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Approximate solutions of boundary value problems for ODEs using Newton interpolating series

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ABSTRACT.

The paper deals with the study of approximate solutions, which are entire functions, of multipoint boundary value problems for differential equations. The solutions are represented as Newton interpolating series.

1. INTRODUCTION

In this paper the power series used in the theory of initial value problems for differential equations are replaced by Newton interpolating series, defined in Section 2, in order to find the solutions of multipoint boundary value problems for differential equations.

We use Newton interpolating series to approximate solutions which are entire functions. Theorem 2.1 gives the connection between an entire Newton interpolating series and its derivative series. In Section 3 we study entire functions which are represented as entire Newton interpolating series (see Theorem 3.2). Two applications of the method based on Newton interpolating series to construct approximate solutions of boundary value problems are given in the last section.

2. NEWTON INTERPOLATING SERIES

Let $S = \{x_k\}_{k\geq 1}$ be a sequence of real numbers. We construct the polynomials

$$u_i = \prod_{k=1}^{i} (X - x_k),\ i = 1, 2, ..., u_0 = 1, \tag{2.1}$$

and we denote also by $u_i = u_i(x)$, where x is a real variable, the polynomial function defined by u_i. We call an infinite series of the form

$$\sum_{i=0}^{\infty} a_i u_i, \tag{2.2}$$

where $a_i \in \mathbb{R}$, a *Newton interpolating series* with coefficients a_i at S.

For any sequence $S = \{x_k\}_{k\geq 1}$ we define the set

$$I_S = \{i \mid x_i \neq x_j \text{ for all } \ j < i\}. \tag{2.3}$$

We call the sequence S *purely periodic* if I_S is a finite set and there exists a positive integer m such that for each positive integer i less or equal to m $x_i = x_{i+jm},\ j = 1, 2, \ldots$. If S is purely periodic the Newton interpolating series at S defined by (2.2) is called also *purely periodic*. More on the properties of purely periodic Newton interpolating series can be found in [2] and [3].

Consider a Newton interpolating series at S given by (2.2). Then for every $i \geq 1$

$$\frac{u_i'(x)}{u_i(x)} = \sum_{k=1}^{i} \frac{1}{x - x_k}. \tag{2.4}$$

Since, for every $i \geq 1$ there exist the real numbers $A_{j,i}$, $j = 1, 2, ..., i$, uniquely determined such that

$$\sum_{k=1}^{i} \frac{1}{x-x_k} = \frac{A_{1,i}}{x-x_i} + \frac{A_{2,i}}{(x-x_i)(x-x_{i-1})} + \ldots + \frac{A_{i,i}}{(x-x_i)(x-x_{i-1})\ldots(x-x_1)} \tag{2.5}$$

it follows that

$$u_i(x) \sum_{k=1}^{i} \frac{1}{x - x_k} = \sum_{k=1}^{i} A_{i-k+1,i} u_{k-1}(x). \tag{2.6}$$

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We suppose that, for every $i \geq 0$, the series $\sum\limits_{k=i+1}^{\infty} A_{k-i,k} a_k$ converges and we denote by $a_i^{(1)}$ its sum. Then the Newton interpolating series

$$\sum_{i=0}^{\infty} a_i^{(1)} u_i, \tag{2.7}$$

where

$$a_i^{(1)} = \sum_{k=i+1}^{\infty} A_{k-i,k} a_k \tag{2.8}$$

is called the *derivative series* of (2.2). If

$$\lim_{i\to\infty} |a_i|^{\frac{1}{i}} = 0, \tag{2.9}$$

the series (2.2) is called an *entire Newton interpolating series*. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function and $S = \{x_k\}_{k\geq 1}$ is a sequence of distinct real numbers. We denote by $f_{i_1,i_2,\ldots,i_s}$ the divided difference with respect to s distinct points $x_{i_1}, \ldots, x_{i_s}$. Thus $f_j = f(x_j)$, $f_{j,k} = \dfrac{f_k - f_j}{x_k - x_j}$ and generally $f_{i_1,\ldots,i_s} = \dfrac{f_{i_2,\ldots,i_s} - f_{i_1,\ldots,i_{s-1}}}{x_{i_s} - x_{i_1}}$.

Now we can prove the following result.

Theorem 2.1. *Let $S = \{x_k\}_{k\geq 1}$ be a bounded sequence of distinct real numbers and let (2.2) be an entire Newton interpolating series at S. Then the series and its derivative series converge absolutely at every $x \in \mathbb{R}$. Moreover, if $f = f(x)$ and respectively $g = g(x)$ are their sums, then $f(x)$ is differentiable and $g = f'$.*

Proof. We choose a positive constant M such that $|x_i| \leq M$ for all i. If x is a real number, then

$$|a_i u_i(x)| \leq |a_i|(|x| + M)^i \tag{2.10}$$

and by (2.9) it follows that the series (2.2) converges absolutely for all real numbers x. By (2.9) we can write for every i $|a_i| = \theta_i^i$, where $\theta_i \geq 0$ and $\lim\limits_{i\to\infty} \theta_i = 0$. Now, by (2.8), $|a_i^{(1)}| \leq \sum\limits_{k=i+1}^{\infty} |A_{k-i,k}|\theta_k^k$. Since the numbers x_i are distinct, by (2.6) and Newton interpolation formula, it follows that

$$A_{i-k+1,i} = h_{1,2,\ldots,k} = \sum_{s=1}^{k} \frac{h_s}{\prod\limits_{j=1,j\neq s}^{k} (x_s - x_j)} = \sum_{s=1}^{k} \prod_{j=k+1}^{i} (x_s - x_j), \tag{2.11}$$

where $h(x) = u_i(x) \sum\limits_{k=1}^{i} \dfrac{1}{x - x_k}$ is a polynomial function which depends of i. If we denote $\delta_k = \max\limits_{j\geq k}\{\theta_j\}$, then $\theta_k \leq \delta_k$, $\delta_{k+1} \leq \delta_k$, $k = 1, 2, \ldots$, and $\lim\limits_{i\to\infty} \delta_i = 0$. We choose i_1 such that for all $i \geq i_1$ $|2M\delta_{i+1}| < 1$. Since $A_{k-i,k} = \sum\limits_{s=1}^{i+1} \prod\limits_{j=i+2}^{k} (x_s - x_j)$ we obtain

$$|A_{k-i,k} a_k| \leq (i + 1)(2M)^{k-i-1}\delta_k^k \tag{2.12}$$

and the series $\sum\limits_{k=i+1}^{\infty} A_{k-i,k} a_k$ converges. Moreover

$$|a_i^{(1)}| \leq \sum_{k=i+1}^{\infty} (i + 1)(2M)^{k-i-1}\delta_k^k \leq \frac{i + 1}{(2M)^{i+1}} \sum_{k=i+1}^{\infty} (2M\delta_{i+1})^k \leq \frac{(i + 1)\delta_{i+1}^{i+1}}{1 - 2M\delta_{i+1}}$$

which implies that $\lim\limits_{i\to\infty} |a_i^{(1)}|^{\frac{1}{i}} = 0$. Hence the derivative series converges absolutely at every $x \in \mathbb{R}$. Now by (2.4), (2.6) and (2.8) we obtain $\sum\limits_{i=0}^{n-1} a_i^{(1)} u_i(x) = \sum\limits_{i=0}^{n-1} \left(\sum\limits_{k=i+1}^{\infty} A_{k-i,k} a_k \right) u_i(x) = \sum\limits_{j=1}^{n} a_j u_j'(x) + \sum\limits_{j=n+1}^{\infty} \left(\sum\limits_{k=0}^{n-1} A_{j-k,j} u_k(x) \right) a_j$. Hence, because $|u_i'(x)| \leq i(|x| + M)^{i-1}$, by (2.10) and (2.12) it follows easily that $f(x)$ is differentiable and $g = f'$. □

Remark 2.1. By the proof of Theorem 2.1 it follows that the derivative series can be obtained by termwise differentiation of the series (2.2) and the series converges uniformly on every compact interval.

3. Functions represented into Newton interpolating series

Consider $S = \{x_k\}_{k\geq 1}$ a sequence of real numbers and a function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval. We say that f *can be represented into Newton interpolating series at* S if there exists a series of the form (2.2) which converges uniformly to f on I. A function $g : \mathbb{R} \to \mathbb{R}$ is called an *entire function* if

$$g(x) = \sum_{i=0}^{\infty} b_i x^i, \tag{3.13}$$

where b_i are real numbers such that $\lim_{i\to\infty} |b_i|^{\frac{1}{i}} = 0$.

For every non-negative integer j and s we construct the set

$$F_{j,s} = \{(\theta_1, .., \theta_{s+1}) \in \mathbb{N}^{s+1} : \theta_1 + ... + \theta_{s+1} = j\} \tag{3.14}$$

and the homogenous polynomials

$$P_{j,s}(X_1, ..., X_{s+1}) = \sum_{(\theta_1,...,\theta_{s+1})\in F_{j,s}} X_1^{\theta_1} ... X_{s+1}^{\theta_{s+1}}. \tag{3.15}$$

Then it follows immediately that for every non-negative integers n and i

$$P_{n,i}(X_1, ..., X_{i+1}) = \sum_{j=0}^{n} P_{n-j,i-1}(X_1, ..., X_i) X_{i+1}^j, \tag{3.16}$$

where $P_{0,i}(X_1, ..., X_{i+1}) = 1$.

Lemma 3.1. *Let $g = \sum_{i=0}^{\infty} b_i x^i$ be an entire function and $S = \{x_k\}_{k\geq 1}$ a sequence of distinct real numbers. Then for each s*

$$g_{1,2,...,s+1} = \sum_{n=s}^{\infty} b_n P_{n-s,s}(x_1, ..., x_{s+1}). \tag{3.17}$$

Proof. Since g is an entire function, the series $\sum_{j=0}^{\infty} b_j x_1^j$ converges and $g_1 = g(x_1) = \sum_{j=0}^{\infty} b_j P_{j,0}(x_1)$, with $P_{j,0}(x_1) = x_1^j$. Then $g_{1,2} = \dfrac{g(x_2) - g(x_1)}{x_2 - x_1} = \sum_{j=1}^{\infty} b_j \dfrac{x_2^j - x_1^j}{x_2 - x_1} = \sum_{j=1}^{\infty} b_j P_{j-1,1}(x_1, x_2)$ with $P_{0,1}(x_1, x_2) = 1$ and for $j \geq 1$ $P_{j,1}(x_1, x_2) = x_1^j + x_1^{j-1} x_2 + ... + x_2^j$. Generally it follows that

$$\begin{aligned} g_{1,2,...,s+1} &= \frac{g_{2,...,s+1} - g_{1,...,s}}{x_{s+1} - x_1} \\ &= \sum_{n=s-1}^{\infty} b_n \frac{P_{n-s+1,s-1}(x_2, ..., x_{s+1}) - P_{n-s+1,s-1}(x_1, ..., x_s)}{x_{s+1} - x_1} \\ &= \sum_{n=s}^{\infty} b_n P_{n-s,s}(x_1, ..., x_{s+1}) \end{aligned}$$

and the lemma holds. □

Theorem 3.2. *Suppose $g : [0, 1] \to \mathbb{R}$ is the restriction of an entire function. If $S = \{x_k\}_{k\geq 1}$ is a sequence of distinct points of $[0, 1]$, then g can be represented uniquely into an entire Newton interpolating series at S.*

Proof. Suppose $g(x) = \sum_{i=0}^{\infty} b_i x^i$ and

$$a_i = g_{1,...,i+1} \tag{3.18}$$

for every $i = 0, 1, ...$. From Lemma 3.1 it follows that the elements a_s are given by (3.17). We show that sequence $\{a_i\}_{i\geq 0}$ verify (2.9).

Consider two positive constant M_1, M_2 such that

$$M_1 > M_2 \geq \sup_j |x_j|. \tag{3.19}$$

Then by (3.16) $|P_{k,0}(x_1)| = |x_1^k| < M_1^k$, $|P_{k,1}(x_1, x_2)| \leq \sum_{j=0}^{k} |P_{k-j,0}(x_1) x_2^j| < \sum_{j=0}^{k} M_1^{k-j} M_2^j < \dfrac{M_1^k}{1 - \frac{M_2}{M_1}}$ and by induction

$$|P_{k,j}(x_1, ..., x_{j+1})| \leq \frac{M_1^k}{\left(1 - \frac{M_2}{M_1}\right)^j}. \tag{3.20}$$

Thus by (3.17) and (3.20)

$$|a_n| \le \sum_{j=n}^{\infty} |b_j||P_{j-n,n}(x_1,...,x_{n+1})| \le \frac{1}{(M_1-M_2)^n} \cdot \sum_{j=n}^{\infty} |b_j| M_1^j. \tag{3.21}$$

Since $\lim_{j\to\infty} |b_j|^{\frac{1}{j}} = 0$, by putting $|b_j| = \theta_j^j$ as in the proof of Theorem 2.1 it follows that $\lim_{n\to\infty} \left(\sum_{j=n}^{\infty} |b_j| M_1^j \right)^{\frac{1}{n}} = 0$ and (3.21) implies (2.9).

Consider f the sum of the series (2.2) and

$$\tilde{f} = \sum_{n=0}^{\infty} c_n x^n, \tag{3.22}$$

where

$$c_n = a_n + \sum_{j=n+1}^{\infty} (-1)^{j-n} a_j S_{n,j}(x_1,...,x_j) \tag{3.23}$$

and $S_{n,j}(X_1,...,X_j)$ is the elementary symmetric function of degree $j-n$ in the variables $X_1,...,X_j$. We prove that the series from (3.23) converges, $\tilde{f}$ is an entire function and $\tilde{f}(x) = g(x)$ for all real numbers x.

As in the proof of Theorem 2.1 we put $|a_n| = \varepsilon_n^n$, where by (2.9) $\lim_{n\to\infty} \varepsilon_n = 0$. If $\delta_n = \max_{j\ge n} \varepsilon_j$, then $\lim_{n\to\infty} \delta_n = 0$ and we choose n_0 such that for all $n \ge n_0$

$$\delta_n < \frac{1}{M_2}. \tag{3.24}$$

Since $\delta_{n+1} \le \delta_n$ and

$$|S_{n,j}(x_1,...,x_j)| \le \begin{pmatrix} j \\ j-n \end{pmatrix} M_2^{j-n} \tag{3.25}$$

the series from (3.23) converges and for all $n \ge n_0$

$$|c_n| \le \sum_{j=n}^{\infty} \delta_n^j \begin{pmatrix} j \\ j-n \end{pmatrix} M_2^{j-n} = \delta_n^n \sum_{k=0}^{\infty} \begin{pmatrix} n+k \\ k \end{pmatrix} (\delta_n M_2)^k = \frac{\delta_n^n}{(1-\delta_n M_2)^{n+1}}.$$

Hence $\lim_{n\to\infty} |c_n|^{\frac{1}{n}} = 0$ and $\tilde{f}$ is an entire function. We choose $n_1 = n_1(x)$ such that for all $n \ge n_1$

$$\delta_{n+1}(M_2 + |x|) < 1. \tag{3.26}$$

Then by (3.23), (3.25) and (3.26) obtain

$$\begin{aligned}
\left| \sum_{i=0}^{n} c_i x^i - \sum_{i=0}^{n} a_i u_i \right| &= \left| \sum_{i=0}^{n} \left(c_i - a_i - \sum_{j=i+1}^{n} (-1)^{j-i} a_j S_{i,j}(\alpha_1,...,\alpha_j) \right) x^i \right| \\
&\le \sum_{k=1}^{\infty} |a_{n+k}| M_2^{n+k} + \sum_{k=1}^{\infty} |a_{n+k}| \binom{n+k}{1} M_2^{n+k-1} |x| \\
&+ \sum_{k=1}^{\infty} |a_{n+k}| \binom{n+k}{2} M_2^{n+k-2} |x|^2 + ... + \sum_{k=1}^{\infty} |a_{n+k}| \binom{n+k}{n} M_2^{k} |x|^n \\
&\le \sum_{k=1}^{\infty} \delta_{n+k}^{n+k} M_2^{n+k} + \frac{1}{M_2} \sum_{k=1}^{\infty} \delta_{n+k}^{n+k} \binom{n+k}{1} M_2^{n+k} |x| + ... \\
&+ \frac{1}{M_2^n} \sum_{k=1}^{\infty} \delta_{n+k}^{n+k} \binom{n+k}{n} M_2^{n+k} |x|^n \le \delta_{n+1}^{n+1} M_2^{n+1} \Bigg(\sum_{k=0}^{\infty} (\delta_{n+1} M_2)^k \\
&+ \frac{|x|}{M_2} \sum_{k=0}^{\infty} \binom{n+k}{1} (\delta_{n+1} M_2)^k + \frac{|x|^2}{M_2^2} \sum_{k=0}^{\infty} \binom{n+k}{2} (\delta_{n+1} M_2)^k + ... \\
&+ \frac{|x|^n}{M_2^n} \sum_{k=0}^{\infty} \binom{n+k}{n} (\delta_{n+1} M_2)^k \Bigg) \le \delta_{n+1}^{n+1} M_2^{n+1} \sum_{k=0}^{\infty} (\delta_{n+1} M_2)^k \left(1 + \frac{|x|}{M_2} \right)^{n+k} \\
&\le \frac{(\delta_{n+1}(M_2 + |x|))^{n+1}}{1 - \delta_{n+1}(M_2 + |x|)}.
\end{aligned}$$

Thus by (3.26) $\lim_{n\to\infty}\left(\sum_{i=0}^{n} c_i x^i - \sum_{i=0}^{n} a_i u_i\right) = 0$ and $\tilde{f}(x) = g(x)$ for all real numbers x. But by Theorem 2.1 it follows the series f given by (2.2) converges for all x and $g(x_k) = \tilde{f}(x_k)$ for all k. Since by (2.2) and (3.18) $g(x_k) = f(x_k)$ it follows that $f(x_k) = \tilde{f}(x_k)$. Because the sequence S has a limit point it follows that $f(x) = \tilde{f}(x)$ and then $f(x) = g(x)$. Now the result follows from Remark 2.1. □

4. Solutions of differential equations representable into Newton interpolating series

Consider a differential equation of the form

$$y^{(n)}(x) = F(x, y(x), y'(x), ..., y^{(n-1)}(x)), \tag{4.27}$$

with $x \in [0,1]$ and $F \in C^\infty(\mathbb{R}^{n+1})$. We construct the following sequence $x_1 = 0$, $x_2 = 1$, $x_3 = \frac{1}{2}$ and for $k \geq 4$

$$x_k = \frac{2s+1}{2^{m+1}}, \text{ where } 2^m + 1 < k \leq 2^{m+1} + 1, \quad s = k - 2^m - 2. \tag{4.28}$$

We present a method based on Newton interpolating series to approximate a solution, which is an entire function, of a boundary value problem for the equation (4.27). There are conditions which implies the existence of analytic solutions of a differential equation (see, for example, [4], Ch. IV). For some other related results, see [6], [7], [5].

For simplicity we take $n = 2$, $V = C^\infty([0,1])$ and we define the operator $L : V \to V$ such that

$$Ly(x) = y''(x) - F(x, y(x), y'(x)). \tag{4.29}$$

Let $\gamma_i,\ i = 0,1$ be linear functionals on V of the form $\gamma_i(y) = y(i)$ such that the system $\{\gamma_0, \gamma_1\}$ is linearly independent over $\ker L$. We want to approximate the solution of the two-point boundary value problem

$$Ly = 0,\ \gamma_0(y) = \alpha_0,\ \gamma_1(y) = \alpha_1,\ \alpha_0, \alpha_1 \in \mathbb{R}. \tag{4.30}$$

Denote by $V_1 = \{y \in V : \gamma_0(y) = \alpha_0, \gamma_1(y) = \alpha_1,\ y \text{ entire function}\}$ and suppose that the restriction of L to V_1 denoted also by L is an one-to-one mapping from V_1 onto $L(V_1) \subset V$.

For every $s \geq 2$, consider the subset of V_1

$$X_{s-2} = \left\{ y_s,\ y_s(x) = \sum_{i=0}^{s} a_i u_i(x),\ \gamma_0(y) = \alpha_0, \gamma_1(y) = \alpha_1, ,\ a_i \in \mathbb{R} \right\},$$

where u_i are defined by (2.1) and x_k by (4.28). Then by Theorem 3.2 for every $\varepsilon > 0$ we can find $s \geq 2$ and an element

$$y_s(x) = \sum_{i=0}^{s} a_i u_i(x) \in X_{s-2} \tag{4.31}$$

such that the absolute error $\|y - y_s\|_\infty < \varepsilon$. Since an entire function is uniquely determined by its values at a set having an accumulation point it follows that y_s and y have the same values at x_k, $k = 1, ..., s+1$ and the solutions of system $y_s''(x_k) - F(x_k, y_s(x_k), y_s'(x_k)) = 0$ give the values of a_i from (2.2).

Example 4.1. Consider the two-point boundary value problem ([1], p. 141)

$$y''(x) - 2500y(x) = 2500\cos^2 \pi x + 2\pi^2 \cos 2\pi x,\ x \in [0,1],\ y(0) = y(1) = 0. \tag{4.32}$$

This two-point boundary value problem has a solution $y \in V_1$ uniquely determined which by Theorem 3.2 can be represented into a Newton interpolating series where x_k are given by (4.28).

We approximate the solution by taking the partial sums $y_{32}(x) = \sum_{i=0}^{32} a_i u_i(x)$. The boundary conditions imply $a_0 = a_1 = 0$.

Table 1 lists the absolute errors in y with respect to the exact solution $y^*(x) = \dfrac{e^{50(x-1)} + e^{-50x}}{1 + e^{-50}} - \cos^2 \pi x$. The computations were performed on a computer with a 30-hexadecimal-digit mantissa. Note that the errors in simple shooting method ([1], p. 141) are clearly unacceptable (see the second column). The third column contains the results by using Newton series with $s = 32$.

Table 1

x	simple shooting	Newton series (Ex. 4.1)	Newton series (Ex. 4.2)
0.1	$.19 \cdot 10^{-7}$	$.14 \cdot 10^{-7}$	$0.965 \cdot 10^{-10}$
0.2	$.28 \cdot 10^{-5}$	$.98 \cdot 10^{-10}$	$0.339 \cdot 10^{-9}$
0.3	$.41 \cdot 10^{-3}$	$.54 \cdot 10^{-12}$	$0.548 \cdot 10^{-9}$
0.4	$.61 \cdot 10^{-1}$	$.2 \cdot 10^{-13}$	$0.578 \cdot 10^{-9}$
0.5	$.90 \cdot 10$	$.22 \cdot 10^{-14}$	$0.643 \cdot 10^{-9}$
0.6	$.13 \cdot 10^{4}$	$.27 \cdot 10^{-12}$	$0.108 \cdot 10^{-8}$
0.7	$.20 \cdot 10^{6}$	$.43 \cdot 10^{-10}$	$0.179 \cdot 10^{-8}$
0.8	$.29 \cdot 10^{8}$	$.64 \cdot 10^{-8}$	$0.203 \cdot 10^{-8}$
0.9	$.44 \cdot 10^{10}$	$.95 \cdot 10^{-6}$	$0.106 \cdot 10^{-8}$
1.0	$.65 \cdot 10^{12}$	.0	.0

Errors associated with Examples 4.1 *and* 4.2

Example 4.2. Consider the two-point boundary value problem

(4.33) $$y''(x) - \sqrt{1 + y'^2(x)} = 0, \quad x \in [0,1], \quad y(0) = 1; \; y(1) = \cosh(1).$$

This two-point boundary value problem has a solution $y \in V_1$ uniquely determined which by Theorem 3.2 can be represented into a Newton interpolating series where x_k are given by (4.28).

We approximate the solution by taking the partial sums $y_9(x) = \sum_{i=0}^{9} a_i u_i(x)$. The boundary conditions imply $a_0 = 1; \; a_1 = \cosh(1) - 1$.

The fourth column of Table 1 lists the absolute errors in y with respect to the exact solution $y^*(x) = \cosh(x)$.

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