CARPATHIAN J. MATH.
25 (2009), No. 2, 141 - 146

Online version available at http://carpathian.ubm.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

A fixed point result for asymptotically nonexpansive mappings on an unbounded set

M. ABBAS and B. E. RHOADES

ABSTRACT.
A result regarding the existence of a fixed point for asymptotically nonexpansive mapping defined on an unbounded subset of a Banach space is established.

1. INTRODUCTION AND PRELIMINARIES

Let $(X, \|.\|)$ be an arbitrary real Banach space, C a nonempty unbounded subset of X, and $T : C \longrightarrow X$. Then T is said to be *nonexpansive* if, for any $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|\,;$$

T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers, with $k_n \geq 1$ and $\lim\limits_{n\to\infty} k_n = 1$ such that,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for any $x, y \in C$ and $n \in N$; T is called *demiclosed* on C if, for any sequence $\{x_n\}$ in C which is weakly convergent to an element x, with $\{Tx_n\}$ norm convergent to an element y, we have $x \in C$ and $Tx = y$.

A sequence $\{x_n\}$ in C said to be approximately convergent with respect to mapping $T : C \to C$ if, for some sequence $\{\alpha_n\}$ in $(0, 1)$ with $\lim\limits_{n\to\infty} \alpha_n = 1$, one has $\|x_n - \alpha_n T^n x_n\| = 0$. For example, define $T : R \to R$ by $Tx = x^2$, $x_n = \dfrac{1}{n}$, and $\alpha_n = 1$ for each $n \in N$. It is obvious to note that $\{x_n\}$ is approximately convergent with respect to the mapping T. Also, note that every bounded sequence in any normed space X is approximately convergent with respect to the identity map on X.

An asymptotically nonexpansive mapping T is said to satisfy condition (A) if, for any bounded approximately convergent sequence $\{x_n\}$ with respect to T, in C we have $\lim\sup\limits_{n\to\infty} \|T^m x_n - x_n\| = 0$ for each $m \in N$.

An asymptotically nonexpansive mapping T is said to satisfy the bounded approximate fixed point property if T satisfies condition (A) for $m = 1$.

A point $x \in C$ is called a fixed point of T if $x = Tx$. We denote the set of all fixed points of a map T by $F(T)$.

Let $u \in C$. A set C is called, *$u-$ starshaped, or starshaped with respect to* u, if $tx + (1-t)u \in C$ for each $x \in X$. Note that C is *convex* if C is starshaped with respect to every $u \in X$; C is boundedly compact if every bounded sequence in C has a convergent subsequence in C. We note that a set C being boundedly compact does not imply that C is bounded; for example take $C = R$. For a bounded sequence $\{x_n\}$ in X, denote $\lim\sup\limits_{n\to\infty} \|x_n - x\|$ by $r(x, \{x_n\})$, where $x \in X$. The number $\inf\limits_{x\in C} r(x, \{x_n\})$ is called the asymptotic radius of $\{x_n\}$ with respect to C and is denoted by $r(C, \{x_n\})$. A point $z \in C$ is called an asymptotic center of the sequence $\{x_n\}$ with respect to C if $r(z, \{x_n\}) = \inf\limits_{x\in C} r(x, \{x_n\})$. The set of all such points is denoted by $A(C, \{x_n\})$. It is well known that every bounded sequence $\{x_n\}$ in a uniformly convex Banach space X has a unique asymptotic centre with respect to any closed convex subset C of X. We have the following lemma from ([?]).

Lemma 1.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $\{x_n\}$ a bounded sequence in X and $A(C, \{x_n\}) = \{x_0\}$. If $\{y_n\}$ is a sequence of points in C such that $\lim\limits_{n\to\infty} r(y_m, \{x_n\}) = r(C, \{x_n\})$, then $\lim\limits_{n\to\infty} y_n = x_0$.*

Let $G : X \times X \longrightarrow R$ be a mapping which is linear in its first coordinate, and, for any $x, y \in X$, satisfies $\|x\|^2 \leq G(x, x)$ and $|G(x, y)| \leq M \|x\| \|y\|$ for some $M > 0$ (**[?]**). These conditions enable us to extend the results of **[?]**, **[?]**, **[?]**, and **[?]**, which have been proved for asymptotically nonexpansive mappings on closed, convex, bounded subsets of a Banach space. For the information of the reader we list several examples of functions G which satisfy condition (2.1). We thank Professor George Isac for communicating these examples to us.

(1) If X is a Hilbert space, the mapping G can be the inner product of X.
(2) If X is a Banach space, the semi inner product in the sense of Lumer **[?]** can play the role of the mapping G.
(3) If X is a Banach space, $B : X \times X \longrightarrow R$ a bilinear mapping, and there is a positive constant k such that $B(x, x) \geq k \|x\|^2$, then $G : X \times X \longrightarrow R$ defined by $G(x, y) = \dfrac{1}{k} B(x, y)$ satisfies all of the above conditions.

Received: 19.11.2008; In revised form: 04.07.2009; Accepted: 27.08.2009
2000 *Mathematics Subject Classification.* 47H10.
Key words and phrases. *Asymptotically nonexpansive mapping, fixed point, Banach space.*

(4) Consider the Banach space $C([0,1],H)$, where H is a Hilbert space. We can take G as,

$$G(x,y)=\int_0^1 <x(t),y(t)>dt,$$

where $<.,.>$ is the inner product defined on H.

The class of asymptotically nonexpansive mappings, which is a natural generalization of the important class of nonexpansive mappings, was introduced by Goebel and Kirk [?], where it was shown that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space and $T: X \longrightarrow X$ is asymptotically nonexpansive, then T has a fixed point. Moreover, the set $F(T)$ is closed and convex. Asymptotically nonexpansive mappings have been studied by many authors (see, for example [?], [?], [?], [?], and the references contained therein). A survey of the literature about asymptotically nonexpansive mappings T shows, however, that most of the results deal with the strong and weak convergence of different iterative processes to a point in $F(T)$ under the assumption that $F(T)\neq\phi$. This paper establishes the existence of a fixed point for an asymptotically nonexpansive mapping defined on a unbounded starshaped set, which in turn generalizes several comparable results valid for bounded convex sets.

Definition 1.1. [?] A normed space is said to satisfy Opial's condition if, whenever a sequence $\{x_n\}$ converges weakly to a point x in X, then, for $y\in X$, $y\neq x$,

$$\liminf\|x_n-x\|<\liminf\|x_n-y\|.$$

It is well known from [?], that all of the l_p spaces for $1<p<\infty$ have this property. However, the L_p spaces, do not, unless $p=2$.

Definition 1.2. [?] Let C be a nonempty unbounded subset of X, and $\phi:[0,\infty)\to[0,\infty)$. A mapping $T: C\to E$ is said to be $\phi-$ asymptotically bounded on C if there exist $r,c>0$ such that

$$\|Tx\|\leq c\phi(\|x\|)$$

for all $x\in C$ with $\|x\|>r$.

2. Fixed point theorem

Theorem 2.1. *Let $(X,\|.\|)$ be a uniformly convex Banach space, and C a nonempty unbounded closed starshaped subset with respect to some $u\in C$. Suppose that T is an asymptotically nonexpansive self map of C. If, for each $n\in N$, $x\in C$*

$$\lim\sup_{\|x\|\to\infty}\frac{G(T^nx-u,x)}{\|x\|^2}<1, \tag{2.1}$$

then T has a fixed point in C if and only if T satisfies condition (A).

Proof. Suppose that T satisfies condition (A). For each $n\geq 1$, define the mapping $T_n: C\longrightarrow X$ by

$$T_nx=\alpha_nT^nx+(1-\alpha_n)u,$$

where $\alpha_n=\dfrac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence of real numbers in $(0,1)$ such that $\lim\limits_{n\to\infty}\lambda_n=1$. Since C is starshaped with respect to u, and $T(C)\subseteq C$, $T_n(C)\subseteq C$. For all $x,y\in C$,

$$\begin{aligned}\|T_nx-T_ny\|&=\alpha_n\|T^nx-T^ny\|\\&\leq\lambda_n\|x-y\|,\end{aligned}$$

which implies that, for each $n\in N$, T_n is a contractive mapping with contractive constant $\lambda_n<1$. Applying the Banach contraction principle , we obtain a unique element $x_n\in C$ such that $T_nx_n=x_n$. We shall show that $\{x_n\}$ is a bounded sequence. Assume, on the contrary, that $\{x_n\}$ is not bounded. Then there exists a subsequence of $\{x_n\}$ whose norm tends to infinity. For notational convenience, denote this subsequence by $\{x_m\}$. By (2.1), there exists an $\alpha\in(0,1)$ and a $\beta>0$ such that $G(T^mx-u,x)\leq\alpha\|x\|^2$ for $x\in C$ and $\|x\|>\beta$. For m large enough, we have

$$\begin{aligned}\|x_m\|^2&\leq G(x_m,x_m)=G(\alpha_m(T^mx_m-u)+u,x_m)\\&\leq\alpha_m(G(T^mx_m-u,x_m)+G(u,x_m))\\&\leq\alpha_m(\alpha\|x_m\|^2+M\|u\|\|x_m\|).\end{aligned}$$

Divide both sides of the above inequality by $\|x_m\|^2$ and take the limit as $m\to\infty$ to obtain $1\leq\alpha$, which is a contradiction. Thus $\|x_n\|$ is bounded. Let x_0 be the asymptotic centre of the sequence $\{x_n\}$. Now define a sequence $\{y_m\}$ in C by $y_m=T^mx_0$. For $m,n\in N$, we have

$$\begin{aligned}\|y_m-x_n\|&\leq\|T^mx_0-T^mx_n\|+\|T^mx_n-x_n\|\\&\leq k_m\|x_n-x_0\|+\|T^mx_n-x_n\|\end{aligned}$$

Now

$$r(\{y_m\},\{x_n\}) = \limsup_{n\to\infty} \|x_n - y_m\| \leq k_m \limsup_{n\to\infty} \|x_n - x_0\| + \limsup_{n\to\infty} \|T^m x_n - x_n\|$$

which approaches $r(x,\{x_n\})$ as $m \to \infty$ and hence $y_m \to x_0$. The continuity of T implies that x_0 is a fixed point of T. To prove the converse, suppose that $\{x_n\}$ converges to a fixed point p of T. Since T is asymptotically nonexpansive, it is continuous. Thus, for each integer m, $\lim_{n\to\infty} T^m x_n = T^m p = p$, and condition (A) is satisfied. □

Example 2.1. Let $X = l^p = \{x = \{x_n\} : \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ with $\|x\| = \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}}$. Define $T : X \to X$ by $Tx = \lambda x$, $\lambda \in (0,1)$. Take $G(x,y)$ as the inner product on X, $u = (1,0,0,....)$. Note that

$$\limsup_{\|x\|\to\infty} \frac{G(T^n x - u, x)}{\|x\|^2} = \limsup_{\|x\|\to\infty} [\lambda^n - \frac{x_1}{\|x\|^2}] = \lambda^n < 1$$

Let $\{x_n\}$ be bounded and approximately convergent with respect to T. For $m \in N$, we have

$$\begin{aligned}\|T^m x_n - x_n\| &= |\lambda^m - 1| \, \|x_n\| \\ &\leq |\lambda^m - 1| \, [\|x_n - \alpha_n T^n x_n\| + \|\alpha_n T^n x_n\|] \\ &= |\lambda^m - 1| \, [\|x_n - \alpha_n T^n x_n\| + \alpha_n \lambda^n \|x_n\|],\end{aligned}$$

thus

$$\limsup_{n\to\infty} \|T^m x_n - x_n\| = 0.$$

T satisfies all of the conditions of Theorem 2.1, and $(0,0,0,....)$ is a fixed point of T.

Corollary 2.1. *Let $(X,\|.\|)$ be a uniformly convex Banach space, and C a nonempty unbounded closed starshaped subset with respect to some $u \in C$. Suppose that T is an asymptotically nonexpansive self map of C. If $f : C \to X$ is a $\phi-$ asymptotically bounded mapping on C such that $\lim_{t\to\infty} \frac{\phi(t)}{t} = 0$ and, for each $n \in N$, $x \in C$,*

$$\limsup_{\|x\|\to\infty} \frac{G(T^n x - fx, x)}{\|x\|^2} < 1,$$

then T has a fixed point in C if and only if T satisfies condition (A).

Theorem 2.2. *Let $(X,\|.\|)$ be a reflexive Banach space, and C a nonempty unbounded closed starshaped subset with respect to some $u \in C$. Suppose T is an asymptotically nonexpansive selfmap of C satisfying the bounded approximate fixed point property such that $I - T$ is demiclosed. If* (**??**) *holds then T has a fixed point in C.*

Proof. Following an argument similar to that in Theorem 2.1, we obtain a bounded sequence $\{x_n\}$ in C. Since T satisfies the bounded approximate fixed point property, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. As X is reflexive and $\{x_n\}$ is a bounded sequence, we may assume that $\{x_n\}$ is weakly convergent to an element $p \in C$. The demiclosedness of $I - T$ implies that p is a fixed point of T. □

Theorem 2.3. *Let $(X,\|.\|)$ be a uniformly convex Banach space. Suppose that T is an asymptotically nonexpansive self map of C, where C is a nonempty unbounded closed starshaped subset with respect to some point u in C. If for each $n \in N$,*

$$\limsup_{\|x\|\to\infty} \frac{\|T^n x - Tu\|}{\|x - u\|} < 1, \text{ for } x \in C,\ x \neq u, \tag{2.2}$$

then T has a fixed point in C if and only if T satisfies condition (A).

Proof. Suppose that T satisfies condition (A). For each $n \geq 1$, define a mapping $T_n : C \longrightarrow X$ by

$$T_n x = \alpha_n T^n x + (1 - \alpha_n) u,$$

where $\alpha_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$. Following an argument similar to that in Theorem 2.1, we obtain a sequence $\{x_n\}$ in C such that $T_n(x_n) = x_n$ for each $n \in N$. Now we show that $\{x_n\}$ is a bounded sequence. Assume, on the contrary, that $\{x_n\}$ is not. Then there exists a subsequence of $\{x_n\}$ whose norm tends to infinity. For notational convenience, denote this subsequence by $\{x_m\}$. By (2.2), there exists an $\alpha \in (0,1)$ and a $\beta > 0$ such that $\|T^n x - Tu\| \leq \alpha \|x - u\|$ for $x \in C$ with $\|x\| > \beta$. For n large enough, we have

$$\begin{aligned}\|x_n\| &= \|\alpha_n T^n x_n + (1 - \alpha_n) u\| \\ &\leq \alpha_n (\|T^n x_n - Tu\| + \|Tu\|) + (1 - \alpha_n) \|u\| \\ &\leq \alpha_n (\alpha \|x_n - u\| + \|Tu\|) + (1 - \alpha_n) \|u\|.\end{aligned}$$

Dividing both sides by $\|x_n\|$ and taking the limit as $n \to \infty$, we obtain $1 \leq \alpha$, which is a contradiction. Thus $\|x_n\|$ is bounded. The rest of the proof is similar to that given in Theorem 2.1.

Note that if we define $T : R \to R$ by $Tx = x + a$ where a is some non zero constant, then obviously T does not satisfy condition (A). Moreover T is a fixed point free mapping. For $X = R^2$ with usual norm. A mapping $T : X \to X$ defined by $T(x, y) = t(x, y)$ where $t \in (0, 1)$ satisfies all condition of Theorem 2.3. Moreover (0,0) is a fixed point of T. □

Remark 2.1. (a) Theorem 2.1 can easily be extended to locally convex spaces and thus contains the result of [**?**] as a special case.

(b) Theorem 2.2 extends Proposition 2 of [**?**] to asymptotically nonexpansive mappings.

Acknowledgement. The authors are thankful to the referees for their precise remarks to improve the presentation of the paper.

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Centre for Advanced Studies in Mathematics and
Department of Mathematics
Lahore University of Management Sciences
54792-Lahore, Pakistan
E-mail address: mujahid@lums.edu.pk

Department of Mathematics
Indiana University, Bloomington
IN 47405-7106
E-mail address: rhoades@indiana.edu