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Nonlinear multigrid methods for solving Richards' equation in two space dimensions

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ABSTRACT.

We propose in this paper a nonlinear multigrid algorithm of Full Approximation Storage (MG-FAS) for efficient numerical solution of two dimensional Richards' equation modelling water infiltration into an isotropic, homogeneous, unsaturated porous medium. Comparisons with the nonlinear Alternating Line Gauss-Seidel iterations (ALGS) ilustrate the much better behaviour of our solver.

1. INTRODUCTION

Richards' equation [1] is the widely used mathematical model for flow in porous media in both saturated and unsaturated regimes. It is obtained by inserting Darcy law into the continuity equation. Considering the pressure of the air in the medium constant, Richards' equation has two unknowns: the saturation and the pressure in the fluid phase. Depending on the regime of the flow (unsaturated, or completely saturated), we have to decide which of the two unknowns is the primary ones. This leads to three main forms of the Richards' equation, i.e. saturation based, pressure based, or mixed.

Independent of the form used, two general strategies were developed in order to solve numerically Richards' equation. In the first [5], the original parabolic equation is reduced to an ODE (DAE) system by discretizing the spatial derivatives with finite difference, finite volume or Galerkin methods. The resulted ODE (DAE) system is integrated in time by an ODE (DAE) solver. In the second, the implicit Euler method is used for time discretization (for the discretization of the spatial derivates the methods mentioned previously were used). The nonlinear system obtained is solved by Newton, Picard or modified Picard methods. The linear solvers employed in a nonlinear iteration step are preconditioned conjugated gradients (PCG) ([6] - [3]) or linear multigrid (MG) ([13] - [9]). For some other related results, see [10].

Nonlinear MG proved to be the most powerful iterative methods to solve numerically nonlinear partial differential equations (Navier-Stokes equations, especially). Nonetheless, this method has not been used until now in numerical solving of the Richards' equation. The aim of this work is to rigorously analyse the numerical performances of the nonlinear MG algorithm in solving Richards' equation. The test problems we simulated describe variable inflows and outflows (e.g. infiltration and drainage of water into) in a homogeneous unsaturated porous medium. Boundary conditions of Dirichlet and/or Robin type were taken into consideration. We must mention that a similar approach was used in [7], where the nonlinear iteration used was SIP.

The present paper is organized as follows: in section 2 we present Richards' equation employed as test problem. The finite differences discretization scheme used together with some considerations about the MG-FAS algorithm are presented in section 3. Section 4 is devoted to the description of the iterative solver. In the last section, we present the numerical experiments performed.

2. UNSATURATED FLOW EQUATIONS

Following the approach presented in [9] and [8], the nonlinear diffusion-con- vection equation used to describe the 2-dimensional water infiltration into an isotropic, homogeneous, unsaturated porous medium (with a constant porosity) is

$$\frac{\partial u}{\partial t} - \nabla \bullet (\beta(u)\nabla u) + \frac{\partial K(u)}{\partial y} = f(x,y) \tag{2.1}$$

where $u(x,y,t)$ is the dimensionless fluid content (reduced saturation) and $(x,y) \in \Omega = [0,1]^2$, $t \in [0,T]$. For the unsaturated regime $u \in [0,1)$. Two models were used for the hydraulic diffusivity:

- Broadbridge and White (see [2])

$$\beta(u) = \frac{c(c-1)}{(c-u)^2}, \quad c \in (1,\infty) \tag{2.2a}$$

- and (see [8])

$$\beta(u) = \frac{1}{(1-u)^{1-p}}, \quad 0 < p < 1 \tag{2.2b}$$

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Relation (2.2b) with $0 < p < 1$ is not exactly a model for unsaturated flow (this would be with $p \leq 0$). The data considered in the numerical tests will be taken small enough such that the medium remain far from saturation. The model considered for the hydraulic conductivity is from [2],

$$K(u) = \frac{(c-1)u^2}{c-u}, \quad c \in (1, \infty). \tag{2.3}$$

We attach to the equation (2.1) the initial data

$$u(x, y, 0) = g(x, y), \quad (x, y) \in \Omega \tag{2.4}$$

and the boundary conditions
Dirichlet

$$u(x, y, t) = \gamma(x, y), \quad (x, y) \in \partial\Omega, \ t > 0 \tag{2.5}$$

and / or
Robin

$$(K(u)i_y - \beta(u)\nabla u) \bullet n = \gamma(x, y), \quad (x, y) \in \partial\Omega \tag{2.6}$$

where i_y is the versor of the y - axis and n the normal to the surface. The dimensionless fluid content describes the local behaviour of the system. The quantity used to describe the global behaviour of the system is

$$\overline{u} = \iint_{\Omega} u(x, y)dxdy. \tag{2.7}$$

3. Numerical solution of the Richards' equation

Equation (2.1) was discretized with the central second order accurate scheme on uniform grids with $N \times N$ points, $N = 33, 65, 129$ and 257. The mesh step size h is equal to $h = 1/(N-1)$). The discrete approximation obtained is

$$\begin{aligned} &\frac{du_{i,j}}{dt} - \frac{\beta_{i,j+1/2}(u_{i,j+1} - u_{i,j}) - \beta_{i,j-1/2}(u_{i,j} - u_{i,j-1})}{h^2} \\ &- \frac{\beta_{i+1/2,j}(u_{i+1,j} - u_{i,j}) - \beta_{i-1/2,j}(u_{i,j} - u_{i-1,j})}{h^2} \\ &+ \frac{dK}{du}(u_{i,j})\frac{u_{i,j+1} - u_{i,j-1}}{2h} = f(x_i, y_j) \end{aligned}$$

where

$$\beta_{i,j+1/2} = \frac{2}{\frac{1}{\beta(u_{i,j+1})} + \frac{1}{\beta(u_{i,j})}}, \quad \beta_{i,j-1/2} = \frac{2}{\frac{1}{\beta(u_{i,j})} + \frac{1}{\beta(u_{i,j-1})}}$$

$$\beta_{i+1/2,j} = \frac{2}{\frac{1}{\beta(u_{i+1,j})} + \frac{1}{\beta(u_{i,j})}}, \quad \beta_{i-1/2,j} = \frac{2}{\frac{1}{\beta(u_{i,j})} + \frac{1}{\beta(u_{i-1,j})}}$$

The time discretization is the fully implicit second order accurate scheme from below

$$\begin{aligned} &\frac{3u_{i,j}^{k+1} - 4u_{i,j}^{k} + u_{i,j}^{k-1}}{2\Delta t} \\ &- \frac{\beta_{i,j+1/2}^{k+1}(u_{i,j+1}^{k+1} - u_{i,j}^{k+1}) - \beta_{i,j-1/2}^{k+1}(u_{i,j}^{k+1} - u_{i,j-1}^{k+1})}{h^2} \\ &- \frac{\beta_{i+1/2,j}^{k+1}(u_{i+1,j}^{k+1} - u_{i,j}^{k+1}) - \beta_{i-1/2,j}^{k+1}(u_{i,j}^{k+1} - u_{i-1,j}^{k+1})}{h^2} \\ &+ \frac{dK}{du}(u_{i,j}^{k+1})\frac{u_{i,j+1}^{k+1} - u_{i,j-1}^{k+1}}{2h} = f(x_i, y_j) \end{aligned} \tag{3.8}$$

During the numerical experiments, the time step was constant and equal to $\Delta t = 10^{-3}$. The Robin boundary conditions (2.6) are discretized as
$y = 0 \quad (j = 1)$

$$\left(-K(u) - \beta(u)\frac{\partial u}{\partial y}\right) = \gamma_0(x)$$

$$-\beta_{i,1}\frac{-3u_{i,1} + 4u_{i,2} - u_{i,3}}{2h} - K_{i,1} = \gamma_{0,i}$$

$y = 1 \quad (j = N)$

$$\left(K(u) - \beta(u)\frac{\partial u}{\partial y} \right) = \gamma_1(x)$$

$$-\beta_{i,N}\frac{3u_{i,N} - 4u_{i,N-1} + u_{i,N-2}}{2h} + K_{i,N} = \gamma_{1,i}$$

$x = 0 \quad (i = 1)$

$$\left(-\beta(u)\frac{\partial u}{\partial y} \right) = 0$$

$$-\beta_{1,j}\frac{-3u_{1,j} + 4u_{2,j} - u_{3,j}}{2h} = 0$$

$x = 1 \quad (i = N)$

$$\left(-\beta(u)\frac{\partial u}{\partial y} \right) = 0 \tag{3.9}$$

$$-\beta_{N,j}\frac{3u_{N,j} - 4u_{N-1,j} + u_{N-2,j}}{2h} = 0$$

4. The nonlinear Iterative Solvers

The first iterative solver considered for (3.8) was the Alternating Line Gauss-Seidel (ALGS) method, a nonlinear iterative algorithm similar to that used for the Navier-Stokes equations. The linearization is only local and of Picard type. In order to describe the method let us consider that the numerical values of the dimensionless water content after (l-1) iterations are known. The values corresponding to the l iteration are calculated by applying one iteration sweep to

$$\frac{3u_{i,j}^{k+1,l} - 4u_{i,j}^{k} + u_{i,j}^{k-1}}{2\Delta t} - \frac{\beta_{i,j+1/2}^{k+1,l-1}(u_{i,j+1}^{k+1,l} - u_{i,j}^{k+1,l}) - \beta_{i,j-1/2}^{k+1,l-1}(u_{i,j}^{k+1,l} - u_{i,j-1}^{k+1,l})}{h^2}$$
$$-\frac{\beta_{i+1/2,j}^{k+1,l-1}(u_{i+1,j}^{k+1,l} - u_{i,j}^{k+1,l}) - \beta_{i-1/2,j}^{k+1,l-1}(u_{i,j}^{k+1,l} - u_{i-1,j}^{k+1,l})}{h^2}$$
$$+\frac{dK}{du}(u_{i,j}^{k+1,l-1})\frac{u_{i,j+1}^{k+1,l} - u_{i,j-1}^{k+1,l}}{2h} = f(x_i, y_j)$$

or in stencil form

$$\begin{bmatrix} & \text{CN} & \\ \text{CW} & \text{CM} & \text{CE} \\ & \text{CS} & \end{bmatrix} u_{i,j}^{k+1,l} = f(x_i, y_j) + \frac{4u_{i,j}^{k,l} - u_{i,j}^{k-1,l}}{2\Delta t},$$

where

$$\text{CM} = \frac{3}{2\Delta t} + \frac{\beta_{i,j+1/2}^{k+1,l-1}}{h^2}\frac{\beta_{i,j-1/2}^{k+1,l-1}}{h^2} + \frac{\beta_{i-1/2,j}^{k+1,l-1}}{h^2} + \frac{\beta_{i+1/2,j}^{k+1,l-1}}{h^2},$$

$$\text{CN} = -\frac{\beta_{i,j+1/2}^{k+1,l-1}}{h^2} + \frac{dK}{du}(u_{i,j}^{k+1,l-1})\frac{1}{2h}, CS = -\frac{\beta_{i,j-1/2}^{k+1,l-1}}{h^2} - \frac{dK}{du}(u_{i,j}^{k+1,l-1})\frac{1}{2h},$$

$$\text{CW} = -\frac{\beta_{i-1/2,j}^{k+1,l-1}}{h^2}, \text{CE} = -\frac{\beta_{i+1/2,j}^{k+1,l-1}}{h^2}.$$

The y-line relaxation sweep can be written as

```
for i=2,N-1
    j=1;
        a(j)=1; ba(j)=1; c(j)=0; v(j)=u(i,j);
        for j=2,N-1
            a(j) = CM
            ba(j) = CN
            c(j) = CS
            v(j) = f(i,j)-CE*u(i+1,j)+CW*u(i-1,j)+....
        endfor
        j=n;
        a(j)=1; ba(j)=1; c(j)=0; v(j)=u(i,j);
```

```
        TRIDT (n, a, ba, c, v)
        for j=1, N
            u(i,j) = v(j)
        endfor
endfor
```

The meaning of the above code sequence is the following: for a fixed column i we obtain a tridiagonal linear system containing the unknown values of u on the column i. This tridiagonal system is solved by Gaussian elimination with TRIDT(n,a,ba,c,v). The x-line relaxation sweep can be written in a similar manner. The ALGS algorithm consists of a y-line relaxation sweep followed by a x-line relaxation sweep.

The second nonlinear iterative method tested is the MG-FAS algorithm from [4]. The structure of the MG cycle is: 1) cycle of type V; 2) smoothing by alternating - line Gauss-Seidel; two smoothing steps are performed before the coarse grid correction and one after; 3) prolongation by bilinear interpolation for corrections; 4) restriction of residuals by full weighting. For two discretization levels the algorithm looks like

$$
\begin{aligned}
u_h &= \text{RELAX}^{\nu_1}(u_h, L_h, f_h)\\
d_h &= L_h u_h - f_h\\
d_H &= I_h^H d_h\\
u_H &= L_H^{-1}(L_H u_h^H - d_H)\\
u_h &= u_h + I_H^h(u_H - u_h^H)\\
u_h &= \text{RELAX}^{\nu_2}(u_h, L_h, f_h)
\end{aligned}
$$

where $L_h u_h = f_h$ is the fine grid equation, d_h and d_H the defects on the fine and coarse grid, respectively and I_H^h,I_h^H the interpolation and restriction operators, L_H the coarse grid operator and u_h^H the restriction of u_h to the coarse grid. As relaxation (RELAX) we used the ALGS iteration.
The stopping criterion used is:

$$
\frac{\|r_i\|}{\|r_0\|} \leq 10^{-6} \tag{4.10}
$$

where r_i is the residual after i iterations and $\|\cdot\|$ the discrete Euclidean norm. The maximum number of iterations allowed is 1000.

Remark 4.1. For the convergence proof of the MG-FAS solver see [4].

5. Numerical experiments

We performed tests for the numerical solution of problem (2.1) with both algorithms ALGS and MG-FAS, the two types of hydraulic diffusivity (2.2a) and (2.2b), the two kind of boundary conditions - Dirichlet (2.5) and Robin (2.6) and for different values of N. In Tables 1 and 2 from below we present the corresponding average numbers of iterations per time step (i.e., total number of ALGS or MG-FAS iterations divided by the total number of time steps, respectively) with respect to the stopping rule (4.10). In Figures 1-7 we illustrate the numerical solution of problem (2.1) (left), and the quantity $\bar{u}$ defined in (2.7) used to describe the global behaviour of the system (right), up to final time $t = 10$ (figures 1-2), and, respectively $t = 5$ (figures 3-7). For the cases presented in figures 1,3 and columns 3,4 of table 1 we considered $g(x, y) = 0.90, f(x, y) \equiv 0$, Dirichlet boundary conditions with $\gamma(x, y) = 0.0$ and $\beta(x, y)$ from (2.2a) (figure 1 and column 3 of table (1) or (2.2b) with $p = 0.5$ (figure 3 and column 4 from table 1). In figures 2,4 and columns 5,6 from table 1 we present the results obtained for $g(x, y) = 0.1, f(x, y) = 0.50$, Dirichlet boundary conditions with $\gamma(x, y) = 0.1$ and $\beta(x, y)$ from (2.2a) (figure 2 and column 5 from table 1) or (2.2b) (figure 4 and column 6 from table 1). The solutions presented in figures 5 and 6 and column 3 from table 2 were computed with $g(x, y) = 0.1, f(x, y) = 0.0, \gamma_0 = 0.01$ and $\beta(x, y)$ from (2.2a) (figure 5) or (2.2b) (figure 6). At $y = 1$ we assumed Dirichlet boundary conditions with $u(x, y, t) = 0$. The data presented in column 3 of table 2 are average values obtained from the cases plotted in figures 5 and 6. In figure 7 and column 4 of table 2 we depicted the results obtained for $g(x, y) = 0.1, f(x, y) = 0.0, \gamma_0 = 0.01$ and $\gamma_1 = 0.0$. In all numerical experiments c was considered equal to $c = 1.01$ (highly nonlinear soil).

As expected, we observe in Tables 1 and 2 the much better behaviour of the MG-FAS algorithm against ALGS.

N	Algorithm	$f(x,y)=0.0$		$f(x,y)=0.5$	
		$\beta(u)$ from (2.2a)	$\beta(u)$ from (2.2b)	$\beta(u)$ from (2.2a)	$\beta(u)$ from (2.2b)
33	ALGS	3.0	9.0	3.45	10.16
	MG-FAS	1.0	2.0	1.0	1.4
65	ALGS	3.01	13.88	5.94	16.74
	MG-FAS	2.0	2.57	2.0	2.1
129	ALGS	4.9	18.0	11.0	18.0
	MG-FAS	3.9	4.0	3.8	4.0
257	ALGS	8.0	307	28.16	311
	MG-FAS	4.0	4.84	4.0	4.76

TABLE 1. Dirichlet boundary conditions (2.5).

N	Algorithm	Dirichlet like boundary condition on $y=1$	Robin like boundary condition on $y=1$
33	ALGS	14.9	14.0
	MG-FAS	4.0	3.7
65	ALGS	32.9	30.7
	MG-FAS	4.9	5.7
129	ALGS	93.3	83.9
	MG-FAS	4.3	7.9
257	ALGS	280.5	268.75
	MG-FAS	10.89	12.67

TABLE 2. Robin boundary conditions (2.6).

Test 1. Computed solution for problem (2.1) with Dirichlet boundary conditions (2.5; $N=129$)

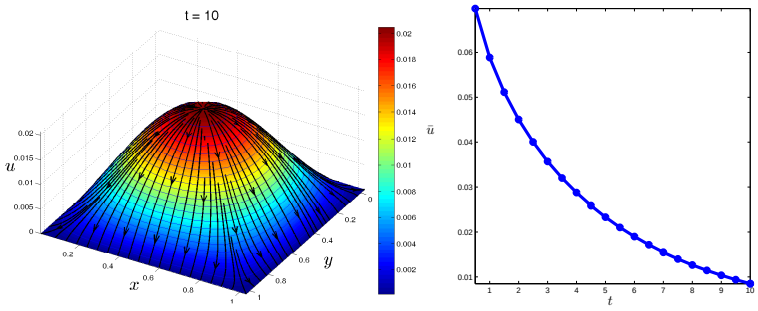

FIGURE 1. $f(x,y)=0.0$, Diffusion 2.2a

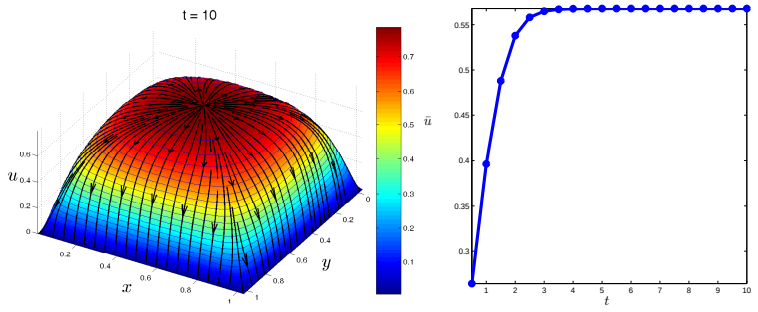

FIGURE 2. $f(x,y)=0.5$, Diffusion 2.2a

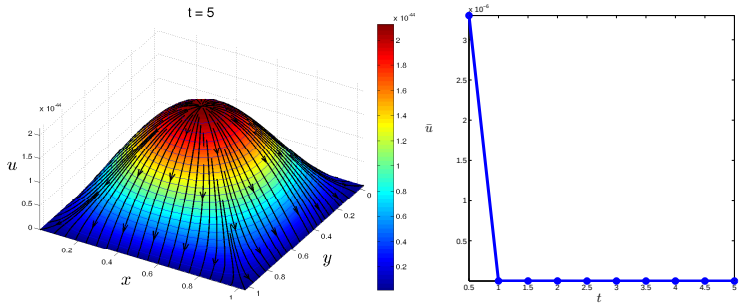

FIGURE 3. $f(x, y) = 0.0$, Diffusion 2.2b

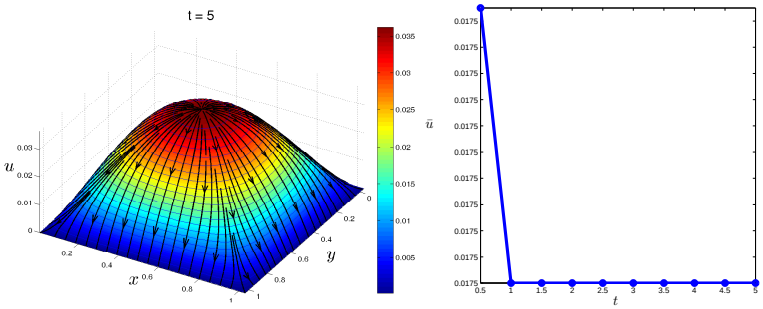

FIGURE 4. $f(x, y) = 0.5$, Diffusion 2.2b

Test 2. Computed solution for problem (2.1) with Robin boundary conditions (2.6; $N = 129$)

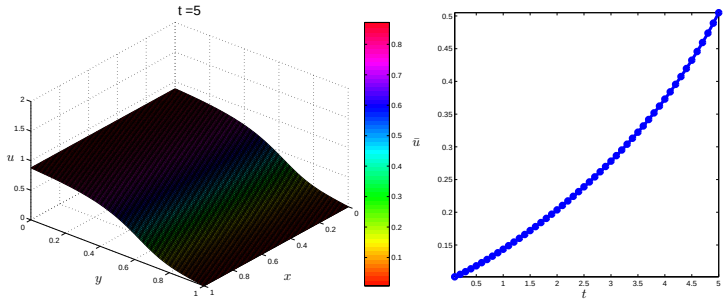

FIGURE 5. Diffusion 2.2a; Dirichlet-like condition on $y = 1$

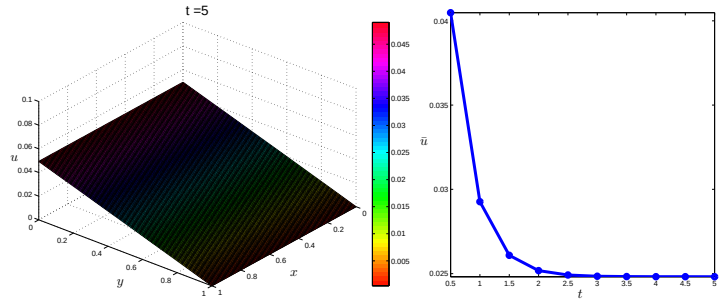

FIGURE 6. Diffusion 2.2b; Dirichlet-like condition on $y = 1$

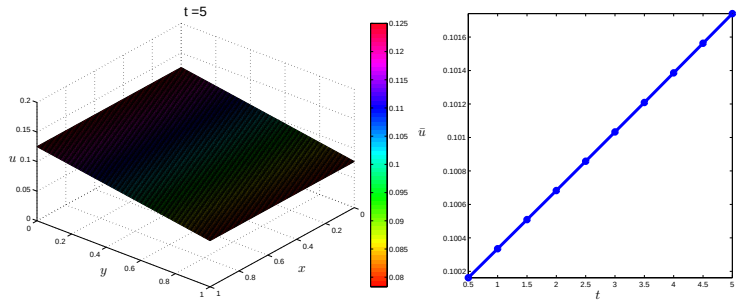

FIGURE 7. Diffusion 2.2b; Robin-like condition on $y = 1$

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