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Diamond-α tangent lines of time scales parametrized regular curves

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ABSTRACT.
We introduce the notion of Δ-regular, ∇-regular and $\Diamond_\alpha$-regular curve, as a generalization of the "classical" regular curve. For each type of curve, we discuss the concept of tangent line on time scales.

1. INTRODUCTION

The study on time scales provides an unification of the discrete theory with the continuous theory and, in the same time, it is an unification and an extension of the traditional differential and difference equations. Also, many applications of this calculus are known, starting from biology, engineering, economics, physics, neural networks, social sciences, computational and numerical algorithms. The standard elements of the time scale calculus are the Δ (delta) and ∇ (nabla) dynamic derivatives and a combined dynamic derivative, called $\Diamond_\alpha$ (diamond-α) dynamic derivative. We refer the reader to [1, 2, 3, 4, 5, 7, 8, 9] for the basic rules of calculus associated with the diamond-α dynamic derivatives.

A time scale can be used as the defining set of the parameter for a parametric equation of a curve. In this manner, some parts of the classical differential geometry can be generalized to obtain a "dynamic" differential geometry.

In Section 2, we review some necessary definitions of the calculus on time scales. In Section 3 we define the Δ-regular, ∇-regular and $\Diamond_\alpha$-regular curve.

2. PRELIMINARIES

A *time scale* is any non-empty closed subset $\mathbb{T}$ of $\mathbb{R}$ (endowed with the topology of subspace of $\mathbb{R}$). In this paper $\mathbb{T}$ will denote a time scale and $[a,b]_\mathbb{T} = [a,b] \cap \mathbb{T}$ a time scaled interval.

For all $t \in \mathbb{T}$, we define the *forward jump operator* σ and the *backward jump operator* ρ by the formulas:

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T}, \quad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \in \mathbb{T}.$$

We make the convention:

$$\inf \emptyset := \sup \mathbb{T}, \quad \sup \emptyset := \inf \mathbb{T}.$$

If $\sigma(t) > t$, then t is said to be *right-scattered*, and if $\rho(t) < t$, then t is said to be *left-scattered*. The points that are simultaneously right-scattered and left-scattered are called *isolated*. If $\sigma(t) = t$, then t is said to be *right dense*, and if $\rho(t) = t$, then t is said to be *left dense*. The points that are simultaneously right-dense and left-dense are called *dense*.

The mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ defined by $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$ are called, respectively the *forward* and *backward graininess functions*.

If $\mathbb{T}$ has a right-scattered minimum m, then define $\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, then define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. Finally, put $\mathbb{T}^\kappa_\kappa = \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$.

Definition 2.1. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, one defines the delta derivative of f in t, to be the number denoted by $f^\Delta(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|,$$

for all $s \in U$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, one defines the nabla derivative of f in t, to be the number denoted by $f^\nabla(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood V of t such that

$$|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]| < \varepsilon|\rho(t) - s|,$$

for all $s \in V$.

We say that f is *delta differentiable* on $\mathbb{T}^\kappa$, provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$ and that f is *nabla differentiable* on $\mathbb{T}_\kappa$, provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$. See [1, 4] for the basic properties of the delta and nabla derivatives.

Definition 2.2. Let $\mathbb{T}$ be a time scale and for $s, t \in \mathbb{T}^\kappa_\kappa$ put $\mu_{ts} = \sigma(t) - s$, and $\nu_{ts} = \rho(t) - s$. One defines the diamond-α dynamic derivative of a function $f : \mathbb{T} \to \mathbb{R}$ in t to be the number denoted by $f^{\Diamond_\alpha}(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that for all $s \in U$,

$$\left|\alpha[f(\sigma(t)) - f(s)]\nu_{ts} + (1-\alpha)[f(\rho(t)) - f(s)]\mu_{ts} - f^{\Diamond_\alpha}(t)\mu_{ts}\nu_{ts}\right| < \varepsilon|\mu_{ts}\nu_{ts}|.$$

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A function is called *diamond-α differentiable* on $\mathbb{T}^\kappa_\kappa$ if $f^{\diamondsuit_\alpha}(t)$ exists for all $t \in \mathbb{T}^\kappa_\kappa$. If $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}$ in the sense of Δ and ∇, then f is diamond-α differentiable at $t \in \mathbb{T}^\kappa_\kappa$, and the diamond-$\alpha$ derivative $f^{\diamondsuit_\alpha}(t)$ is given by

$$f^{\diamondsuit_\alpha}(t) = \alpha f^\Delta(t) + (1-\alpha) f^\nabla(t), \quad 0 \le \alpha \le 1.$$

As it was proved in [9, Theorem 3.9], if f is diamond-α differentiable for $0 < \alpha < 1$ then f is both Δ and ∇ differentiable. It is obvious that for $\alpha = 1$ the diamond-α derivative reduces to the standard Δ derivative and for $\alpha = 0$ the diamond-α derivative reduces to the standard ∇ derivative. For $\alpha \in (0,1)$ it represents a "weighted dynamic derivative". See [8, 9] for some operations with the diamond-α derivative.

Let $a, b \in \mathbb{R}^n$, $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ be any two vectors. The Euclidean scalar product is the number defined as

$$< a, b >= \sum_{i=1}^{n} a_i b_i.$$

The norm (or length) of a vector $a \in \mathbb{R}^n$, denoted by $\|a\|$, is given by

$$\|a\| = \sqrt{< a, a >} = \sqrt{\sum_{i=1}^{n} a_i^2}.$$

3. Main results

We introduce the delta-regular, nabla-regular and diamond-α-regular notions, thus generalizing [6, Definition 3.1].

Definition 3.3. Let $\mathbb{T}$ a time scale and let $a, b \in \mathbb{T}$. A *delta-regular curve* γ is a map $\gamma = (\gamma_1, ..., \gamma_n) : [a,b]_\mathbb{T} \to \mathbb{R}^n$, where $\gamma_1, ..., \gamma_n$ are delta diferentiable real function on $[a,b]^\kappa_\mathbb{T}$ with rd-continuous Δ-derivatives and

$$\sum_{i=1}^{n} |\gamma_i^\Delta(t)|^2 \neq 0, \text{ for all } t \in [a,b]^\kappa_\mathbb{T}.$$

A *nabla-regular curve* γ is a map $\gamma = (\gamma_1, ..., \gamma_n) : [a,b]_\mathbb{T} \to \mathbb{R}^n$, where $\gamma_1, ..., \gamma_n$ are nabla diferentiable real function on $[a,b]^\kappa_\mathbb{T}$ with ld-continuous ∇-derivatives and

$$\sum_{i=1}^{n} |\gamma_i^\nabla(t)|^2 \neq 0, \text{ for all } t \in [a,b]_{\mathbb{T}_\kappa}.$$

A *diamond-α-regular curve* γ is a map $\gamma = (\gamma_1, ..., \gamma_n) : [a,b]_\mathbb{T} \to \mathbb{R}^n$, where $\gamma_1, ..., \gamma_n$ are diamond-α diferentiable real function on $[a,b]^\kappa_\mathbb{T}$ with continuous $\diamondsuit_\alpha$-derivatives and

$$\sum_{i=1}^{n} |\gamma_i^{\diamondsuit_\alpha}(t)|^2 \neq 0, \text{ for all } t \in [a,b]^\kappa_{\mathbb{T}_\kappa}.$$

Considering the vector $x = (x_1, ..., x_n)$, then a curve can be given in a parametric form

$$x_i = \gamma_i(t), \text{ for all } i \in \{1, ..., n\}, \tag{3.1}$$

or in a vector form

$$x = \gamma(t), \quad t \in \mathbb{T}, \tag{3.2}$$

while the conditions fulfilled by the derivatives are, respectively

$$\begin{aligned} &\|\gamma^\Delta(t)\| \neq 0, \ t \in [a,b]^\kappa_\mathbb{T}, \\ &\|\gamma^\nabla(t)\| \neq 0, \ t \in [a,b]_{\mathbb{T}_\kappa}, \\ &\|\gamma^{\diamondsuit_\alpha}(t)\| \neq 0, \ t \in [a,b]^\kappa_{\mathbb{T}_\kappa}. \end{aligned} \tag{3.3}$$

Let γ be a curve given in the parametric form (3.1), $t_0, t \in [a,b]^\kappa_{\mathbb{T}_\kappa}$ and $\alpha \in [0,1]$. We denote by $P_0 = (\gamma_1(t_0), ..., \gamma_n(t_0))$ and $P = (\gamma_1(t), ..., \gamma_n(t))$. We define the point

$$P_0^\alpha = (\alpha\gamma_1(\sigma(t_0)) + (1-\alpha)\gamma_1(\rho(t_0)), ..., \alpha\gamma_n(\sigma(t_0)) + (1-\alpha)\gamma_n(\rho(t_0))).$$

If $\alpha = 1$, then $P_0^1 = P_0^\sigma$, while if $\alpha = 0$, then $P_0^0 = P_0^\rho$ and P_0^σ, P_0^ρ are points belonging to the curve γ. We denote by $d(P, \cdot)$ the distance from the point P to any object from $\mathbb{R}^n$ (point, lines, etc.). Using this notations, we can introduce the notion of diamond-α tangent of a curve.

Definition 3.4. Let γ be a curve, $\alpha \in [0,1]$, $t_0, t \in [a,b]^\kappa_{\mathbb{T}_\kappa}$ and d a line trough the point P_0^α. One says that d is *the diamond-α tangent line* to the curve γ at the point P_0, if the following relation takes place

$$\lim_{P \to P_0,\ P \neq P_0^\alpha} \frac{d(P,d)}{d(P, P_0^\alpha)} = 0. \tag{3.4}$$

If $\alpha = 1$, then the line d is called *delta tangent* or *forward tangent line*, while if $\alpha = 0$, then the line d is called *nabla tangent* or *backward tangent line* to the curve γ at the point P_0.

The following theorem, that generalizes [6, Theorem 3.1], will allow to express the direction vectors of this diamond-α tangent lines, for $n = 3$ and $n = 2$.

Theorem 3.1. *All the diamond-α-regular curves from $\mathbb{R}^3$ and $\mathbb{R}^2$ have, at any point $P_0 = P_0(t_0)$, with $\mu(t_0) = \nu(t_0)$, a diamond-α tangent line that has the direction vector $\gamma^{\diamond_\alpha}(t_0)$.*

Proof. Let γ be a curve from $\mathbb{R}^3$, and the line $d \subset \mathbb{R}^3$ is its diamond-α tangent curve at the point $P_0 = P_0(t_0)$, with $\mu(t_0) = \nu(t_0)$. Let v a direction vector of the line d. The distance from the point $P = P(t)$ to the point P_0^α is equal $\|\gamma(t) - \alpha\gamma(\sigma(t_0)) - (1-\alpha)\gamma(\rho(t_0))\|$, while the distance from P to the line d is equal $\|[\gamma(t) - \alpha\gamma(\sigma(t_0)) - (1-\alpha)\gamma(\rho(t_0))] \times v\|$, where $\times$ denotes the vector product of those two vectors.

We have two cases, according to the density of t_0.

(1) If $\mu(t_0) = \nu(t_0) = 0$, then t_0 is dense and $P_0^\alpha = P_0$, for every $\alpha \in [0,1]$, by Definition 3.4,

$$\begin{aligned}
&\lim_{P \to P_0,\ P \neq P_0} \frac{d(P,d)}{d(P,P_0^\alpha)} \\
&\quad = \lim_{t \to t_0, t \neq t_0} \frac{\|(\gamma(t) - \gamma(t_0)) \times v\|}{\|\gamma(t) - \gamma(t_0)\|} = 0.
\end{aligned}$$

But

$$\begin{aligned}
&\lim_{t \to t_0, t \neq t_0} \frac{\|(\gamma(t) - \gamma(t_0)) \times v\|}{\|\gamma(t) - \gamma(t_0)\|} \\
&\quad = \lim_{t \to t_0, t \neq t_0} \frac{\left\| \frac{\gamma(t) - \gamma(t_0)}{t - t_0} \times v \right\|}{\left\| \frac{\gamma(t) - \gamma(t_0)}{t - t_0} \right\|} \\
&\quad = \frac{\|\gamma'(t_0) \times v\|}{\gamma'(t_0)} = 0.
\end{aligned}$$

As $\gamma'(t_0) = \gamma^{\diamond_\alpha}(t_0) \neq 0$ and $\gamma'(t_0) \times v = 0$, then the vectors $\gamma'(t_0)$ and v are collinear.

(2) If $\mu(t_0) = \nu(t_0) \neq 0$, then t_0 is isolated and, since d is diamond-α tangent line of the curve γ, we have

$$\begin{aligned}
&\lim_{P \to P_0,\ P \neq P_0^\alpha} \frac{d(P,d)}{d(P,P_0^\alpha)} \\
&\quad = \lim_{t \to t_0,\ P \neq P_0^\alpha} \frac{\|[\gamma(t) - \alpha\gamma(\sigma(t_0)) - (1-\alpha)\gamma(\rho(t_0))] \times v\|}{\|\gamma(t) - \alpha\gamma(\sigma(t_0)) - (1-\alpha)\gamma(\rho(t_0))\|} = 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{t \to t_0,\ P \neq P_0^\alpha} \frac{\|[\gamma(t) - \alpha\gamma(\sigma(t_0)) - (1-\alpha)\gamma(\rho(t_0))] \times v\|}{\|\gamma(t) - \alpha\gamma(\sigma(t_0)) - (1-\alpha)\gamma(\rho(t_0))\|} \\
&\quad = \lim_{t \to t_0,\ P \neq P_0^\alpha} \frac{\|\{\alpha[\gamma(t) - \gamma(\sigma(t_0))] + (1-\alpha)[\gamma(t) - \gamma(\rho(t_0))]\} \times v\|}{\|\alpha[\gamma(t) - \gamma(\sigma(t_0))] + (1-\alpha)[\gamma(t) - \gamma(\rho(t_0))]\|} \\
&\quad = \lim_{t \to t_0,\ P \neq P_0^\alpha} \frac{\left\| \left[\frac{\alpha}{t - \rho(t_0)} \frac{\gamma(t) - \gamma(\sigma(t_0))}{t - \sigma(t_0)} + \frac{(1-\alpha)}{t - \sigma(t_0)} \frac{\gamma(t) - \gamma(\rho(t_0))}{t - \rho(t_0)} \right] \times v \right\|}{\left\| \frac{\alpha}{t - \rho(t_0)} \frac{\gamma(t) - \gamma(\sigma(t_0))}{t - \sigma(t_0)} + \frac{(1-\alpha)}{t - \sigma(t_0)} \frac{\gamma(t) - \gamma(\rho(t_0))}{t - \rho(t_0)} \right\|} \\
&\quad = \frac{\left\| \left[\frac{\alpha}{t_0 - \rho(t_0)} \gamma^\Delta(t_0) + \frac{(1-\alpha)}{t_0 - \sigma(t_0)} \gamma^\nabla(t_0) \right] \times v \right\|}{\left\| \frac{\alpha}{t_0 - \rho(t_0)} \gamma^\Delta(t_0) + \frac{(1-\alpha)}{t_0 - \sigma(t_0)} \gamma^\nabla(t_0) \right\|} \\
&\quad = \frac{\|\gamma^{\diamond_\alpha}(t_0) \times v\|}{\|\gamma^{\diamond_\alpha}(t_0)\|}.
\end{aligned}$$

As in the previous case, the fact that $\gamma^{\diamond_\alpha}(t_0) \neq 0$ and $\gamma^{\diamond_\alpha}(t_0) \times v = 0$, gives us the collinearity of the vectors $\gamma^{\diamond_\alpha}(t_0)$ and v. Both cases imply that the vector $\gamma^{\diamond_\alpha}(t_0)$ is the direction vector of the line d. Thus, the equation of the diamond-α tangent line at the point P_0 to the curve γ is

$$\frac{x - \gamma_1(t_0)}{\gamma_1^{\diamond_\alpha}(t_0)} = \frac{y - \gamma_2(t_0)}{\gamma_2^{\diamond_\alpha}(t_0)} = \frac{z - \gamma_3(t_0)}{\gamma_3^{\diamond_\alpha}(t_0)}. \tag{3.5}$$

In order to obtain the equation of a plane curve, we consider $\gamma_3 = c$, constant, in (3.5), and we get

$$\frac{x-\gamma_1(t_0)}{\gamma_1^{\diamondsuit_\alpha}(t_0)} = \frac{y-\gamma_2(t_0)}{\gamma_2^{\diamondsuit_\alpha}(t_0)}. \tag{3.6}$$

From (3.6), we get the equation of diamond-α tangent line, at the point $P_0(x_0)$, with $\mu(x_0) = \nu(x_0)$ to the curve given by the equation

$$y = \gamma(x), \; x \in [a,b]_{\mathbb{T}}.$$

That is the line given by the equation

$$y - \gamma(x_0) = \gamma^{\diamondsuit_\alpha}(x - x_0).$$

$\square$

Remark 3.1. During the proof of Theorem 3.1, we have used the syntax "$\lim\limits_{t\to t_0, P\neq P_0^\alpha}$" instead of "$\lim\limits_{t\to t_0, t\neq t_0}$" since P_0^α is not situated on the curve γ, in general, and the point t for which $P(t) = P_0^\alpha$, if it exists, it does not only depend on t_0, but also on γ.

The following result completes Theorem 3.1, and its proof follows in the same manner, and thus we omit it.

Theorem 3.2. *All the diamond-α-regular curves in $\mathbb{R}^3$ and $\mathbb{R}^2$ have, at any point $P_0 = P_0(t_0)$, a forward tangent line that has the direction vector $\gamma^{\Delta}(t_0)$ and a backward tangent line that has the direction vector $\gamma^{\nabla}(t_0)$.*

Remark 3.2. If $P_0 \neq P_0^\sigma$, then the forward tangent line at the point P_0 of the curve γ is the line through the points P_0 and P_0^σ, and if $P_0 \neq P_0^\rho$, then the backward tangent line at the point P_0 of the curve γ is the line through the points P_0 and P_0^ρ.

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