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Differential subordinations and superordinations for analytic functions defined by an integral operator

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ABSTRACT.
By using the integral operator $I^n f(z)$, $z \in U$ (Definition 1.2) we give in this paper some results and examples for differential subordinations and superordinations for analytic functions and we will determine some properties on admissible functions defined with the integral operator I^n.

1. INTRODUCTION

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + \dots\}.$$

We also consider the class

$$\mathcal{A} = \{f \in \mathcal{H} : f(z) = z + a_2 z^2 + \dots\}.$$

We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Since we use the terms of subordination and superordination, we review here these definitions.

Definition 1.1. Let $f, F \in \mathcal{H}$. The function f is said to be subordinate to F or F is said to be superordinate to f, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$.

In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$. Since most of the functions considered in this paper and conditions on them are defined uniformly in the unit disk U, we shall omit the requirement "$z \in U$".

Let $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let h be univalent in U and $q \in Q$. In [2] the authors considered the problem of determining conditions on admissible function ψ such that

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z) \tag{1.1}$$

implies $p(z) \prec q(z)$, for all functions $p \in \mathcal{H}[a, n]$ that satisfy the differential subordination (1.1).

Moreover, they found conditions so that the function q is the "smallest" function with this property, called the best dominant of the subordination (1.1).

Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let $h \in \mathcal{H}$ and $q \in \mathcal{H}[a, n]$. Recently, in [3] the authors studied the dual problem and determined conditions on φ such that

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z) \text{ implies } q(z) \prec p(z), \tag{1.2}$$

for all functions $p \in Q$ that satisfy the above differential superordination.

Definition 1.2. Let $f \in \mathcal{A}$. The integral operator I^n of f is defined in [5] as:

(i) $I^0 f(z) = f(z)$;

(ii) $I^1 f(z) = If(z) = \int_0^z f(t) t^{-1} dt$;

(iii) $I^n f(z) = I(I^{n-1} f(z))$, where $z \in U$ and $n \in \mathbb{N}$.

A holomorphic function $f : U \to C$ is called convex if f is univalent on U and $f(U)$ is a convex domain.

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2. Preliminaries

In our present investigation we shall need the following results.

Theorem 2.1. [1] *Let q be an univalent function in U and $\gamma \in \mathbb{C}^*$ such that*

$$\operatorname{Re}\frac{zq''(z)}{q'(z)}+1\geq \max\left\{0,-\operatorname{Re}\frac{1}{\gamma}\right\}.$$

If p is an analytic function in U, with $p(0)=q(0)$ and

$$p(z)+\gamma zp'(z)\prec q(z)+\gamma zq'(z), \tag{2.3}$$

then $p(z)\prec q(z)$ and q is the best dominant of (2.3).

Corollary 2.1. [1] *Let q be convex function in U, with $q(0)=a$ and $\gamma\in\mathbb{C}$ such that $\operatorname{Re}\gamma>0$. If $p\in\mathcal{H}[a,1]\cap Q$ and $p(z)+\gamma zp'(z)$ is univalent in U, then*

$$q(z)+\gamma zq'(z)\prec p(z)+\gamma zp'(z)\ \Rightarrow\ q(z)\prec p(z)$$

and q is the best subordinant.

In this paper we shall use the operator I^n, to obtain certain special subordinations and superordinations for analytic functions on the unit disc U.

3. Main results

Theorem 3.2. *Let q be an univalent function in U with $q(0)=1$, $\gamma\in\mathbb{C}^*$ such that*

$$\operatorname{Re}\left[1+\frac{zq''(z)}{q'(z)}\right]>\max\left\{0,-\operatorname{Re}\frac{1}{\gamma}\right\}.$$

If $f\in\mathcal{A}$ and

$$\frac{I^{n+1}f(z)}{I^nf(z)}+\gamma\left\{1-\frac{I^{n-1}f(z)I^{n+1}f(z)}{[I^nf(z)]^2}\right\}\prec q(z)+\gamma zq'(z), \tag{3.4}$$

then

$$\frac{I^{n+1}f(z)}{I^nf(z)}\prec q(z) \tag{3.5}$$

and q is the best dominant of subordination (3.2).

Proof. We define the function

$$p(z):=\frac{I^{n+1}f(z)}{I^nf(z)}.$$

By calculating the logarithmic derivative of p, we obtain

$$\frac{zp'(z)}{p(z)}=z\frac{[I^{n+1}f(z)]'}{I^{n+1}f(z)}-z\frac{[I^nf(z)]'}{I^nf(z)}. \tag{3.6}$$

Taking account of the identity

$$z[I^{n+1}f(z)]'=I^nf(z) \tag{3.7}$$

and replace in relation (3.6), we have

$$\frac{zp'(z)}{p(z)}=\frac{I^nf(z)}{I^{n+1}f(z)}-\frac{I^{n-1}f(z)}{I^nf(z)}=\frac{1}{p(z)}-\frac{I^{n-1}f(z)}{I^nf(z)}$$

and

$$p(z)+\gamma zp'(z)=\frac{I^{n+1}f(z)}{I^nf(z)}+\gamma\left\{1-\frac{I^{n-1}f(z)\cdot I^{n+1}f(z)}{[I^nf(z)]^2}\right\}.$$

The subordination (3.4) becomes

$$p(z)+\gamma zp'(z)\prec q(z)+\gamma zq'(z).$$

The conclusion of this theorem result from Theorem 2.1. □

We will give an application of Theorem 3.1 feeling the convex function

$$q(z)=\frac{1+Az}{1+Bz}.$$

Example 3.1. Let $A, B, \gamma \in \mathbb{C}$, $A \neq B$, such that $|B| \leq 1$ and $\operatorname{Re}\gamma > 0$. If for $f \in \mathcal{A}$

$$\frac{I^{n+1}f(z)}{I^n f(z)} + \gamma\left\{1 - \frac{I^{n-1}f(z)\cdot I^{n+1}f(z)}{[I^n f(z)]^2}\right\} \prec \frac{1+Az}{1+Bz} + \gamma\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\frac{I^n f(z)}{I^{n+1}f(z)} \prec \frac{1+Az}{1+Bz} \tag{3.8}$$

and $q(z) = \dfrac{1+Az}{1+Bz}$ is the best dominant of subordination (3.8).

Theorem 3.3. *Let q be a convex function in U, with $q(0) = 1$ and $\gamma \in \mathbb{C}$ such that $\operatorname{Re}\gamma > 0$. If $f \in \mathcal{A}$,*

$$\frac{I^{n+1}f(z)}{I^n f(z)} \in \mathcal{H}[1,1] \cap Q,$$

$$\frac{I^{n+1}f(z)}{I^n f(z)} + \gamma\left\{1 - \frac{I^{n-1}f(z)\cdot I^{n+1}f(z)}{[I^n f(z)]^2}\right\}$$

is univalent in U and

$$q(z) + \gamma z q'(z) \prec \frac{I^{n+1}f(z)}{I^n f(z)} + \gamma\left\{1 - \frac{I^{n-1}f(z)\cdot I^{n+1}f(z)}{[I^n f(z)]^2}\right\}, \tag{3.9}$$

then

$$q(z) \prec \frac{I^{n+1}f(z)}{I^n f(z)} \tag{3.10}$$

and q is the best subordinant of superordination (3.10).

Proof. Let

$$p(z) := \frac{I^{n+1}f(z)}{I^n f(z)}.$$

The superordination (3.9) may be write:

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z).$$

If we apply Corollary 2.2 we obtain the conclusion of this theorem. □

Now, we give a result of "sandwich" type.

Theorem 3.4. *Let q_1 and q_2 by convex function in the unit disk U, with $q_1(0)=q_2(0)=1$, $\gamma \in \mathbb{C}$ such that $\operatorname{Re}\gamma > 0$. If $f \in \mathcal{A}$,*

$$\frac{I^{n+1}f(z)}{I^n f(z)} \in \mathcal{H}[1,1] \cap Q, \quad \frac{I^{n+1}f(z)}{I^n f(z)} + \gamma\left\{1 - \frac{I^{n-1}f(z)\cdot I^{n+1}f(z)}{[I^n f(z)]^2}\right\}$$

is univalent in U and

$$q_1(z) + \gamma z q_1'(z) \prec \frac{I^{n+1}f(z)}{I^n f(z)} + \gamma\left\{1 - \frac{I^{n-1}f(z)\cdot I^{n+1}f(z)}{[I^n f(z)]^2}\right\} \prec q_2(z) + \gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{I^{n+1}f(z)}{I^n f(z)} \prec q_2(z), \tag{3.11}$$

and q_1 and q_2 are the best subordinant and the best dominant respectively of (3.11).

Theorem 3.5. *Let q be an univalent function in the unit disk U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ and suppose that*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\operatorname{Re}\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$(1+\gamma)z\frac{I^n f(z)}{[I^{n+1}f(z)]^2} + \gamma z\frac{I^{n-1}f(z)}{[I^{n+1}f(z)]^2} - 2\gamma z\frac{[I^n f(z)]^2}{[I^{n+1}f(z)]^3} \prec q(z) + \gamma z q'(z), \tag{3.12}$$

then

$$z\frac{I^n f(z)}{[I^{n+1}f(z)]^2} \prec q(z) \tag{3.13}$$

and q is the best dominant of subordination (3.13).

Proof. Let

$$p(z) := z\frac{I^n f(z)}{[I^{n+1} f(z)]^2}.$$

By calculating the logarithmic derivative of p, we obtain

$$\frac{zp'(z)}{p(z)} = 1 + \frac{I^{n-1} f(z)}{I^n f(z)} - 2\frac{I^n f(z)}{I^{n+1} f(z)}. \tag{3.14}$$

It follows that

$$p(z) + \gamma zp'(z) = (1+\gamma)z\frac{I^n f(z)}{[I^{n+1} f(z)]^2} + \gamma z\frac{I^{n-1} f(z)}{[I^{n+1} f(z)]^2} - 2\gamma z\frac{[I^n f(z)]^2}{[I^{n+1} f(z)]^3}.$$

The subordination (3.12) may be written

$$p(z) + \gamma zp'(z) \prec q(z) + \gamma zq'(z).$$

The conclusion of this theorem results by using Theorem 2.1. □

We will give an example where we choose dominant q a convex function given by

$$q(z) = \frac{1+Az}{1+Bz}.$$

Example 3.2. Let $A, B, \gamma \in \mathbb{C}$, $A \neq B$ such that $|B| \leq 1$ and $\operatorname{Re}\gamma > 0$. If $f \in \mathcal{A}$ and

$$(1+\gamma)z\frac{I^n f(z)}{[I^{n+1} f(z)]^2} + \gamma z\frac{I^{n-1} f(z)}{[I^{n+1} f(z)]^2} - 2\gamma z\frac{[I^n f(z)]^2}{[I^{n+1} f(z)]^3} \prec \frac{1+Az}{1+Bz} + \gamma\frac{(A-B)z}{(1+Bz)^2},$$

then

$$z\frac{I^n f(z)}{[I^{n+1} f(z)]^2} \prec \frac{1+Az}{1+Bz} \tag{3.15}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant of subordination (3.15).

Theorem 3.6. *Let q be a convex function in the unit disk U, $q(0) = 1$, $\gamma \in \mathbb{C}$ such that $\operatorname{Re}\gamma > 0$. If $f \in \mathcal{A}$,*

$$z\frac{I^n f(z)}{[I^{n+1} f(z)]^2} \in \mathcal{H}[1,1] \cap Q,$$

$$(1+\gamma)z\frac{I^n f(z)}{[I^{n+1} f(z)]^2} + \gamma z\frac{I^{n-1} f(z)}{[I^{n+1} f(z)]^2} - 2\gamma z\frac{[I^n f(z)]^3}{[I^{n+1} f(z)]^3}$$

is univalent in U and

$$q(z) + \gamma zq'(z) \prec (1+\gamma)z\frac{I^n f(z)}{[I^{n+1} f(z)]^2} + \gamma z\frac{I^{n-1} f(z)}{[I^{n+1} f(z)]^2} - 2\gamma z\frac{[I^n f(z)]^2}{[I^{n+1} f(z)]^3}, \tag{3.16}$$

then

$$q(z) \prec z\frac{I^n f(z)}{[I^{n+1} f(z)]^2} \tag{3.17}$$

and q is the best subordinant of superordination (3.17).

Proof. We define

$$p(z) := z\frac{I^n f(z)}{[I^{n+1} f(z)]^2}.$$

If we make some similar to in proof of Theorem 3.4, the superordination (3.16) becomes

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z).$$

The conclusion of this theorem results by applying Corollary 2.2. □

We will give a theorem of ”sandwich” type.

Theorem 3.7. *Let q_1, q_2 be convex function in U, with $q_1(0) = q_2(0) = 1$, $\gamma \in \mathbb{C}$, such that $\operatorname{Re}\gamma > 0$. If $f \in \mathcal{A}$,*

$$z\frac{I^n f(z)}{[I^{n+1} f(z)]^2} \in \mathcal{H}[1,1] \cap Q,$$

$$(1+\gamma)z\frac{I^n f(z)}{[I^{n+1} f(z)]^2} + \gamma z\frac{I^{n-1} f(z)}{[I^{n+1} f(z)]^2} - 2\gamma z\frac{[I^n f(z)]^2}{[I^{n+1} f(z)]^3}$$

is univalent in U and

$$q_1(z) + \gamma z q_1'(z) \prec (1+\gamma)z\frac{I^n f(z)}{[I^{n+1} f(z)]^2} + \gamma z\frac{I^{n-1} f(z)}{[I^{n+1} f(z)]^2} - 2\gamma z\frac{[I^n f(z)]^2}{[I^{n+1} f(z)]^3}$$
$$\prec q_2(z) + \gamma z q_2'(z),$$

then

(3.18) $$q_1(z) \prec z\frac{I^n f(z)}{[I^{n+1} f(z)]^2} \prec q_2(z)$$

and q_1 and q_2 are the best subordinant and the best dominant respectively of (3.15).

Special results related to differential subordinations were obtained in [4] by S. S. Miller and P. T. Mocanu.

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