1, 177–184
[8] Berinde V., *Error estimates for approximating fixed points of discontinuous quasi-contractions*, General Mathematics **13** (2005), No. 2, 23–34

[9] Berinde, V. and Berinde, Mădălina, *On Zamfirescu's fixed point theorem*, Rev. Roumaine Math. Pures Appl., **50** (2005), No. 5-6, 443–453
[10] Berinde, V., *Iterative Approximation of Fixed Points*, 2nd Ed., Springer Verlag, Berlin Heidelberg New York, 2007
[11] Berinde, V., *On the iterative solution of some decomposable nonlinear operator equations*, Creat. Math. Inf. **17** (2008), No. 2, 1–7
[12] Berinde, V., *General constructive fixed point theorems for C iric-type almost contractions in metric spaces*, Carpathian J. Math., **24** (2008), No. 2, 10–19
[13] Chatterjea, S. K., *Fixed-point theorems*, C. R. Acad. Bulgare Sci. **25** (1972), 727–730
[14] Jungck, G., *Commuting maps and fixed points*, Amer. Math Monthly **83** (1976), 261–263
[15] Jungck, G. , *Common fixed points for noncontinuous nonself maps on non-metric spaces*, Far East J. Math. Sci. **4** (1996), 199–215
[16] Kannan, R., *Some results on fixed points*, Bull. Calcutta Math. Soc. **10** (1968), 71–76
[17] Meszaros, J., *A comparison of various definitions of contractive type mappings*, Bull. Calcutta Math. Soc. **84** (1992), No. 2, 167–194
[18] Păcurar, Mădălina, *Viscosity approximation of fixed points with $\phi$-contractions*, Carpathian J. Math., **24** (2008), No. 1, 88–93
[19] Păcurar, Mădălina, *Sequences of almost contractions and fixed points*, Carpathian J. Math., **24** (2008), No. 2, 101–109
[20] Păcurar, Mădălina, *Approximating common fixed points of Presić-Kannan type operators by a multi-step iterative method*, An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat., **17** (2009) (in print)
[21] Rhoades, B. E., *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226** (1977) 257–290
[22] Rhoades, B. E., *Contractive definitions revisited*, Contemporary Mathematics **21** (1983) 189–205
[23] Rhoades, B. E., *Contractive definitions and continuity*, Contemporary Mathematics **72** (1988) 233–245
[24] Rus, I. A., *Principles and Applications of the Fixed Point Theory* (in Romanian), Editura Dacia, Cluj-Napoca, 1979
[25] Rus, I. A., *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001
[26] Rus I. A., *Picard operators and applications*, Scientiae Math. Japon. **58** (2003), No. 1, 191–219
[27] Rus, I. A., Petruşel, A. and Petruşel, Gabriela, *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, 2008
[28] Zamfirescu, T., *Fix point theorems in metric spaces*, Arch. Math. (Basel), **23** (1972), 292–298
[29] Walter, W., *Remarks on a paper by F. Browder about contraction*, Nonlinear Anal. TMA **5** (1981), 21-25

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