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Counting maximal chains of subgroups of finite nilpotent groups

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ABSTRACT.

The main goal of our paper is to determine the total number of maximal chains of subgroups in a finite nilpotent group. This counting problem is reduced to finite p-groups. Explicit formulas are obtained in some particular cases.

1. INTRODUCTION

One of the most important problems of combinatorial group theory is to count the chains of subgroups of a finite group. This topic has enjoyed a rapid deve-lopment in the last few years. Thus, in [4]-[6], the set of k-chains and the set of all chains of such groups are investigated in the context of subsets of multisets and partitions of a set, while [3] deals with the cardinality of these sets by using the Inclusion-Exclusion Principle. Note also that the well-known Delannoy numbers, studied in several papers as [3] and [11], count in fact all chains of subgroups of a finite cyclic group which satisfy a certain property. Another more recent problem which involves some combinatorial aspects on chains of subgroups is the classifying of distinct fuzzy subgroups of finite groups (for example, see [12]).

A chain of subgroups of a group is called a *maximal chain* if it is not properly included in another chain. The total number of maximal chains of subgroups for a finite cyclic group was determined in different ways in [3] (see Theorem 1 for $k = k_{\max}$) or in [12] (see Proposition 11).

In the present paper we extend this study to finite nilpotent groups. Because any maximal subgroup of a finite nilpotent group is normal, the maximal chains of these groups will coincide with their composition series. In particular, we infer that all maximal chains of subgroups of such a group are of the same length.

In the following let $(G, \cdot, e)$ be a finite nilpotent group (where e denotes the identity of G) of order $p_1^{n_1} p_2^{n_2} ... p_m^{n_m}$ ($p_1, p_2, ..., p_m$ are distinct primes) and $L(G)$ be the subgroup lattice of G. Recall that $L(G)$ is a complete bounded lattice with respect to set-inclusion, having an initial element, $\{e\}$, and a final element, G. It is well-known that G can be written as the direct product of its Sylow subgroups

$$G = \bigtimes_{i=1}^{m} G_i,$$

where $|G_i| = p_i^{n_i}$, for all $i = 1, 2, ..., m$. Since the subgroups of a direct product of groups having coprime orders are also direct products (see Corollary of (4.19), [9], I), one obtains that

$$L(G) = \bigtimes_{i=1}^{m} L(G_i).$$

The above lattice direct decomposition is often used in order to reduce many problems on $L(G)$ to the subgroup lattices of finite p-groups. It will play an essential role in proving the key result of Section 2. In Section 3 we shall give explicit formulas of the number of maximal chains of subgroups for three classes of finite p-groups: finite elementary abelian p-groups, finite abelian p-groups of type $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ and finite p-groups possessing a maximal subgroup which is cyclic. Some conclusions and further research directions are indicated in the last section.

Most of our notation is standard and will not be repeated here. Basic definitions and results on groups can be found in [9]. For subgroup lattice concepts we refer to [7].

2. THE KEY RESULT

In this section we present how one can reduce the counting of the number $n(G)$ of maximal chains of subgroups in a finite nilpotent group G to the more convenient case of finite p-groups.

First of all, let $G = G_1 \times G_2$, where $|G_i| = p_i^{n_i}$, $i = 1, 2$ (p_1, p_2 distinct primes), and consider π_1, π_2 to be the canonical projections onto the direct factors G_1, G_2. Denote also by $\mathcal{C}, \mathcal{C}_1$ and $\mathcal{C}_2$ the sets consisting of all maximal chains of subgroups in G, G_1 and G_2, respectively. Then we get the surjective function

$$\pi : \mathcal{C} \to \mathcal{C}_1 \times \mathcal{C}_2,\ \pi(C) = (\pi_1(C), \pi_2(C)),\ \text{ for any } C \in \mathcal{C}.$$

In order to compute $|\mathcal{C}|$ we need to determine the number of elements in an equivalence class modulo Ker π (it is clear that all these classes have the same cardinality). Take $C_1 \times C_2 \in \mathcal{C}_1 \times \mathcal{C}_2$, where

$$C_i : \{e\} = G_{i0} \subset G_{i1} \subset ... \subset G_{in_i} = G_i,\ i = 1, 2,$$

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and denote by $f(n_1,n_2)$ the number of chains $C\in\mathcal{C}$ satisfying $\pi(C)=C_1\times C_2$. For $n_2=1$, there are n_1+1 such chains, namely

$$C^i:\{e\}=G_{10}\subset ...\subset G_{1i}\subset G_{1i}\times G_2\subset G_{1i+1}\times G_2\subset ...\subset G_{1n_1}\times G_2=G,\ i=\overline{0,n_i},$$

and therefore we have

$$f(n_1,1)=n_1+1. \tag{2.1}$$

Since the set $\pi^{-1}(C_1\times C_2)$ consists of maximal chains C containing $G_{1n_1-1}\times G_2$ or $G_1\times G_{2n_2-1}$ and these subgroups are maximal in G, we infer that f satisfies the following recurrence relation

$$f(n_1,n_2)=f(n_1-1,n_2)+f(n_1,n_2-1). \tag{2.2}$$

Now, by using (2.1), (2.2) and the fact that f is symmetric, it results

$$f(n_1,n_2)=\binom{n_1+n_2}{n_1}=\binom{n_1+n_2}{n_1,n_2},$$

which implies that

$$|\mathcal{C}|=\binom{n_1+n_2}{n_1,n_2}|\mathcal{C}_1|\,|\mathcal{C}_2|.$$

In this way, one obtains

$$n(G)=\binom{n_1+n_2}{n_1,n_2}n(G_1)n(G_2).$$

This equality can naturally be extended to the general case of an arbitrary finite nilpotent group, by induction on the number of its direct components.

Theorem 2.1. *Let G be a finite nilpotent group of order $p_1^{n_1}p_2^{n_2}...p_m^{n_m}$ ($p_1,p_2,...,p_m$ distinct primes) and $G=\bigtimes_{i=1}^{m}G_i$ be the decomposition of G as a direct product of its Sylow subgroups. Then the numbers $n(G)$ and $n(G_i)$, $i=1,2,...,m$, are connected by the following equality:*

$$n(G)=\binom{n_1+n_2+...+n_m}{n_1,n_2,...,n_m}\prod_{i=1}^{m}n(G_i).$$

If the group G is cyclic, then so is each G_i and we have $n(G_i)=1,\ i=1,2,...,m$. Therefore, the following corollary holds.

Corollary 2.1. *The number of maximal chains of subgroups in a finite cyclic group of order $p_1^{n_1}p_2^{n_2}...p_m^{n_m}$ ($p_1,p_2,...,p_m$ distinct primes) is equal to the multinomial coefficient*

$$\binom{n_1+n_2+...+n_m}{n_1,n_2,...,n_m}=\frac{(n_1+n_2+...+n_m)!}{n_1!n_2!...n_m!}\,.$$

Note that the previous corollary is nothing else than Proposition 11 of [12], therefore Theorem 2.1 generalizes this result. It is also clear that the quantity

$$\binom{n_1+n_2+...+n_m}{n_1,n_2,...,n_m}$$

represents the number of all maximal lattice paths in the lattice $L(n_1,n_2,...,n_m)$ studied in [3] and [11].

3. Maximal chains of subgroups of finite p-groups

Since by Theorem 2.1 our counting problem is reduced to finite p-groups, in this section we focus on these groups. The following simple remark is very useful: any maximal chain of subgroups of a finite p-group contains a unique mini-mal/maximal subgroup. So, for every group investigated here we shall count first the number of its mini-mal/maximal subgroups and then we shall add the numbers of maximal chains of each such subgroup. We now study three classes of finite p-groups.

3.1. **Finite elementary abelian p-groups.** A finite elementary abelian p-group has a direct decomposition of type

$$\mathbb{Z}_p^k=\underbrace{\mathbb{Z}_p\times\mathbb{Z}_p\times...\times\mathbb{Z}_p}_{k\text{ factors}},$$

where p is a prime and $k\in\mathbb{N}^*$. The total number of subgroups of a given order in such a group is well-known (for example, see Proposition 2 of Tărnăuceanu [10], § 2.2).

Lemma 3.1. *For $r \in \{0, 1, ..., k\}$, the number of all subgroups of order p^r in the finite elementary abelian p-group $\mathbb{Z}_p^k$ is 1 if $r = 0$ or $r = k$, and it is*

$$\sum_{1 \le i_1 < i_2 < ... < i_r \le k} p^{i_1 + i_2 + ... + i_r - \frac{r(r+1)}{2}}$$

if $1 \le r \le k - 1$.

In particular, the number of minimal subgroups (as well as the number of ma-ximal subgroups) of $\mathbb{Z}_p^k$ is

$$\sum_{i=1}^{k} p^{i-1} = \frac{p^k - 1}{p - 1} \cdot$$

For all these subgroups M, the factor group $\dfrac{\mathbb{Z}_p^k}{M}$ is isomorphic to $\mathbb{Z}_p^{k-1}$. We infer that $n(\mathbb{Z}_p^k)$ satisfies the following recurrence relation

$$n(\mathbb{Z}_p^k) = \frac{p^k - 1}{p - 1} n(\mathbb{Z}_p^{k-1}).$$

Obviously, this leads to an explicit formula of $n(\mathbb{Z}_p^k)$.

Proposition 3.1. *The number $n(\mathbb{Z}_p^k)$ of maximal chains of subgroups in the finite elementary abelian p-group $\mathbb{Z}_p^k$ is given by the following equality:*

$$n(\mathbb{Z}_p^k) = \frac{1}{(p-1)^k} \prod_{i=1}^{k} (p^i - 1).$$

3.2. **Finite abelian p-groups of type $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$.** By the fundamental theorem of finitely generated abelian groups, a finite abelian p-group has a direct decomposition of type

$$\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times ... \times \mathbb{Z}_{p^{\alpha_k}},$$

where p is a prime and $1 \le \alpha_1 \le \alpha_2 \le ... \le \alpha_k$. The subgroups of such a group have been studied in [1] and [2]. It is well-known that it possesses also $\dfrac{p^k - 1}{p - 1}$ minimal subgroups, but the corresponding factor groups have not the same structure, as in § 3.1. In the following we shall look at its maximal subgroups (whose number is $\dfrac{p^k - 1}{p - 1}$, too). In order to simplify our study, we shall also suppose that $k = 2$.

According to Suzuki [9], vol. I, (4.19), a subgroup M of $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ is uniquely determined by two subgroups $H_1 \subseteq H_1'$ of $\mathbb{Z}_{p^{\alpha_1}}$, two subgroups $H_2 \subseteq H_2'$ of $\mathbb{Z}_{p^{\alpha_2}}$ and a group isomorphism $\varphi : \dfrac{H_1'}{H_1} \to \dfrac{H_2'}{H_2}$ (more exactly, $M = \{(a_1, a_2) \in H_1' \times H_2' \mid \varphi(a_1 H_1) = a_2 H_2\}$). Moreover, we have $|M| = |H_1'||H_2| = |H_2'||H_1|$. Imposing the condition that M is maximal (i.e. $|M| = p^{\alpha_1 + \alpha_2 - 1}$), we distinguish the next three cases.

Case 1. $|H_1'| = |H_1| = p^{\alpha_1},\ |H_2'| = |H_2| = p^{\alpha_2 - 1}$.

Then, between $\dfrac{H_1'}{H_1}$ and $\dfrac{H_2'}{H_2}$, there exists only the trivial isomorphism and therefore $M \cong \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2 - 1}}$.

Case 2. $|H_1'| = |H_1| = p^{\alpha_1 - 1},\ |H_2'| = |H_2| = p^{\alpha_2}$.

Similarly to Case 1, we get $M \cong \mathbb{Z}_{p^{\alpha_1 - 1}} \times \mathbb{Z}_{p^{\alpha_2}}$.

Case 3. $|H_1'| = p^{\alpha_1},\ |H_1| = p^{\alpha_1 - 1},\ |H_2'| = p^{\alpha_2},\ |H_2| = p^{\alpha_2 - 1}$.

In this case there are $p - 1$ distinct isomorphisms φ from $\dfrac{H_1'}{H_1}$ to $\dfrac{H_2'}{H_2}$ $\cdot$ Put $\dfrac{H_1'}{H_1} = \langle x_1 H_1 \rangle$ and $\dfrac{H_2'}{H_2} = \langle x_2 H_2 \rangle$. Then $\varphi(x_1 H_1) = x_2^q H_2$ for some $q \in \{1, 2, ..., p-1\}$. Because the elements $(x_1^p, e), (x_1, x_2^q) \in M$ have the orders $p^{\alpha_1 - 1}$ and p^{α_2}, respectively, and $\langle (x_1^p, e) \rangle \cap \langle (x_1, x_2^q) \rangle = \{(e, e)\}$, we infer again that $M \cong \mathbb{Z}_{p^{\alpha_1 - 1}} \times \mathbb{Z}_{p^{\alpha_2}}$.

Thus, we have shown that $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ has $p + 1$ maximal subgroups, one isomorphic to $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2 - 1}}$ and p isomorphic to $\mathbb{Z}_{p^{\alpha_1 - 1}} \times \mathbb{Z}_{p^{\alpha_2}}$. Let us denote by $f_p(\alpha_1, \alpha_2)$ the number of maximal chains of subgroups in $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$. Then the function f_p is symmetric and we have $f_p(\alpha_1, 0) = f_p(0, \alpha_2) = 1$, for all α_1, α_2. Moreover, f_p verifies the following recurrence relation

(3.3) $$f_p(\alpha_1, \alpha_2) = p f_p(\alpha_1 - 1, \alpha_2) + f_p(\alpha_1, \alpha_2 - 1), \text{ for all } 1 \le \alpha_1 \le \alpha_2.$$

Fix $\alpha_2 \in \mathbb{N}^*$. For "small" values of α_1 we are able to compute directly $f_p(\alpha_1, \alpha_2)$:

$$\begin{aligned}
f_p(1,\alpha_2) &= 1+\alpha_2 p,\\
f_p(2,\alpha_2) &= 1+(\alpha_2+1)p+\frac{(\alpha_2-1)(\alpha_2+2)}{2}\,p^2,\\
f_p(3,\alpha_2) &= 1+(\alpha_2+2)p+\frac{\alpha_2(\alpha_2+3)}{2}\,p^2+\frac{(\alpha_2-2)(\alpha_2+2)(\alpha_2+3)}{6}\,p^3,\\
f_p(4,\alpha_2) &= 1+(\alpha_2+3)p+\frac{(\alpha_2+1)(\alpha_2+4)}{2}\,p^2+\frac{(\alpha_2-1)(\alpha_2+3)(\alpha_2+4)}{6}\,p^3+\\
&+\frac{(\alpha_2-3)(\alpha_2+2)(\alpha_2+3)(\alpha_2+4)}{24}\,p^4,\\
&\vdots
\end{aligned}$$

and so on. The above equalities show that we must search $f_p(\alpha_1, \alpha_2)$ of type

$$f_p(\alpha_1,\alpha_2)=\sum_{i=0}^{\alpha_1} a_i^{\alpha_1}(\alpha_2)p^i.$$

By identifying the coefficients in the relation (3.3), an explicit formula for $f_p(\alpha_1, \alpha_2)$ is obtained, namely

$$f_p(\alpha_1,\alpha_2)=1+\sum_{i=1}^{\alpha_1}\frac{(\alpha_1+\alpha_2-2i+1)(\alpha_1+\alpha_2-i+2)(\alpha_1+\alpha_2-i+3)...(\alpha_1+\alpha_2)}{i!}\,p^i.$$

Remark that we have

$$\frac{(\alpha_1+\alpha_2-2i+1)(\alpha_1+\alpha_2-i+2)(\alpha_1+\alpha_2-i+3)...(\alpha_1+\alpha_2)}{i!}=$$

$$=\binom{\alpha_1+\alpha_2}{i}-\binom{\alpha_1+\alpha_2}{i-1},\ \text{ for all }\ i=\overline{1,\alpha_1}.$$

Hence the next proposition holds.

Proposition 3.2. *The number $n(\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{p^{\alpha_2}})$ of maximal chains of subgroups in the finite abelian p-groups $\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{p^{\alpha_2}}$ is:*

$$n(\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{p^{\alpha_2}})=1+\sum_{i=1}^{\alpha_1}\left(\binom{\alpha_1+\alpha_2}{i}-\binom{\alpha_1+\alpha_2}{i-1}\right)p^i.$$

Remark 3.1. 1) Let n be a fixed nonnegative integer and $A_n=(a_{ij})\in\mathcal{M}_{n+1}(\mathbb{N})$ be the matrix defined by $a_{ij}=f_p(i,j)$, for all $i,j=0,1,...,n$. Then A_n is symme-tric and defines a quadratic form $\sum_{i,j=0}^{n} a_{ij}X^iY^j$. By using (3.3), a direct calculation shows that all principal minors in the top left corner of A_n are >0, and so the previous quadratic form is positively defined. In particular, all eigenvalues of the matrix A_n are positive.

2) We consider the central numbers $f_p(n,n)$, $n\in\mathbb{N}$. From (3.3), we have

$$f_p(n,n)=pf_p(n-1,n)+f_p(n,n-1)=(p+1)f_p(n-1,n),$$

which implies that

$$(p+1)\mid f_p(n,n),\ \text{ for any } n\geq 1.$$

Moreover, $f_p(n,n)$, $n\in\mathbb{N}$, can be written in the more convenient form

$$f_p(n,n)=\sum_{i=0}^{n}a_ip^i,$$

where

$$a_i=\frac{2n-2i+1}{2n-i+1}\binom{2n}{i},\ \text{for all } i=0,1,...,n.$$

We also obtain the following inequality

$$f_p(n,n)<\sum_{i=1}^{n}\binom{2n}{i}p^i.$$

Finally, we note that the method developed above can successfully be applied for any $k\geq 2$, and this leads to an explicit formula for the number of maximal chains of subgroups of an arbitrary finite abelian group.

3.3. **Finite p-groups possessing a cyclic maximal subgroup.** Let p be a prime, $r \geq 3$ be an integer and denote by $\mathcal{G}$ the class consisting of all finite p-groups of order p^r having a maximal subgroup which is cyclic. Clearly, $\mathcal{G}$ contains finite abelian p-groups of type $\mathbb{Z}_p \times \mathbb{Z}_{p^{r-1}}$ (studied in § 3.2), but in contrast with § 3.1 and § 3.2 which deal only with finite abelian p-groups, it also contains some finite nonabelian p-groups. They are completely described by Theorem 4.1, [9], II: a nonabelian group G belongs to $\mathcal{G}$ if and only if it is isomorphic to

– $M(p^r) = \left\langle x, y \mid x^{p^{r-1}} = y^p = 1,\ y^{-1}xy = x^{p^{r-2}+1} \right\rangle$,

when p is odd, or to one of the next groups

– $M(2^r)$ $(r \geq 4)$,

– the dihedral group

$$D_{2^r} = \left\langle x, y \mid x^{2^{r-1}} = y^2 = 1,\ yxy^{-1} = x^{2^{r-1}-1} \right\rangle,$$

– the generalized quaternion group

$$Q_{2^r} = \left\langle x, y \mid x^{2^{r-1}} = y^4 = 1,\ yxy^{-1} = x^{2^{r-1}-1} \right\rangle,$$

– the quasi-dihedral group

$$S_{2^r} = \left\langle x, y \mid x^{2^{r-1}} = y^2 = 1,\ y^{-1}xy = x^{2^{r-2}-1} \right\rangle \ (r \geq 4),$$

when $p = 2$.

We determine in turn $n(G)$ for all these groups.

We shall focus first on $M(p^r)$. It is well-known that its commutator subgroup $D(M(p^r))$ has order p and is generated by x^q, where $q = p^{r-2}$. We also have $\Omega_1(M(p^2)) = \langle x^q, y \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, therefore $M(p^r)$ contains $p+1$ minimal subgroups: $M_1 = D(M(p^r))$, $M_2 = \langle y \rangle$, $M_3 = \langle x^q y \rangle$,..., $M_{p+1} = \langle x^q y^{p-1} \rangle$. For each $i \in \{2, 3, ..., p+1\}$, it is obvious that $\dfrac{M(p^r)}{M_i} \cong \mathbb{Z}_{p^{r-1}}$, and so there exists only one maximal chain in $M(p^r)$ which contains M_i. On the other hand, $\dfrac{M(p^r)}{D(M(p^r))}$ is an abelian group of order p^{r-1}. Denote by x_1, y_1 the classes of x, y modulo $D(M(p^r))$. Then $x_1^q = y_1^p = 1$ and $y_1^{-1}x_1y_1 = y^{-1}xyD(M(p^r)) = x^{q+1}D(M(p^r)) = xD(M(p^r)) = x_1$ (that is, x_1 and y_1 commute), which show that

$$\frac{M(p^r)}{D(M(p^r))} \cong \mathbb{Z}_p \times \mathbb{Z}_{p^{r-2}}.$$

One obtains $n\left(\dfrac{M(p^r)}{D(M(p^r))}\right) = n(\mathbb{Z}_p \times \mathbb{Z}_{p^{r-2}}) = 1 + (r-2)p$, in view of Proposition 3.2. Since $n(M(p^r)) = \displaystyle\sum_{i=1}^{p+1} n\left(\frac{M(p^r)}{M_i}\right)$, we get an explicit formula for this number.

Proposition 3.3. *The number $n(M(p^r))$ of maximal chains of subgroups in the finite p-group $M(p^r)$ is*

$$n(M(p^r)) = 1 + (r-1)p.$$

Now we study the groups $D_{2^r}, Q_{2^r}, S_{2^r}$. An important property of these groups is that their center is of order 2 (it is generated by x^q, where $q = 2^{r-2}$). Moreover, we have

$$\frac{G}{Z(G)} \cong D_{2^{r-1}},$$

which leads us to a recurrence relation satisfied by $n(G)$.

The minimal subgroups of D_{2^r} are: $M_1 = Z(D_{2^r})$, $M_2 = \langle y \rangle$, $M_3 = \langle xy \rangle$, ..., $M_{2^{r-1}+1} = \left\langle x^{2^{r-1}-1}y \right\rangle$. Again, it results $\dfrac{D_{2^r}}{M_i} \cong \mathbb{Z}_{2^{r-1}}$, for all $i = 2, 3, ..., 2^{r-1}+1$.

Since $\dfrac{D_{2^r}}{M_1} \cong D_{2^{r-1}}$, we infer that the number $n(D_{2^r})$ of maximal chains of subgroups in D_{2^r} verifies the recurrence relation

$$n(D_{2^r}) = n(D_{2^{r-1}}) + 2^{r-1}. \tag{3.4}$$

Writing (3.4) for $r = 2, 3, ...$ and summing up these equalities, we find an explicit formula of $n(D_{2^r})$.

Proposition 3.4. *The following equality holds:*

$$n(D_{2^r}) = 2^r - 1.$$

The simplest situation is that of the generalized quaternion group Q_{2^r}. This possesses a unique minimal subgroup, namely $Z(Q_{2^r})$. Then $n(Q_{2^r}) = n(D_{2^{r-1}})$ and this number is obtained by Proposition 3.4.

Proposition 3.5. *The following equality holds:*

$$n(Q_{2^r}) = 2^{r-1} - 1.$$

Finally, we count the maximal chains of S_{2^r}, $r \geq 4$. For each $i \in \{0, 1, ..., 2^{r-1}-1\}$, we have $(x^i y)^2 = x^{iq(q-1)} = x^{-iq}$. Therefore $\mathrm{ord}(x^i y) = 2$ when i is even, while $\mathrm{ord}(x^i y) = 4$ when i is odd. So, the minimal subgroups of S_{2^r} are of the form $\langle x^q \rangle$ and $\langle x^{2j} y \rangle$, $j = 0, 1, ..., 2^{r-2} - 1$. As in the previous situations, we obtain $\frac{S_{2^r}}{\langle x^{2j} y \rangle} \cong \mathbb{Z}_{2^{r-1}}$, for all $j = \overline{0, 2^{r-2} - 1}$. Since $\frac{S_{2^r}}{\langle x^q \rangle} \cong D_{2^{r-1}}$, it results that $n(S_{2^r}) = n(D_{2^{r-1}}) + 2^{r-2}$, which proves the next result.

Proposition 3.6. *The number $n(S_{2^r})$ of maximal chains of subgroups in the finite p-group S_{2^r} is given by the following equality:*

$$n(S_{2^r}) = 3 \cdot 2^{r-2} - 1.$$

We finish this section by mentioning that the number of maximal chains of subgroups in any finite nilpotent group whose Sylow subgroups are of types des-cribed in Section 3 can precisely be determined, according to Theorem 2.1.

4. Conclusions and further research

All our previous results show that the counting of maximal chains of subgroups of finite groups is an interesting combinatorial aspect of group theory. Clearly, it can successfully be made for other finite nilpotent groups, whose structure of minimal/maximal subgroups leads to a certain recurrence relation. This study can be also extended to more large classes of finite groups, as supersolvable or solvable groups. It will surely constitute the subject of some further researches.

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