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Some characterizations of submaximality

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ABSTRACT.
A topological space $(X, \mathcal{T})$ is submaximal if $\mathcal{T}$ is the maximal element of $[\mathcal{T}_s]$. Submaximality was first defined and characterized by Bourbaki. Since then, some mathematicians presented several characterizations of submaximal spaces. In this paper, we will attempt to develop the concept of submaximality and offer some new results. Furthermore, some results concerning α-scattered space will be obtained.

1. INTRODUCTION

Submaximality plays a very significiant role in topology. It is an important necessary condition for maximal topologies of many topological invariants. The orginal meaning of submaximal property on spaces was introduced by Bourbaki [5]. After that, many mathematicians presented several characterizations of submaximal spaces. In this work, we will attempt to develop the concept of submaximality and offer some new results.Various definitions, relations between them and basic results in the literature that will be used throughout this paper are introduced in the first section. Section 2 begins by giving equivalent conditions for the relation $\mathcal{T} \leq \sigma$ [21] to hold between topologies $\mathcal{T}$,σ on X. Then different notations for the family of nowhere dense sets in a topological space $(X, \mathcal{T})$ are given. Finally, new characterizations and fundamental properties of submaximality have been offered. In the last section, some results concerning α-scattered space will be obtained.

Throughout this work we assume no seperation axioms unless they are explicitly stated. The family of all topologies on a nonempty set X is a lattice (ordered by inclusion) and is denoted by LT(X). We will use the abbreviation "iff" for " if and only if ".

For any subset A of X, its closure and its interior with respect to $\mathcal{T}$ will be denoted by $\mathcal{T}clA$ and $\mathcal{T}intA$, respectively. Given $\mathcal{T} \in$ LT(X) and a subset A of X, recall that A is semi-open (preopen, α-open, regular open, semi-preopen) if $A \subset \mathcal{T}cl(\mathcal{T}intA)$ (resp. $A \subset \mathcal{T}int(\mathcal{T}clA)$, $A \subset \mathcal{T}int(\mathcal{T}cl(\mathcal{T}intA))$, $A = \mathcal{T}int(\mathcal{T}clA)$, $A \subset \mathcal{T}cl(\mathcal{T}int(\mathcal{T}clA))$). The complements of the sets mentioned above are called semi-closed (preclosed, α-closed, regular closed, semi-preclosed). In what follows, we denote the families of semi-open, preopen, regular open, semi-preopen, semi-closed, preclosed, regular closed, semi-preclosed subsets of a space $(X, \mathcal{T})$ respectively by $SO(X, \mathcal{T})$, $PO(X, \mathcal{T})$, $RO(X, \mathcal{T})$, $SPO(X, \mathcal{T})$, $SC(X, \mathcal{T})$,
$PC(X, \mathcal{T})$, $RC(X, \mathcal{T})$, $SPC(X, \mathcal{T})$.

Given a subset A of X, the semi-closure(preclosure, semi-preclosure) of A, denoted by sclA (pclA, spclA), is the intersection of all semi-closed (preclosed, semi-preclosed) subsets of X containing A. Dually, the semi-interior (preinterior, semi-preinterior) of A, denoted by sintA (pintA, spintA), is the union of all semi-open (preopen, semi-open) sets contained in A.

It is clear that, $x \in$ pclA(sclA, spclA) iff for any preopen(semi-open, semi-preopen) set U containing x, $U \cap A \neq \emptyset$. And observe that, pclA=X iff pint(X-A)=$\emptyset$. Similar results can be obtained for sclA and spclA.

The family of all α-open sets in $(X, \mathcal{T})$ is a topology on X finer than $\mathcal{T}$ and denoted by $\mathcal{T}^\alpha$. The topology generated by the family of $RO(X, \mathcal{T})$ is called the semi-regularisation of $\mathcal{T}$ and denoted by $\mathcal{T}_s$. It is clear that, $\mathcal{T}_s \subset \mathcal{T} \subset \mathcal{T}^\alpha$. Given $\mathcal{T} \in$ LT(X), the family $\{A \subset X : A \cap B \in PO(X, \mathcal{T})$ whenever $B \in PO(X, \mathcal{T})\}$ will be denoted by $\mathcal{T}_{PO(X,\mathcal{T})}$. Note that $\mathcal{T}_{PO(X,\mathcal{T})}$ is a topology on X satisfying $\mathcal{T}^\alpha \subset \mathcal{T}_{PO(X,\mathcal{T})} \subset PO(X, \mathcal{T})$ [2].
A subset A of a space (X, τ) is said to be dense (co-dense, nowhere dense) if $\mathcal{T}clA = X$ ($\mathcal{T}intA = \emptyset$, $\mathcal{T}int(\mathcal{T}clA) = \emptyset$). We denote the families of dense sets, co-dense sets, nowhere dense sets of a space $(X, \mathcal{T})$ by $D(X, \mathcal{T})$, $CD(X, \mathcal{T})$, $\mathcal{I}_n(\mathcal{T})$, respectively (If it is unnecessary, we don't specify the topology).

Given $\mathcal{T}, \sigma \in$ LT(X),

- $\mathcal{T}$ and σ are RO-equivalent iff $RO(X, \mathcal{T}) = RO(X, \sigma)$ iff $\sigma_s = \mathcal{T}_s$ iff $\sigma \in [\mathcal{T}_s]$
- $\mathcal{T}$ and σ are α-equivalent iff $\mathcal{T}^\alpha = \sigma^\alpha$ iff $\sigma \in [\mathcal{T}^\alpha]$.

The following theorem can be written from [16] and [18].

Theorem 1.1. *Let $\mathcal{T}, \sigma \in LT(X)$ and $\mathcal{T} \subset \sigma$. Then the following statements are equivalent:*

(1) $RO(X, \sigma) \subset \mathcal{T}$,
(2) $\mathcal{T}_s = \sigma_s$,
(3) For each $U \in \sigma$, $\mathcal{T}clU = \sigma clU$.

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A topological property R is called **contractive(expansive)** when, a space $(X, \mathcal{T})$ possesses the property R, then for any $\sigma \in LT(X)$ such that $\sigma \subset \mathcal{T}$ $(\mathcal{T} \subset \sigma)$, (X, σ) possesses the property R. A topological property R is called **semi-regular (semi-topological) property** when a space $(X, \mathcal{T})$ possesses the property R iff for any $\sigma \in [\mathcal{T}_s]$ $(\sigma \in [\mathcal{T}^\alpha])$, (X, σ) possesses the property R.

A topological space $(X, \mathcal{T})$ with property R is **maximal R** if whenever $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{T} \neq \mathcal{T}'$, $(X, \mathcal{T}')$ does not have property R. It is known that, if R is a contractive, semi-regular property, then $(X, \mathcal{T})$ cannot be maximal R without submaximal. So, submaximality is important in the investigation of some maximal topologies.

Some mathematicians presented several charecterizations of submaximality of $(X, \mathcal{T})$ and submaximality of $(X, \mathcal{T}^\alpha)$ ([1], [2], [3], [5], [6], [7], [8], [10], [13], [14], [15], [19]).

Definition 1.1. [5]For any $\mathcal{T} \in LT(X)$, maximal members of $[\mathcal{T}_s]$ is called **submaximal**.

Results in the following theorem are well knowns.

Theorem 1.2. *For any $\mathcal{T} \in LT(X)$, the following statements are equivalent:*

(1) $(X, \mathcal{T})$ is submaximal,
(2) Each dense set is open,
(3) $\mathcal{T} = PO(X, \mathcal{T})$.

It is known that, if $\mathcal{T}$ and σ topologies are RO-equivalent, then the sets $C(X, \mathcal{T})$ and $C(X, \sigma)$ of real-valued continuous functions on X with respect to $\mathcal{T}$ and σ, respectively are the same. In the investigation to be weakened the condition of RO-equivalentness for having $C(X, \mathcal{T}) = C(X, \sigma)$, a relation was defined on LT(X) as follows.

Definition 1.2. [21] Given $\mathcal{T}, \sigma \in LT(X)$, $\mathcal{T} \leq \sigma$ if $\mathcal{T} \subset \sigma$ and for each $U \in \mathcal{T}$, $\mathcal{T}clU = \sigma clU$.

Let us give some basic examples.

Example 1.1. Let $\mathcal{T} \in LT(X)$. Then
a. For each $\sigma \in [\mathcal{T}_s]$, $\mathcal{T}_s \leq \sigma$,
b. For each $\sigma \in [\mathcal{T}_s]$ such that $\mathcal{T} \subset \sigma$, $\mathcal{T} \leq \sigma$.

Definition 1.3. Given $\mathcal{T} \in LT(X)$,
a. $(X, \mathcal{T})$ is called $\mathbf{T_D}$ if for each $x \in X$, $\mathcal{T}cl\{x\} - \{x\}$ is closed,
b. $(X, \mathcal{T})$ is called **semi** $-$ $\mathbf{T_D}$ if for each $x \in X$, $\mathcal{T}cl\{x\} - \{x\}$ is semi-closed.

Theorem 1.3. [12] *For any topology $\mathcal{T} \in LT(X)$, the following statements are equivalent:*

(1) $(X, \mathcal{T})$ is semi-T_D,
(2) $(X, \mathcal{T}^\alpha)$ is T_D.

2. Submaximal Spaces

In this section, we give equivalent conditions for the relation $\mathcal{T} \leq \sigma$ to hold between topologies $\mathcal{T}, \sigma$ on X. Then different notations for the family of nowhere dense sets in a topological space $(X, \mathcal{T})$ are given, followed by equivalent conditions for a topological space to be an α-space, a submaximal space.

Theorem 2.4. *Let $\mathcal{T}, \sigma \in LT(X)$ and $\mathcal{T} \subset \sigma$. Then the following statements are equivalent:*

a. $\mathcal{T} \leq \sigma$,
b. $PO(X, \sigma) \subset PO(X, \mathcal{T})$,
c. $RO(X, \sigma) \subset PO(X, \mathcal{T})$,
d. $\sigma \subset PO(X, \mathcal{T})$,
e. *The family $PO(X, \mathcal{T})$ contains a base of σ,*
f. *For each $U \in SO(X, \mathcal{T})$, $\mathcal{T}clU = \sigma clU$,*
g. *For each $U \in \mathcal{T}^\alpha$, $\mathcal{T}clU = \sigma clU$.*

Proof. The equivalences of (a) and (d) is given in [9] without proof. Sufficiency of (a) for (b), (f), (g) are clear from [22]. But all of the proofs will be given.

$(a) \Rightarrow (b)$. Let $\mathcal{T} \leq \sigma$ and $A \in PO(X, \sigma)$. Then $A \subset \sigma int(\sigma clA)$. Since $\mathcal{T} \leq \sigma$, we have $A \subset \sigma int(\sigma clA) \subset \sigma int(\mathcal{T}clA) = \mathcal{T}int(\mathcal{T}clA)$. Thus, $A \in PO(X, \mathcal{T})$.

$(b) \Rightarrow (c)$. Let $PO(X, \sigma) \subset PO(X, \mathcal{T})$. Since $RO(X, \sigma) \subset PO(X, \sigma)$, we have $RO(X, \sigma) \subset PO(X, \mathcal{T})$.

$(c) \Rightarrow (a)$. Let $RO(X, \sigma) \subset PO(X, \mathcal{T})$ and $U \in \mathcal{T}$. Since $U \in \sigma$ and $\sigma clU \in RC(X, \sigma)$, we have $\sigma clU \in PC(X, \mathcal{T})$. Then $\mathcal{T}clU \subset \mathcal{T}cl(\mathcal{T}int(\sigma clU)) \subset \sigma clU$. Also, $\mathcal{T} \subset \sigma$ implies $\sigma clU \subset \mathcal{T}clU$, so that $\mathcal{T}clU = \sigma clU$.
Both $(d) \Rightarrow (c)$ and $(b) \Rightarrow (d)$ are clear.

$(d) \Leftrightarrow (e)$. This follows immediately, since arbitrary union of pre-open sets is also pre-open.

$(d) \Rightarrow (f)$. Let $\sigma \subset PO(X, \mathcal{T})$ and $U \in SO(X, \mathcal{T})$. Then $U \subset \mathcal{T}cl(\mathcal{T}intU)$. Since $\sigma clU \in PC(X, \mathcal{T})$, we have $\mathcal{T}clU = \mathcal{T}cl(\mathcal{T}intU) \subset \mathcal{T}cl(\mathcal{T}int(\sigma clU)) \subset \sigma clU$. Also, $\mathcal{T} \subset \sigma$ implies $\sigma clU \subset \mathcal{T}clU$, so that $\mathcal{T}clU = \sigma clU$.

$(f) \Rightarrow (g)$ is obvious, since $\mathcal{T}^\alpha \subset SO(X, \mathcal{T})$.

$(g) \Rightarrow (a)$ is obvious, since $\mathcal{T} \subset \mathcal{T}^\alpha$. □

An immediate consequences are the following.

Corollary 2.1. *Let $\mathcal{T}, \sigma \in LT(X)$ such that $\mathcal{T} \leq \sigma$. Then we have,*

a. $SO(X, \mathcal{T}) \subset SO(X, \sigma)$,
b. $SPO(X, \sigma) \subset SPO(X, \mathcal{T})$,
c. $\mathcal{T}^{\alpha} \subset \sigma^{\alpha}$,
d. $\mathcal{I}_n(\mathcal{T}) \subset \mathcal{I}_n(\sigma)$,
e. $RO(X, \mathcal{T}) \subset RO(X, \sigma)$

Proof. This follows immediately from [22]. □

Corollary 2.2. *Let $\mathcal{T}, \sigma \in LT(X)$. If $\mathcal{T} \subset \sigma \subset \mathcal{T}_{PO(X,\mathcal{T})}$, then $\mathcal{T} \leq \sigma$.*

Proof. Since $\mathcal{T} \subset \sigma \subset \mathcal{T}_{PO(X,\mathcal{T})} \subset PO(X, \mathcal{T})$, proof is clear from Theorem 2.4. □

Corollary 2.3. *For any $\mathcal{T} \in LT(X)$, the following statements are equivalent:*

a. *$(X, \mathcal{T})$ is submaximal,*
b. *For any $\sigma \in LT(X)$ such that $\mathcal{T} \leq \sigma$, $\mathcal{T} = \sigma$.*

Proof. Let $(X, \mathcal{T})$ be submaximal and $\mathcal{T} \leq \sigma$. By Theorem 1.2, $\mathcal{T} = PO(X, \mathcal{T})$. Since $\mathcal{T} \leq \sigma$, $\mathcal{T} \subset \sigma \subset PO(X, \mathcal{T})$ (by Theorem 2.4). Thus, $\mathcal{T} = \sigma$.
Conversely, assume that for each $\sigma \in LT(X)$ such that $\mathcal{T} \leq \sigma$, $\mathcal{T} = \sigma$. If $\sigma \in [\mathcal{T}_s]$ and $\mathcal{T} \subset \sigma$, by Example 1.1, $\mathcal{T} \leq \sigma$. And then, $\mathcal{T} = \sigma$. Hence, $\mathcal{T}$ is the maximal element of $[\mathcal{T}_s]$, i.e. $(X, \mathcal{T})$ is submaximal space. □

Corollary 2.4. *Let $\mathcal{T}, \sigma \in LT(X)$ such that $\mathcal{T} \leq \sigma$. Then $\mathcal{T}$ and σ are α-equivalent iff $\sigma \subset SO(X, \mathcal{T})$.*

Proof. Let $\mathcal{T}$ and σ be α-equivalent. Then $\sigma^{\alpha} = \mathcal{T}^{\alpha}$ and $\sigma \subset \mathcal{T}^{\alpha} \subset SO(X, \mathcal{T})$.
Conversely, let $\sigma \subset SO(X, \mathcal{T})$. Then by Theorem 2.4, $\mathcal{T} \subset \sigma \subset PO(X, \mathcal{T})$. And we have, $\mathcal{T} \subset \sigma \subset \mathcal{T}^{\alpha} = SO(X, \mathcal{T}) \cap PO(X, \mathcal{T})$. Hence, $\sigma \in [\mathcal{T}^{\alpha}]$ [17] □

Proposition 2.1. *Let $\mathcal{T}, \sigma, \omega \in LT(X)$ and $\mathcal{T}, \sigma$ be RO-equivalent topologies. Then the following statements are equivalent:*

a. $RO(X, \tau) \subset RO(X, \omega) \subset \sigma$,
b. $RO(X, \tau) \subset \omega_s \subset \sigma$,
c. $\mathcal{T}_s \subset \omega_s \subset \sigma$,
d. $\omega \in [\mathcal{T}_s]$.

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ is clear.
$(c) \Rightarrow (d)$. Let $\mathcal{T}_s \subset \omega_s \subset \sigma$. Since $\mathcal{T}$ and σ are RO-equivalent, $\mathcal{T}_s = \sigma_s \subset \omega_s \subset \sigma$. Hence $RO(X, \sigma) \subset \omega_s \subset \sigma$. Since $(\omega_s)_s = \omega_s$, $\sigma_s = (\omega_s)_s = \omega_s$ (by Theorem 1.1). Thus, $\mathcal{T}_s = \omega_s$ and $\omega \in [\mathcal{T}_s]$.
$(d) \Rightarrow (a)$ is clear. □

Corollary 2.5. *Let $\mathcal{T}, \omega \in LT(X)$. If $RO(X, \mathcal{T}) \subset \omega_s \subset \mathcal{T}^{\alpha}$, then $\omega \in [\mathcal{T}_s]$.*

Proof. Note that, $\mathcal{T}^{\alpha}$ and $\mathcal{T}$ are RO-equivalent. And so, by Proposition 2.1, we have $\omega \in [\mathcal{T}_s]$. □

Corollary 2.6. *Let $\mathcal{T}, \sigma \in LT(X)$ such that $\mathcal{T} \leq \sigma$. Then the following statements are equivalent:*

a. $\sigma \in [\mathcal{T}_s]$,
b. $RO(X, \sigma) \subset SC(X, \mathcal{T})$,
c. $RO(X, \sigma) \subset SO(X, \mathcal{T})$,
d. $RO(X, \sigma) \subset SR(X, \mathcal{T})$ *(here $SR(X, \mathcal{T}) = SO(X, \mathcal{T}) \cap SC(X, \mathcal{T})$),*
e. $SR(X, \mathcal{T}) = SR(X, \sigma)$.

Proof. Since $RO(X, \mathcal{T}) \subset SR(X, \mathcal{T})$, sufficiency of (a) for (b), (c) and (d) are clear.
Obviously, (d) implies (b) and (c).
Under the condition (b) and $\mathcal{T} \leq \sigma$, by using Theorem 2.4 and Corollary 2.1 we have $RO(X, \mathcal{T}) \subset RO(X, \sigma) \subset PO(X, \mathcal{T}) \cap SC(X, \mathcal{T}) = RO(X, \mathcal{T})$. So, it will be $RO(X, \mathcal{T}) = RO(X, \sigma)$. Under the condition (c) and $\mathcal{T} \leq \sigma$, we have $RO(X, \mathcal{T}) \subset RO(X, \sigma) \subset PO(X, \mathcal{T}) \cap SO(X, \mathcal{T}) = \mathcal{T}^{\alpha}$. Hence, it will be $\sigma \in [\mathcal{T}_s]$ (by Proposition 2.1).
Sufficiency of (a) for (e) is clear from [12]. Since $RO(X, \sigma) \subset SR(X, \sigma)$ under the condition (e), (d) is satisfied. □

Theorem 2.5. [22] *For $\mathcal{T} \in LT(X)$, the following statements are equivalent:*
$I \in \mathcal{I}_n$ *iff* $pintI = \emptyset$ *iff* $spintI = \emptyset$.

In what follows, we give another represantations of the family of nowhere dense sets.

Theorem 2.6. *Let $\mathcal{T} \in LT(X)$.*

$$\mathcal{I}_n = \{\mathcal{T}cl(\mathcal{T}intA) - A : A \in P(X)\} \tag{2.1}$$
$$= \{A - \mathcal{T}int(\mathcal{T}clA) : A \in P(X)\} \tag{2.2}$$
$$= \{A - pintA : A \in P(X)\} \tag{2.3}$$
$$= \{\mathcal{T}clA - A : A \in SO(X,\mathcal{T})\} \tag{2.4}$$
$$= \{A - \mathcal{T}cl(\mathcal{T}int(\mathcal{T}clA)) : A \in P(X)\} \tag{2.5}$$
$$= \{A - spintA : A \in P(X)\} \tag{2.6}$$

Proof. (2.1). Let $I \in \mathcal{I}_n$. Since $\mathcal{T}int(\mathcal{T}clI) = \emptyset$, we have $\mathcal{T}cl(\mathcal{T}int(X - I)) = X$.
Hence $\mathcal{T}cl(\mathcal{T}int(X - I)) - (X - I) = I$ and $I \in \{\mathcal{T}cl(\mathcal{T}intA) - A : A \in P(X)\}$. Conversely, it is clear that for each $A \in P(X)$, $\mathcal{T}cl(\mathcal{T}intA) - \mathcal{T}intA \in \mathcal{I}_n$.
And so, by the heredity property of $\mathcal{I}_n$, we have $\mathcal{T}cl(\mathcal{T}intA) - A \in \mathcal{I}_n$.
(2.2). If $I \in \mathcal{I}_n$, then $\mathcal{T}int(\mathcal{T}clI) = \emptyset$ and $I - \mathcal{T}int(\mathcal{T}clI) = I$.
Thus, $I \in \{A - \mathcal{T}int(\mathcal{T}clA) : A \in P(X)\}$. Conversely, it is clear that for each $A \in P(X)$, $\mathcal{T}clA - \mathcal{T}int(\mathcal{T}clA) \in \mathcal{I}_n$.
And so, by the heredity property of $\mathcal{I}_n$, we have $A - \mathcal{T}int(\mathcal{T}clA) \in \mathcal{I}_n$.
(2.3). Clear that for each subset A, $pint(A - pintA) = \emptyset$ and hence from Theorem 2.5, we have $A - pintA \in \mathcal{I}_n$.
Conversely, if $I \in \mathcal{I}_n$ then, $pintI = \emptyset$ and $I = I - pintI$. In this case, $I \in \{A - pintA : A \in P(X)\}$.
(2.4). Let $I \in \mathcal{I}_n$. Since $\mathcal{T}cl(\mathcal{T}int(X - I)) = X$, we have $X - I \in SO(X,\mathcal{T})$. On the other hand, $\mathcal{T}cl(X - I) - (X - I) = I$. Thus, $I \in \{\mathcal{T}clA - A : A \in SO(X,\mathcal{T})\}$. Conversely, let $A \in SO(X,\mathcal{T})$. Then $\mathcal{T}clA = \mathcal{T}cl(\mathcal{T}intA)$.
Since $\mathcal{T}cl(\mathcal{T}intA) - A \in \mathcal{I}_n$ (by (1)), we have $\mathcal{T}clA - A \in \mathcal{I}_n$.
(2.5). If $I \in \mathcal{I}_n$, then $\mathcal{T}cl(\mathcal{T}int(\mathcal{T}clI)) = \emptyset$ and $I - \mathcal{T}cl(\mathcal{T}int(\mathcal{T}clI)) = I$. Thus, $I \in \{A - \mathcal{T}cl(\mathcal{T}int(\mathcal{T}clA)) : A \in P(X)\}$. Conversely, by (2.2) we have for each $A \in P(X)$, $A - \mathcal{T}cl(\mathcal{T}int(\mathcal{T}clA)) \in \mathcal{I}_n$.
(2.6). By using Theorem 2.5, proof can be made in a similar way of the proof of (2.3). □

Definition 2.4. Let $\mathcal{T} \in LT(X)$ and $A \subset X$.
a. A is called **p-dense**, if pclA=X,
b. A is called **s-dense**, if sclA=X,
c. A is called **sp-dense**, if spclA=X.

In the following note, (a) is clear from Theorem 2.5 and (b) is clear from [2].

Remark 2.1. Let $\mathcal{T} \in LT(X)$ and $A \subset X$.
a. A is p-dense iff $\mathcal{T}cl(\mathcal{T}intA) = X$ iff A is sp-dense iff $\mathcal{T}int(\mathcal{T}cl(\mathcal{T}intA)) = X$,
b. A is s-dense iff A is dense.

If R is contractive semi-topological property, then $(X,\mathcal{T})$ cannot be maximal R without $\mathcal{T} = \mathcal{T}^\alpha$. In literature, a space $(X,\mathcal{T})$ is called **nodec** or **α-spaces** if $\mathcal{T} = \mathcal{T}^\alpha$.

Theorem 2.7. [17] *For any $\mathcal{T} \in LT(X)$, $\mathcal{T} = \mathcal{T}^\alpha$ iff each nowhere dense set is closed.*

In what follows, we give another equivalent conditions for a space in order to be nodec.

Theorem 2.8. *Let $\mathcal{T} \in LT(X)$. Then the following statements are equivalent:*
a. *$\mathcal{T} = \mathcal{T}^\alpha$,*
b. *Every p-dense set is $\mathcal{T}$-open,*
c. *Every sp-dense set is $\mathcal{T}$-open,*
d. *Every subset A of X, for which $pintA = \emptyset$ is $\mathcal{T}$-closed,*
e. *Every subset A of X, for which $spintA = \emptyset$ is $\mathcal{T}$-closed,*
f. *For every subset A of X, $(\mathcal{T}cl(\mathcal{T}intA)) - A$ is $\mathcal{T}$-closed,*
g. *For every semi-open subset A of X, $\mathcal{T}clA - A$ is $\mathcal{T}$-closed,*
h. *For every subset A of X, $A - (\mathcal{T}int(\mathcal{T}clA))$ is $\mathcal{T}$-closed,*
i. *For every subset A of X, $A - pintA$ is $\mathcal{T}$-closed,*
j. *For every subset A of X, $A - (\mathcal{T}cl(\mathcal{T}int(\mathcal{T}clA)))$ is $\mathcal{T}$-closed,*
k. *For every subset A of X, $A - spintA$ is $\mathcal{T}$-closed.*

Proof. Proof is clear from Definition 2.4,Theorem 2.5, Theorem 2.6 and Theorem 2.7. □

Theorem 2.9.
a. *Every subset of X can be written as the disjoint union of a preopen set and a nowhere dense set.*
b. *Every subset of X can be written as the disjoint of a semi-preopen set and a nowhere dense set.*

Proof. Let $A \subset X$. Since $A = pintA \cup (A - pintA)$ and $A = spintA \cup (A - spintA)$, then results are immediately from Theorem 2.6. □

As a consequence, for any $A \subset X$ we have the following.

- $A \in PO(X,\mathcal{T})$ iff $A - pintA = \emptyset$, $A \in \mathcal{I}_n$ iff pintA=$\emptyset$,
- $A \in SPO(X,\mathcal{T})$ iff $A - spintA = \emptyset$, $A \in \mathcal{I}_n$ iff spintA=$\emptyset$,
- pintA=$\emptyset$ iff spintA=$\emptyset$.

Theorem 2.10. *For any $\mathcal{T} \in LT(X)$, the following statements are equivalent:*
a. *$(X,\mathcal{T}^\alpha)$ is submaximal,*
b. *Every dense set is p-dense,*
c. *Every dense set is sp-dense,*
d. *Every s-dense set is p-dense,*
e. *$CD(X,\mathcal{T}) = \{(\mathcal{T}cl(\mathcal{T}intA)) - A : A \in P(X)\}$,*
f. *$CD(X,\mathcal{T}) = \{A - (\mathcal{T}int(\mathcal{T}clA)) : A \in P(X)\}$,*
g. *$CD(X,\mathcal{T}) = \{\mathcal{T}clA - A : A \in SO(X,\mathcal{T})\}$,*
h. *$CD(X,\mathcal{T}) = \{A - (\mathcal{T}cl(\mathcal{T}int(\mathcal{T}clA))) : A \in P(X)\}$,*
i. *$CD(X,\mathcal{T}) = \{A - spintA : A \in P(X)\}$,*
j. *$CD(X,\mathcal{T}) = \{A - pintA : A \in P(X)\}$,*

Proof. $(a) \Rightarrow (b)$. Let $A \subset X$ and $\mathcal{T}clA = X$. Then $A \in PO(X,\mathcal{T})$. Since $(X,\mathcal{T}^\alpha)$ is submaximal, $PO(X,\mathcal{T}) = PO(X,\mathcal{T}^\alpha) = \mathcal{T}^\alpha \subset SO(X,\mathcal{T})$. Then $A \in SO(X,\mathcal{T})$ and we have $\mathcal{T}clA = \mathcal{T}cl(\mathcal{T}intA) = X$. Hence, A is p-dense.

Both $(b) \Leftrightarrow (c)$ and $(b) \Leftrightarrow (d)$ are clear from Remark 2.1.

$(c) \Rightarrow (a)$. Let $A \in \mathcal{D}(X,\mathcal{T}^\alpha)$. Since $\mathcal{D}(X,\mathcal{T}^\alpha) = \mathcal{D}(X,\mathcal{T})$, $A \in \mathcal{D}(X,\mathcal{T})$. By the hypothesis, $spclA = X$ and by Remark 2.1, $\mathcal{T}int(\mathcal{T}cl(\mathcal{T}intA)) = X$. Thus, $A \in \mathcal{T}^\alpha$. It is known that, $(X,\mathcal{T}^\alpha)$ is submaximal iff $CD(X,\mathcal{T}) = \mathcal{I}_n$ [10]. And so, if we use Theorem 2.6, proofs of $(a) \Leftrightarrow (e) \Leftrightarrow (f) \Leftrightarrow (g) \Leftrightarrow (h) \Leftrightarrow (i) \Leftrightarrow (j)$ are clear. □

It is clear from Theorem 1.2 and Remark 2.1 that $(X,\mathcal{T})$ is submaximal iff each s-dense set is open.

We can obtain other equivalent conditions for submaximality of $(X,\mathcal{T})$, if we use equivalent conditions for nodec spaces and equivalent conditions for submaximality of $(X,\mathcal{T}^\alpha)$.

Theorem 2.11. *Let $\mathcal{T} \in LT(X)$ and $(X,\mathcal{T})$ be semi-regular. Then we have,*
a. *If $(X,\mathcal{T})$ is submaximal, then $[\mathcal{T}_s] = \{\mathcal{T}\}$,*
b. *If $(X,\mathcal{T}^\alpha)$ is submaximal, then $\mathcal{T}^\alpha$ is the finest member of $[\mathcal{T}_s]$.*

Proof. Firstly we note that, for any $\sigma \in [\mathcal{T}_s]$, $\mathcal{T}_s \leq \sigma$. And since $\mathcal{T}$ is semi-regular, $\mathcal{T} \leq \sigma$ and $PO(X,\sigma) \subset PO(X,\mathcal{T})$.

a. Let $(X,\mathcal{T})$ be submaximal. Then $PO(X,\mathcal{T}) = \mathcal{T}$. If $\sigma \in [\mathcal{T}_s]$, then by using $\mathcal{T} \leq \sigma$, we have $\mathcal{T} \subset \sigma$ and $\sigma \subset PO(X,\sigma) \subset PO(X,\mathcal{T}) = \mathcal{T}$. Thus $\mathcal{T} = \sigma$.

b. Let $(X,\mathcal{T}^\alpha)$ be submaximal. Since $PO(X,\mathcal{T}) = PO(X,\mathcal{T}^\alpha) = \mathcal{T}^\alpha$, $\sigma \subset PO(X,\sigma) \subset PO(X,\mathcal{T}) = \mathcal{T}^\alpha$ for each $\sigma \in [\mathcal{T}_s]$. □

Proposition 2.2. *Let $\mathcal{T} \in LT(X)$. If the family $PO(X,\mathcal{T})$ is a topology, then $(X,\mathcal{T}_{PO(X,\mathcal{T})})$ is submaximal.*

Proof. Let the topology generated by the family $PO(X,\mathcal{T})$ be denoted by σ. Firstly, we want to show that σ is submaximal. Let $D \in \mathcal{D}(X,\sigma)$. Since $\mathcal{T} \subset \sigma$, we have $\mathcal{T}clD = X$. Then $D \in PO(X,\mathcal{T})$ and $D \in \sigma$.
On the other hand, it is known that if $PO(X,\mathcal{T})$ is a topology, then $\mathcal{T}_{PO(X,\mathcal{T})} = \sigma$. Thus, $(X,\mathcal{T}_{PO(X,\mathcal{T})})$ is submaximal. □

Theorem 2.12. *For any $\mathcal{T} \in LT(X)$, the following statements are equivalent:*
a. *$(X,\mathcal{T})$ is semi-T_D and the family $PO(X,\mathcal{T})$ is a topology,*
b. *$(X,\mathcal{T}^\alpha)$ is submaximal.*

Proof. Let $(X,\mathcal{T}^\alpha)$ be submaximal. Then $\mathcal{T}^\alpha = PO(X,\mathcal{T}^\alpha) = PO(X,\mathcal{T})$, so that $PO(X,\mathcal{T})$ is topology. Since every submaximal space is T_D, $(X,\mathcal{T}^\alpha)$ is T_D. Therefore, $(X,\mathcal{T})$ is semi-T_D (by Theorem 1.3).

Conversely, let $(X,\mathcal{T})$ be semi-T_D and $PO(X,\mathcal{T})$ be topology. Since $(X,\mathcal{T})$ is semi-T_D, $\mathcal{T}^\alpha = \mathcal{T}_{PO(X,\mathcal{T})}$ [4]. On the other hand, since $PO(X,\mathcal{T})$ is topology, $\mathcal{T}_{PO(X,\mathcal{T})} = PO(X,\mathcal{T}) = \mathcal{T}^\alpha$. And so, by Proposition 2.2, $(X,\mathcal{T}^\alpha)$ is submaximal. □

3. α-SCATTERED SPACES

Recall that a space X is **scattered** if every nonempty subspace has an isolated point. It is known that, $(X,\mathcal{T}^\alpha)$ is scattered iff for each nonempty open subset has an isolated point [20]. $(X,\mathcal{T})$ is called **α-scattered** space if $(X,\mathcal{T}^\alpha)$ is scattered. The aim of this section is to give some results on α-scattered spaces.
$I(X,\mathcal{T})$ will stand for the set of isolated points of $(X,\mathcal{T})$.

Before presenting new results let us give the following theorem and note.

Theorem 3.13. [20] *For any $\mathcal{T} \in LT(X)$, the following statements are equivalent:*
a. *$(X,\mathcal{T}^\alpha)$ is α-scattered,*
b. *$\mathcal{T}clI(X,\mathcal{T}) = X$.*

Remark 3.2. [20] Let $\mathcal{T} \in LT(X)$.
a. If $(X, \mathcal{T})$ is α-scattered, then $(X, \mathcal{T}^\alpha)$ is submaximal,
b. α-scatteredness is a semi-topological property.

Now we may state our results.

Theorem 3.14. *Let $\mathcal{T}, \sigma \in LT(X)$.*
a. *If $\mathcal{T} \leq \sigma$ and $(X, \mathcal{T})$ is α-scattered, then (X, σ) is α-scattered.*
b. *If $(X, \mathcal{T}_s)$ is α-scattered, then for each $\sigma \in [\mathcal{T}_s]$, (X, σ) is α-scattered.*
c. *If $(X, \mathcal{T})$ is T_1, $(X, \mathcal{T})$ is α-scattered iff $(X, \mathcal{T}_s)$ is α-scattered.*
d. *If $(X, \mathcal{T})$ is T_1 and α-scattered, then for each $\sigma \in [\mathcal{T}_s]$, (X, σ) is α-scattered. And also, $(X, \mathcal{T}_s)$ is the minimal α-scattered member of the class $\{(X, \omega) : \omega \in [\mathcal{T}_s]\}$.*
e. *If $(X, \mathcal{T})$ is α-scattered, then for each $A \in SPO(X, \mathcal{T})$, $(A, \mathcal{T}_A)$ is also α-scattered.*

Proof. **a.** Let $\mathcal{T} \leq \sigma$ and $(X, \mathcal{T})$ be α-scattered. Since $\mathcal{T} \leq \sigma$, we have $\mathcal{T} \subset \sigma \subset PO(X, \mathcal{T})$. By Remark 3.2, $(X, \mathcal{T}^\alpha)$ is submaximal, so that $PO(X, \mathcal{T}) = PO(X, \mathcal{T}^\alpha) = \mathcal{T}^\alpha$. Thus, $\mathcal{T} \subset \sigma \subset \mathcal{T}^\alpha$ and then $\sigma \in [\mathcal{T}^\alpha]$. From Remark 3.2, (X, σ) is α-scattered.

b. Let $(X, \mathcal{T}_s)$ be α-scattered. If $\sigma \in [\mathcal{T}_s]$, then clearly $\mathcal{T}_s \leq \sigma$. Therefore, (X, σ) is α-scattered from (a).

c. Firstly we note that, if $(X, \mathcal{T})$ is T_1, then $I(X, \mathcal{T}) = I(X, \mathcal{T}_s)$ [5].
Let $(X, \mathcal{T})$ be α-scattered. By Theorem 3.13, $\mathcal{T}clI(X, \mathcal{T}) = X$. On the other hand, since $I(X, \mathcal{T}) \in \mathcal{T}$, $\mathcal{T}clI(X, \mathcal{T}) = \mathcal{T}_s clI(X, \mathcal{T}) = X$. Thus, $\mathcal{T}_s clI(X, \mathcal{T}) = \mathcal{T}_s clI(X, \mathcal{T}_s) = X$ and $(X, \mathcal{T}_s)$ is α-scattered.
The other part is obvious from (b).

d. Let $(X, \mathcal{T})$ be T_1 and α-scattered. By (c), $(X, \mathcal{T}_s)$ is α-scattered. And also, for any $\sigma \in [\mathcal{T}_s]$ we have (X, σ) is α-scattered. On the other hand, if $\sigma \in [\mathcal{T}_s]$, then $\mathcal{T}_s \subset \sigma$. Hence $(X, \mathcal{T}_s)$ is the minimal α-scattered member of the class $\{(X, \omega) : \omega \in [\mathcal{T}_s]\}$.

e. Let $(X, \mathcal{T})$ be α-scattered and $A \in SPO(X, \mathcal{T})$. Then $(X, \mathcal{T}^\alpha)$ is scattered. Since every subspace of scattered spaces is also scattered, $(A, \mathcal{T}^\alpha|A)$ is scattered. Furthermore, since $A \in SPO(X, \mathcal{T})$, we have $\mathcal{T}^\alpha|A = (\mathcal{T}|A)^\alpha$ [11]. Therefore, $(A, (\mathcal{T}|A)^\alpha)$ is scattered, so that $(A, \mathcal{T}|A)$ is α-scattered. □

Corollary 3.7. *For T_1 spaces, α-scatteredness is a semi-regular property.*

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