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Oscillation theorems for non-linear difference equation of the second order

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ABSTRACT.

We obtain some oscillation criteria for the solutions of the non-linear difference equation of the form

$$\Delta\left(r_n\psi\left(x_n\right)f\left(\Delta x_n\right)\right)+q_n\varphi\left(g\left(x_{n+1}\right),r_{n+1}\psi\left(x_{n+1}\right)f\left(\Delta x_{n+1}\right)\right)=0,\ n=0,1,2,...,$$

where $u\,\varphi\left(u,v\right)>0$ for all $u\neq 0$, $x\,g\left(x\right)>0$ and $xf\left(x\right)>0$ for all $x\neq 0$, $\psi\left(x\right)>0$ for all $x\in R$, $\{r_n\}_{n=0}^{\infty}$ is sequence of positive real numbers and $\{q_n\}_{n=0}^{\infty}$ is sequence of real values. The relevance of our theorems becomes clear due to a carefully selected examples.

1. INTRODUCTION

This paper concerned with oscillation of the solution to the second order nonlinear difference equation of the form

$$(E)\qquad \Delta\left(r_n\psi\left(x_n\right)f\left(\Delta x_n\right)\right)+q_n\varphi\left(g\left(x_{n+1}\right),r_{n+1}\psi\left(x_{n+1}\right)f\left(\Delta x_{n+1}\right)\right)=0,$$

$n=0,1,...$, where Δ denotes the forward difference operator $\Delta x_n=x_{n+1}-x_n$ for any sequence $\{x_n\}$of real numbers, the function φ is defined and continuous on $R\times R$ with $u\varphi\left(u,v\right)>0$ and $\dfrac{\partial\varphi\left(u,v\right)}{\partial v}\leq 0$ for all $u\neq 0$ and $v\in R$ and $\varphi\left(\lambda u,\lambda v\right)=\lambda\varphi\left(u,v\right)$, where $\lambda\in(0,\infty)$, the function $g:R\to R$ satisfies $xg\left(x\right)>0$ for all $x\neq 0$ and $g\left(u\right)-g\left(v\right)=g_1\left(u,v\right)\left(u-v\right)^{\delta}$ for $u,v\neq 0$, $\delta>0$ is the ratio of odd positive integers, $g_1\left(u,v\right)\geq 0$ and $g\left(u\right)\geq g\left(v\right)$ iff $u\geq v$, ψ and f are continuous functions on R with $\psi\left(x\right)>0$ for all $x\in R$ and $xf\left(x\right)>0$ for all $x\neq 0$, $\{r_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers and $\{q_n\}_{n=0}^{\infty}$ is a sequence of real valued.

A solution of (E) is a nontrivial real a sequence $\{x_n\}$ satisfying Equation (E) for $n\geq 0$. A solution $\{x_n\}$ of (E) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation(E) is said to be oscillatory if all its solutions are oscillatory.

A prototype of equation (E) is the equation

$$(E_1)\qquad \Delta\left(r_n\psi\left(x_n\right)\left(\Delta x_n\right)^{\gamma}\right)+q_n\varphi\left(g\left(x_{n+1}\right),r_{n+1}\psi\left(x_{n+1}\right)\left(\Delta x_{n+1}\right)^{\gamma}\right)=0,$$

$n=0,1,2,...$, where $\gamma>0$ is the ratio of odd positive integers.

In recent years, the asymptotic behavior of second order non linear difference equations has been the subject of investigations by many authors, see e. g. [1-2, 4-7, 10-27].

A lot of work has been done on the following particular cases of (E)

$$(E_2)\qquad \Delta\left(r_n\left(\Delta x_n\right)^{\gamma}\right)+q_nx_{n+1}^{\gamma}=0,\ n=0,1,...,$$

where $\gamma>0$ is the ratio of odd positive integers,

$$(E_3)\qquad \Delta\left(r_n\Delta x_n\right)+q_ng(x_{n+1})=0,\ n=0,1,...,$$

and

$$(E_4)\qquad \Delta\left(r_n\psi\left(x_n\right)\Delta x_n\right)+q_ng(x_{n+1})=0,\ n=0,1,...\quad .$$

For the equation(E_2), E. Thandapani and K. Ravi [22, Lemma 2], proved that, if there exist positive integers N_0 and N, $N\geq N_0$ such that

$$(1.1)\qquad \sum_{i=N_0}^{\infty}q_i\geq 0\ \text{ and }\ \sum_{i=N}^{\infty}q_i>0\ \forall N\geq N_0,$$

$$(1.2)\qquad \sum_{n=0}^{\infty}\left(\frac{1}{r_n}\right)^{\frac{1}{\gamma}}=\infty,$$

and $\{x_n\}$ is a non-oscillatory solution of equation (E_2) such that $x_n>0$ for all $n\geq N$, then there exists an integer $N_1\geq N$ such that $\Delta x_n>0$ for all $n\geq N_1$.

The oscillatory behavior of the equation(E_3) and particular cases of it were studied by many authors (e. g. see [13, 15, 16, 19, 25, 26]).

El-Sheikh et al. [6], studied the oscillatory and nonoscillatory solutions of the equation (E_4).

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We remark that qualitative properties of the differential equation (E)

$$\left[r(t)\,\psi(x(t))\,f(\dot{x}(t))\right]^{\cdot} + q(t)\,\varphi(g(x(t)), r(t)\,\psi(x(t))\,f(\dot{x}(t))) = 0, \tag{1.3}$$

when $r(t) = \psi(x) = 1$ and $f(x) = g(x) = x$ have been considered by many authors. We mention in particular to Bihari [3] and Kartsatos [9].

In this paper, we intend to use the Riccati transformation technique for obtaining several new oscillation criteria for (E), which can be considered as the discrete analogues of the results in [3, 9].

2. Main Results

In this section, we will use the Riccati technique to establish sufficient conditions for the oscillation of (E).

Theorem 2.1. *Assume that there exists a constant $C_1 \in R_+$ such that*

$$\Phi(m) = \int_0^m \frac{dv}{\varphi(1,v)} \geq -C_1 \text{ for every } m \in R, \tag{2.4}$$

and

$$\limsup_{n\to\infty} \sum_{i=n_0}^{n-1} q_i = \infty. \tag{2.5}$$

Then every solution of equation (E) oscillates.

Proof. Suppose to the contrary that $\{x_n\}$ is a nonoscillatory solution of (E). Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E) such that $x_n > 0$, $n \geq n_0$.
Define the sequence $\{w_n\}$ by

$$w_n = \frac{r_n\psi(x_n)\,f(\Delta x_n)}{g(x_n)}, n \geq n_0.$$

Then, for all $n \geq n_0$, we have

$$\Delta w_n = \frac{\Delta(r_n\psi(x_n)\,f(\Delta x_n))}{g(x_{n+1})} - r_n\psi(x_n)\,f(\Delta x_n)\frac{\Delta(g(x_n))}{g(x_n)\,g(x_{n+1})}.$$

This and (E) imply

$$\Delta w_n = -\varphi(1, w_{n+1})\,q_n - r_n\psi(x_n)\,f(\Delta x_n)\frac{g_1(x_{n+1}, x_n)(\Delta x_n)^{\delta}}{g(x_n)\,g(x_{n+1})}.$$

Hence, for all $n \geq n_0$, we obtain

$$\Delta w_n \leq -\varphi(1, w_{n+1})\,q_n,$$

or

$$\varphi(1, w_{n+1})\,q_n \leq -\Delta w_n,\ n \geq n_0.$$

Dividing this inequality by $\varphi(1, w_{n+1}) > 0$, we obtain

$$q_n \leq -\frac{\Delta w_n}{\varphi(1, w_{n+1})},\ n \geq n_0. \tag{2.6}$$

Summing (2.6) from n_0 to $n-1$, we have

$$\sum_{m=n_0}^{n-1} q_m \leq -\sum_{l=n_0}^{n-1}\frac{\Delta w_l}{F(w_{l+1})}, \text{ where } F(w_n) = \varphi(1, w_n). \tag{2.7}$$

Define $\delta(t) = w_l + (t-l)\,\Delta w_l$, $t \in [l, l+1]$. Then, we have one of the following two cases:

Case (1). If $\Delta w_l \geq 0$, then $w_l \leq \delta(t) \leq w_{l+1}$. Thus in view of the definition of the function φ, we get

$$\frac{\Delta w_l}{F(w_l)} \leq \frac{\delta'(t)}{F(\delta(t))} \leq \frac{\Delta w_l}{F(w_{l+1})}. \tag{2.8}$$

Case (2). If $\Delta w_l \leq 0$, then $w_{l+1} \leq \delta(t) \leq w_l$. So we can directly obtain (2.8). Now, by (2.7) and (2.8) we get

$$\begin{aligned}\sum_{m=n_0}^{n-1} q_m &\leq -\int_{n_0}^{n}\frac{d(\delta(t))}{F(\delta(t))} = -\int_{\delta(n_0)}^{\delta(n)}\frac{du}{\varphi(1,u)} \\ &\leq -[\Phi(\delta(n)) - \Phi(\delta(n_0))] \leq C_1 + \Phi(\delta(n_0)) = C_1 + \Phi(w_{n_0}).\end{aligned} \tag{2.9}$$

Taking the limit superior on both sides for (2.9), we obtain

$$\limsup_{n\to\infty} \sum_{i=n_0}^{n-1} q_i < \infty,$$

which contradicts (2.5). Hence, the proof is completed. □

Theorem 2.2. *Assume that $f(x) \geq bx$ for all $x \in R$ and for some constant $b > 0$. Furthermore, assume that*

$$\liminf_{|w|\to\infty} \varphi(1, w) = C > 0, \tag{2.10}$$

$$\text{the function } \left(\frac{\psi}{g}\right) \text{ is nonincreasing for all } x \neq 0, \tag{2.11}$$

$$\int_0^{\pm\varepsilon} \left(\frac{\psi(u)}{g(u)}\right) du < \infty \text{ for all } \varepsilon > 0, \tag{2.12}$$

$$\limsup_{n\to\infty} \sum_{m=n_0}^{n-1} \frac{1}{r_m} < \infty, \tag{2.13}$$

and

$$\limsup_{n\to\infty} \sum_{m=n_0}^{n-1} \left(\frac{1}{r_m}\left(\sum_{i=n_0}^{m-1} q_i\right)\right) = \infty. \tag{2.14}$$

Then every solution of equation (E) oscillates.

Proof. Suppose to the contrary that $\{x_n\}$ is a nonoscillatory solution of (E). Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E) such that $x_n > 0$, $n \geq n_0$. Define the sequence $\{w_n\}$ as in the proof of the previous theorem.

Following the same procedure, we get

$$q_n \leq -\frac{\Delta w_n}{\varphi(1, w_{n+1})}, \; n \geq n_0. \tag{2.15}$$

Now, we have one of the following two cases

Case (1). If $\Delta w_n \geq 0$, then $w_{n+1} \geq w_n \geq w_{n_0}$.
Thus in view of the definition of the function φ, we get

$$-\frac{\Delta w_n}{F(w_{n+1})} \leq -\frac{\Delta w_n}{F(w_{n_0})}, \; n \geq n_0. \tag{2.16}$$

Case (2). If $\Delta w_l \leq 0$, then $w_{n+1} \leq w_n \leq w_{n_0}$.
So, by the definition of the function φ and the condition (2.10) we can directly obtain (2.16). Now, by (2.15) and (2.16), we get

$$\sum_{l=n_0}^{n-1} q_l \leq -\frac{1}{F(w_{n_0})} \sum_{l=n_0}^{n-1} \Delta w_l.$$

Then, for all $n \geq n_0$, we have

$$\sum_{l=n_0}^{n-1} q_l \leq -\frac{1}{C_0}(w_n - w_{n_0}), \quad \text{where} \quad F(w_{n_0}) = C_0 > 0.$$

Hence, for all $n \geq n_0$, we obtain

$$\frac{w_n}{C_0} \leq \frac{w_{n_0}}{C_0} - \sum_{l=n_0}^{n-1} q_l = C_2 - \sum_{l=n_0}^{n-1} q_l, \quad \text{where} \quad C_2 = \frac{w_{n_0}}{C_0}.$$

Then,

$$C_0^{-1} \frac{r_n \psi(x_n) f(\Delta x_n)}{g(x_n)} - C_2 \leq -\sum_{l=n_0}^{n-1} q_l.$$

Hence, for all $n \geq n_0$, we obtain

$$\frac{b}{C_0}\left(\frac{\psi(x_n)}{g(x_n)}\right)\Delta x_n - \frac{C_2}{r_n} \leq C_0^{-1}\left(\frac{\psi(x_n)}{g(x_n)}\right) f(\Delta x_n) - \frac{C_2}{r_n} \leq -\frac{1}{r_n}\sum_{l=n_0}^{n-1} q_l.$$

Summing the above inequality from n_0 to $n-1$, we have

$$C_3 \sum_{l=n_0}^{n-1} \left(\frac{\psi(x_l)}{g(x_l)}\right)\Delta x_l - C_2 \sum_{l=n_0}^{n-1} \frac{1}{r_l} \leq -\sum_{l=n_0}^{n-1}\left(\frac{1}{r_l}\sum_{m=n_0}^{l-1} q_m\right), \quad \text{where} \quad C_3 = \frac{b}{C_0}. \tag{2.17}$$

Define $\delta(t) = x_l + (t-l)\Delta x_l$, $t \in [l, l+1]$. Then we have one of the following two cases:

Case (1). If $\Delta x_l \geq 0$, then $x_l \leq \delta(t) \leq x_{l+1}$. Thus in view of the assumption (2.8) we get

$$\left(\frac{\psi(x_l)}{g(x_l)}\right)\Delta x_l \geq \frac{\psi(\delta(t))}{g(\delta(t))}\delta'(t) \geq \left(\frac{\psi(x_{l+1})}{g(x_{l+1})}\right)\Delta x_l. \tag{2.18}$$

Case (2). If $\Delta x_l \leq 0$, then $x_{l+1} \leq \delta(t) \leq x_l$. So we can directly obtain (2.18). Now, by (2.17) and (2.18) we get

$$C_3 \int_{n_0}^{n} \left(\frac{\psi}{g}\right)(\delta(t))\, d(\delta(t)) \leq C_2 \sum_{l=n_0}^{n-1} \frac{1}{r_l} - \sum_{l=n_o}^{n-1}\left(\frac{1}{r_l}\sum_{m=n_0}^{l-1} q_m\right).$$

Then, for all $n \geq n_0$, we obtain

$$C_3 \int_{\delta(n_0)}^{\delta(n)} \frac{\psi(u)}{g(u)}du \leq C_2 \sum_{l=n_0}^{n-1} \frac{1}{r_l} - \sum_{l=n_o}^{n-1}\left(\frac{1}{r_l}\sum_{m=n_0}^{l-1} q_m\right),$$

which implies that

$$\int_{\delta(n_0)}^{\delta(n)} \frac{\psi(u)}{g(u)}du \to -\infty \text{ as } n \to \infty.$$

Now, if $\delta(n) \geq \delta(n_0)$ for large n, then $\int_{\delta(n_0)}^{\delta(n)} \frac{\psi(u)}{g(u)}du \geq 0$, which a contradiction.

Hence, for large n, $\delta(n) \leq \delta(n_0)$, so

$$-\int_{\delta(n)}^{\delta(n_0)} \frac{\psi(u)}{g(u)}du \geq -\int_{0}^{\delta(n_0)} \frac{\psi(u)}{g(u)}du > -\infty,$$

which is again a contradiction. This completes the proof of Theorem 2.2. □

In the following, we state and prove some lemmas which will be needed later on.

Lemma 2.1. *Assume that there exist positive integers N_0 and N, $N \geq N_0$ such that*

$$\sum_{i=N_0}^{\infty} q_i \geq 0 \quad \text{and} \quad \sum_{i=N}^{\infty} q_i > 0\,,\ \forall\, N \geq N_0. \tag{2.19}$$

Then there exist an integer $N_1 \geq N$ such that

$$\sum_{i=N_1}^{n} q_i \geq 0,\ \ \forall\, n > N_1.$$

The proof of the above lemma can be found in [7, lemma 2.1].

Lemma 2.2. *In addition to the conditions (2.8) and (2.19) assume that*

$$\begin{aligned} F(u) - F(v) = F_1(u,v)(u-v)\,, \text{ for } u,v \neq 0,\ F_1(u,v) \leq K_2 < 0 \text{ and} \\ F(u) \geq F(v) \text{ iff } u \leq v, \text{ where } F(w) = \varphi(1,w)\,. \end{aligned} \tag{2.20}$$

Also, assume that

$$\int_{0}^{\pm\varepsilon} \left(\frac{\psi(u)}{g(u)}\right)^{\frac{1}{\gamma}} du < \infty \text{ for all } \varepsilon > 0, \tag{2.21}$$

and

$$\sum_{n=0}^{\infty} \left(\frac{1}{r_n}\right)^{\frac{1}{\gamma}} = \infty. \tag{2.22}$$

If $\{x_n\}$ is a non-oscillatory solution of equation (E_1) such that $x_n > 0$ for all $n \geq N$, then there exists an integer $N_1 \geq N$ such that $\Delta x_n > 0$ for all $n \geq N_1$.

Proof. If not, assume first that $\triangle x_n < 0$ for all large n, say $n \geq N_1 \geq N$. Without loss of generality, we may assume that (2.19) holds for $n \geq N_1$ and $q_{N_1} \geq 0$. Define

$$Q_n = \sum_{l=N_1}^{n} q_l \quad \text{for } n \geq N_1 \text{ and } Q_{N_1-1} = 0. \tag{2.23}$$

We have then,

$$\begin{aligned}\sum_{l=N_1}^{n} q_l F(w_{l+1}) &= \sum_{l=N_1}^{n} F(w_{l+1})\,\Delta Q_{l-1} = \sum_{l=N_1}^{n}\left[\Delta\left(F(w_l)\,Q_{l-1}\right) - Q_{l-1}\Delta F(w_l)\right]\\
&= F(w_{n+1})\,Q_n - F(w_{N_1})\,Q_{N_1-1} - \sum_{l=N_1}^{n} Q_{l-1}\Delta F(w_l)\\
&= F(w_{n+1})\,Q_n - \sum_{l=N_1}^{n}\left(\left(F(w_{l+1}) - F(w_l)\right)Q_{l-1}\right)\\
&= F(w_{n+1})\,Q_n - \sum_{l=N_1}^{n}\left(F_1(w_{l+1}, w_l)\,\Delta w_l Q_{l-1}\right)\\
&\geq -\sum_{l=N_1}^{n}\left(F_1(w_{l+1}, w_l)\,\Delta w_l Q_{l-1}\right), \text{ where } w_l = \frac{r_l\psi(x_l)(\Delta x_l)^{\gamma}}{g(x_l)}.\end{aligned}$$

From equation (E_1), we get

$$\sum_{l=N_1}^{n}\frac{\Delta\left(r_l\,\psi(x_l)(\Delta x_l)^{\gamma}\right)}{g(x_{l+1})} - \sum_{l=N_1}^{n}\left(F_1(w_{l+1}, w_l)\,\Delta w_l Q_{l-1}\right) \leq 0.$$

Hence,

$$\sum_{l=N_1}^{n}\Delta w_l - \sum_{l=N_1}^{n}\left(F_1(w_{l+1}, w_l)\,\Delta w_l Q_{l-1}\right) \leq 0.$$

Thus,

$$w_{n+1} - w_{N_1} - \sum_{l=N_1}^{n}\left(F_1(w_{l+1}, w_l)\,\Delta w_l Q_{l-1}\right) \leq 0.$$

Then,

$$w_{n+1} - \sum_{l=N_1}^{n}\left(F_1(w_{l+1}, w_l)\,\Delta w_l Q_{l-1}\right) \leq 0. \tag{2.24}$$

We define

$$h_{n+1} = w_{n+1} - \sum_{l=N_1}^{n} F_1(w_{l+1}, w_l)\,\Delta w_l Q_{l-1}. \tag{2.25}$$

Then,

$$\Delta h_{n+1} = \Delta w_{n+1} - F_1(w_{n+2}, w_{n+1})\,\Delta w_{n+1} Q_n + F_1(w_{N_1+1}, w_{N_1})\,\Delta w_{N_1} Q_{N_1-1}.$$

Thus,

$$\Delta h_{n+1} = \Delta w_{n+1}\left(1 - F_1(w_{n+2}, w_{n+1})\,Q_n\right). \tag{2.26}$$

Assume that, $\Delta h_{n+1} \leq 0$ for all $n \geq N_1$. Since, $(1 - F_1(w_{n+2}, w_{n+1})\,Q_n) > 0$ for all $n \geq N_1$,

$$\Delta w_{n+1} \leq 0,\; n \geq N_1. \tag{2.27}$$

Summing (2.27) from N_1 to $n-1$, we obtain

$$w_{n+1} \leq w_{N_1+1} < 0,\; n \geq N_1.$$

Then, for all $n \geq N_1$, we have

$$\left(\frac{\psi(x_{n+1})}{g(x_{n+1})}\right)^{\frac{1}{\gamma}}\Delta x_{n+1} \leq -\delta_1\left(\frac{1}{r_{n+1}}\right)^{\frac{1}{\gamma}}, \text{ where } \delta_1 = -(w_{N_1+1})^{\frac{1}{\gamma}} > 0.$$

Summing the above inequality from N_1 to $n-1$, we have

$$\sum_{l=N_1}^{n-1}\left(\frac{\psi(x_{l+1})}{g(x_{l+1})}\right)^{\frac{1}{\gamma}}\Delta x_{l+1} \leq -\delta_1\sum_{l=N_1}^{n-1}\left(\frac{1}{r_{l+1}}\right)^{\frac{1}{\gamma}} \leq -\delta_1\sum_{l=N_1+1}^{n}\left(\frac{1}{r_l}\right)^{\frac{1}{\gamma}}. \tag{2.28}$$

Define $\delta(t) = x_{l+1} + (t-l)\,\Delta x_{l+1},\; t \in [l+1, l+2]$. Since $\Delta x_{l+1} < 0$, $x_{l+2} \leq \delta(t) \leq x_{l+1}$. Thus in view of the assumption (2.11), we get

$$\frac{\psi(x_{l+2})}{g(x_{l+2})} \geq \frac{\psi(\delta(t))}{g(\delta(t))} \geq \frac{\psi(x_{l+1})}{g(x_{l+1})}.$$

Then

$$\frac{\psi(x_{l+2})}{g(x_{l+2})}\Delta x_{l+1} \le \frac{\psi(\delta(t))}{g(\delta(t))}\delta(t) \le \frac{\psi(x_{l+1})}{g(x_{l+1})}\Delta x_{l+1}. \tag{2.29}$$

Now, by (2.28) and (2.29), we get

$$\int_{n_0}^{n}\left(\frac{\psi(\delta(t))}{g(\delta(t))}\right)^{\frac{1}{\gamma}} d(\delta(t)) \le -\delta_1 \sum_{l=N_1+1}^{n}\left(\frac{1}{r_l}\right)^{\frac{1}{\gamma}}.$$

Which implies that

$$\int_{\delta(n_0)}^{\delta(n)}\left(\frac{\psi(u)}{g(u)}\right)^{\frac{1}{\gamma}} du \to -\infty \text{ as } n \to \infty.$$

Since for all large n, $\Delta x_n < 0$, $\delta(n) < \delta(n_0)$ for all large n, so

$$-\int_{\delta(n)}^{\delta(n_0)}\left(\frac{\psi(u)}{g(u)}\right)^{\frac{1}{\gamma}} du \ge -\int_{0}^{\delta(n_0)}\left(\frac{\psi(u)}{g(u)}\right)^{\frac{1}{\gamma}} du > -\infty,$$

which a contradiction.

Assume that $\Delta h_{n+1} \ge 0$ for all $n \ge N_1$. Since, $(1 - F_1(w_{n+2}, w_{n+1}) Q_n) > 0$ for all $n \ge N_1$,

$$\Delta w_{n+1} \ge 0, \ n \ge N_1.$$

Hence,

$$-\varphi(1, w_{n+2}) q_{n+1} - r_{n+1}\psi(x_{n+1})(\Delta x_{n+1})^{\gamma}\frac{\Delta g(x_{n+1})}{g(x_{n+1}) g(x_{n+1})} \ge 0.$$

Then,

$$-\varphi(1, w_{n+2}) q_{n+1} \ge 0 \quad \text{for all } n \ge N_1.$$

Since $\varphi(1, w_{n+2}) > 0$,

$$q_{n+1} \le 0 \ \text{ for all } n \ge N_1.$$

Summing the above inequality from N_1 to $n-1$, we have $\sum\limits_{l=N_1}^{n-1} q_{l+1} \le 0$.

Thus, $\sum\limits_{l=N_1+1}^{n} q_l \le 0$, which contradicts (2.19).

Next, assume that Δx_n is oscillatory for $n \ge N_2 \ge N_1 \ge N_0$. Then there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ with $\lim\limits_{k\to\infty} n_k = \infty$ and such that $\Delta x_{n_k} = 0$, $k = 1, 2, 3, ...,$
Letting

$$w_n = \frac{r_n\psi(x_n)(\Delta x_n)^{\gamma}}{g(x_n)}, n \ge N_2.$$

Then, for all $n \ge N_2$, we obtain

$$\Delta w_n = \frac{\Delta(r_n\psi(x_n)(\Delta x_n)^{\gamma})}{g(x_{n+1})} - \frac{r_n\psi(x_n)(\Delta x_n)^{\gamma}\Delta g(x_n)}{g(x_n) g(x_{n+1})}.$$

This and (E_1) imply

$$\Delta w_n \le -\varphi(1, w_{n+1}) q_n, \ n \ge N_2.$$

Dividing this inequality by $\varphi(1, w_{n+1}) > 0$, we obtain

$$q_n \le -\frac{\Delta w_n}{\varphi(1, w_{n+1})}, \ n \ge N_2. \tag{2.30}$$

Summing (2.30) from n_1 to $n_k - 1$, we have

$$\sum_{l=n_1}^{n_k-1} q_l \le -\sum_{l=n_1}^{n_k-1}\frac{\Delta w_l}{\varphi(1, w_{l+1})}. \tag{2.31}$$

By (2.8) and (2.31), we get

$$\sum_{l=n_1}^{n_k-1} q_l \le -\int_{n_1}^{n_k} \frac{d\left(\delta\left(t\right)\right)}{F\left(\delta\left(t\right)\right)} \le -\int_{\delta(n_1)}^{\delta(n_k)} \frac{du}{\varphi\left(1,u\right)} = -\int_{w_{n_1}}^{w_{n_k}} \frac{du}{\varphi\left(1,u\right)}$$
$$\le -\int_{0}^{0} \frac{du}{\varphi\left(1,u\right)} \le 0,$$

which contradicts (2.19). Hence $\Delta x_n > 0$ for all $n \ge N_1$. □

Theorem 2.3. *Suppose that* (2.11), (2.19), (2.20), (2.21) *and* (2.22) *hold. Furthermore, assume that, there exists* $\lambda \ge 1$ *such that*

$$\limsup_{m\longrightarrow\infty} \frac{1}{m^\lambda} \sum_{n=n_0}^{m-1} (m-n)^\lambda q_n = \infty. \tag{2.32}$$

Then every solution of Eq. (E_1) *oscillates.*

Proof. Suppose to the contrary that $\{x_n\}$ is a non oscillatory solution of (E_1). Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E_1), such that $x_n > 0$ for all large n. In view of Lemma 2.2, we see that, there is some $n_1 \ge n_0$ such that

$$x_n > 0,\ \Delta x_n > 0,\ n \ge n_1.$$

Define the sequence $\{w_n\}$ by

$$w_n = \frac{r_n \psi(x_n)(\Delta x_n)^\gamma}{g(x_n)},\ n \ge n_1.$$

Then $w_n > 0$ and

$$\Delta w_n \le -\varphi(1, w_{n+1}) q_n.$$

Dividing this inequality by $\varphi(1, w_{n+1}) > 0$, we obtain

$$q_n \le -\frac{\Delta w_n}{\varphi(1, w_{n+1})},\ n \ge n_1.$$

As in the proof of Theorem 2.2, we can obtain the following inequality

$$C_0 \sum_{n=n_1}^{m-1} (m-n)^\lambda q_n \le -\sum_{n=n_1}^{m-1} (m-n)^\lambda \Delta w_n. \tag{2.33}$$

But

$$-\sum_{n=n_1}^{m-1} (m-n)^\lambda \Delta w_n = (m-n_1)^\lambda w_{n_1} - \sum_{n=n_1}^{m-1} w_{n+1}\left[(m-n)^\lambda - (m-n-1)^\lambda\right].$$

By means of the well-known inequality [8]

$$x^\beta - y^\beta \ge \beta y^{\beta-1}(x-y) \text{ for all } x \ge y > 0 \text{ and } \beta \ge 1,$$

we have

$$-\sum_{n=n_1}^{m-1} (m-n)^\lambda \Delta w_n \le (m-n_1)^\lambda w_{n_1} - \sum_{n=n_1}^{m-1} \lambda w_{n+1}(m-n-1)^{\lambda-1} \tag{2.34}$$
$$\le (m-n_1)^\lambda w_{n_1}.$$

Then, by (2.33) and (2.34), we get

$$C_0 \sum_{n=n_1}^{m-1} (m-n)^\lambda q_n \le (m-n_1)^\lambda w_{n_1},$$

which implies that

$$C_0 \frac{1}{m^\lambda} \sum_{n=n_1}^{m-1} (m-n)^\lambda q_n \le \left(\frac{m-n_1}{m}\right)^\lambda w_{n_1}.$$

Hence,

$$\limsup_{m\longrightarrow\infty} \frac{1}{m^\lambda} \sum_{n=n_1}^{m-1} (m-n)^\lambda q_n < \infty,$$

which is contrary to (2.32). The proof is complete. □

3. Examples

In this section, we give some examples which illustrate our main results.

Example 3.1. Consider the difference equation

$$\Delta\left(n\left(1+|x_n|\right)\left(\Delta x_n\right)^3\right)+16\left(2n+1\right)e^{16n}x_{n+1}^3e^{-\frac{n(1+|x_n|)(\Delta x_n)^3}{x_{n+1}^3}}=0,\ n\geq 1. \tag{3.35}$$

Here, $r_n=n$, $q_n=16\left(2n+1\right)e^{16n}$, $\psi\left(x\right)=1+|x|$, $f\left(x\right)=x^3$, $g\left(x\right)=x^3$ and $\varphi\left(u,v\right)=ue^{-\frac{v}{u}}$. All conditions of Theorem 2.1 are satisfied, and hence, all solutions of equation (3.35) are oscillatory. In fact, $x_n=\left(-1\right)^n$ is such a solution of equation (3.35).

Example 3.2. Consider the difference equation

$$\Delta\left(n^2(\Delta x_n)\right)+2(2n^2+2n+1)x_{n+1}^{\frac{1}{3}}=0,\ n\geq 1. \tag{3.36}$$

Here, $r_n=n^2$, $q_n=2(2n^2+2n+1)$, $\psi\left(x\right)=1$, $f\left(x\right)=x$, $g\left(x\right)=x^{\frac{1}{3}}$ and $\varphi\left(u,v\right)=u$ and $[\inf_{|w|\to\infty}\varphi\left(1,w\right)=1]$. All conditions of Theorem 2.2 are satisfied, and hence, all solutions of equation (3.36) are oscillatory. In fact, $x_n=\left(-1\right)^n$ is such a solution of equation (3.36).

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