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Permutation groups with the same movement

M. ALAEIYAN and H. A. TAVALLAEE

ABSTRACT.
Let G be a permutation group on a set Ω with no fixed point in Ω. If for each subset Γ of Ω the size $|\Gamma^g \setminus \Gamma|$ is bounded, for $g \in G$, we define the movement of g as the max$|\Gamma^g \setminus \Gamma|$ over all subsets Γ of Ω, and the movement of G is defined as the maximum of move(g) over all non-identity elements of $g \in G$. In this paper we classify all permutation groups with maximum possible degree in which every non-identity element has the same movement.

1. INTRODUCTION

Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. If for each subset Γ of Ω and each element $g \in G$, the size $|\Gamma^g \setminus \Gamma|$ is bounded, we define the *movement* of Γ as move(Γ) =max$_{g\in G}|\Gamma^g \setminus \Gamma|$. If move$(\Gamma) \leq m$ for all $\Gamma \subseteq \Omega$, then G is said to have *bounded movement* and the *movement* of G is defined as the maximum of move(Γ) over all subsets Γ. This notion was introduced in [10]. Similarly, for each $1 \neq g \in G$, we define the movement of g as the max$|\Gamma^g \setminus \Gamma|$ over all subsets Γ of Ω. If all non-identity elements of G have the same movement, then we say that G has *constant movement*.

Clearly every permutation group with constant movement has bounded movement. Further by [10, Theorem 1], if G has movement equal to m, then Ω is finite, and its size is bounded by a function of m.

For transitive permutation groups of movement m, the following bounds on Ω were obtained in [6] and [10].

Lemma 1.1. *Let G be a transitive permutation group on a set Ω such that G has movement m. Then,* **(a)** *if G is a 2-group then $|\Omega| \leq 2m$,* **(b)** *if G is not a 2-group and p is the least odd prime dividing $|G|$, then $|\Omega| \leq \lfloor 2mp/(p-1)\rfloor$. (For $x \in R$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)*

There are various types of transitive permutation groups in which every non-identity element has the same movement, and the bounds in Lemma 1.1 are attained. For example, let G be either a p-group of exponent p or a 2-group. If we consider G as a permutation group in its regular representation, then we see that all non-identity elements have the same movement, and so G has constant movement. We denote by $K : P$ a semi-direct product of K by P with normal subgroup K.

The first purpose of this paper is to classify all transitive permutation groups G in which every non-identity element has the same movement m as follows:

Theorem 1.1. *Let m be a positive integer, and let G be a transitive permutation group on a set Ω which has constant movement equal to m. Then G has the maximum possible degree as described in Lemma 1.1, and G is either a p-group in its regular representation, where p is a prime or one of the following holds, when p is the least odd prime dividing the order of G:*

1. *$|\Omega| = p$, $m = (p-1)/2$ and G is the semi-direct product $Z_p : Z_{2^a}$ where $2^a|(p-1)$ for some $a \geq 1$;*

2. *$G := A_4$, A_5, $|\Omega| = 6$, and $m = 2$.*

3. *$|\Omega| = 2^s p$ where p is a Mersenne prime, $m = 2^{s-1}(p-1)$, and $1 < 2^s < p$, and G is the semi-direct product $K : P$ with K a 2-group and $P = Z_p$ is fixed point free on Ω; K has p-orbits of length 2^s, and each non-identity element of K moves exactly $2^s(p-1)$ points of Ω.*

Moreover, all permutation groups listed above have constant movement.

Now we consider the intransitive case and then we have the following lemma so that it gives an upper bound for $|\Omega|$ (see [2,10]):

Lemma 1.2. *Let G be a permutation group on a set Ω with no fixed point in Ω, and suppose that, for some positive integers m and t, move$(G) = m$ and G has t orbits in Ω. Then,*

(a) *If G is a 2-group, then $|\Omega| \leq t + 2m - 1 \leq 4m - 2$.*

(b) *If G is not a 2-group and if p is the least odd prime dividing $|G|$, then $|\Omega| \leq 2mp/(p-1) + t - 1 \leq (9m-3)/2$ for $p = 3$, and $|\Omega| \leq 4m - p$ for $p \geq 5$.*

The second main result is as follows:

Theorem 1.2. *Let m be a positive integer, let G be an intransitive permutation group on a set Ω of maximum possible size n, as described in Lemma 1.3, with constant movement equal to m. Then either*

1. *$G = Z_2^r$, $m = 2^{r-1} \geq 1$ and G has $2m-1$ orbits of length 2, for some positive integer r, and in addition let p be a least odd prime dividing $|G|$, then we have:*

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2. *For an odd prime p, $G = Z_p^d$, $m = p^{d-1}.(p-1)/2$ and G has $(p^d-1)/(p-1)$ orbits of length p, for some positive integer d.*

3. *$G = K : H$ with normal 2-subgroup K, where H is isomorphic to $Z_{2^s} : Z_p$, $s \geq 0$.*

Moreover if every non-identity element of G has prime order, then G is the semi-direct product of $Z_2^2 : Z_3$ with normal subgroup Z_2^2, $m = 2$ and G has 2 orbits Ω_1 and Ω_2 of length 3 and 4 respectively.

4. *$G = P : H$ with normal p-subgroup P, where H is either isomorphic to Z_{2^a}, or a generalized quaternion group.*

All the permutation groups in parts (1) and (2) are examples (see Example 2.2). Also there exist examples in part (3) for $p = 3$, $s = 0$, and $K = Z_2^2$ (see Example 2.3). But we do not know any other examples in part (3), or any examples in part (4).

All the groups in Theorem 1.1 and Theorem 1.2, are given in Section 2 and proved to satisfy the hypotheses of these theorems. In Section 4, we prove Theorem 1.1, and also we prove Theorem 1.1 in Section 5. Thus we complete the classification of all permutation groups with constant movement which have maximum possible degree. In the first step we have a classification of all transitive permutation groups with constant movement, and in the second step we consider intransitive permutation group with maximum possible degree, and then we have a classification of such groups which have constant movement.

2. Examples

Let G be a transitive permutation group on a finite set Ω. Then by [12, Theorem 3.26], which we shall refer to as Burnside's Lemma, the average number of fixed points in Ω of elements of G is equal to the number of G-orbits in Ω, namely 1, and since 1_G fixes $|\Omega|$ points and $|\Omega| > 1$, it follows that there is some element of G which has no fixed points in Ω. We shall say that such elements are fixed point free on Ω.

Let $1 \neq g \in G$ and suppose that g in its disjoint cycle representation has t nontrivial cycles of lengths $l_1, ..., l_t$, say. We might represent g as

$$g = (a_1 a_2 ... a_{l_1})(b_1 b_2 ... b_{l_2}) ... (z_1 z_2 ... z_{l_t}).$$

Let $\Gamma(g)$ denote a subset of Ω consisting of $\lfloor l_i/2 \rfloor$ points from the i^{th} cycle, for each i, chosen in such a way that $\Gamma(g)^g \cap \Gamma(g) = \emptyset$. For example we could choose

$$\Gamma(g) = \{a_2, a_4, \ldots, a_{k_1}, b_2, b_4, \ldots, b_{k_2}, ..., z_2, z_4, \ldots, z_{k_t}\},$$

where $k_i = l_i - 1$ if l_i is odd and $k_i = l_i$ if l_i is even. Note that $\Gamma(g)$ is not uniquely determined as it depends on the way each cycle is written down. For any set $\Gamma(g)$ of this kind we say that $\Gamma(g)$ consists of *every second point of every cycle of g*. From the definition of $\Gamma(g)$ we see that $|\Gamma(g)^g - \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^{t} \lfloor l_i/2 \rfloor$. In [6] we have shown that this quantity is an upper bound for $|\Gamma^g - \Gamma|$ for an arbitrary subset Γ. Thus the movement of g is $|\Gamma(g)|$.

First we give examples of groups with constant movement that are transitive and of exponent p.

Lemma 2.3. (a) *Let $m := p^{a-1}(p-1)/2$ for some $a \geq 1$, where p is an odd prime and suppose that G is a regular permutation group of exponent p on a set Ω of size $p^a = 2mp/(p-1)$. Then all elements of G have the same movement equal to m.*

(b) *Let m be a power of 2, and suppose that G is a 2-group of order $2m$. Then the regular representation of G on Ω is a permutation group in which every non-identity element has the same movement m.*

Proof. Let $1 \neq g \in G$ and let $\Gamma \subseteq \Omega$. By [6, Lemma 2.1], $|\Gamma^g - \Gamma| \leq m$. Since G is regular, g is fixed point free on Ω. Suppose that $\Gamma(g)$ consists of every second point of every cycle of g. Then by definition $\Gamma(g)^g \cap \Gamma(g) = \emptyset$. If p is an odd prime, then

$$|\Gamma(g)^g - \Gamma(g)| = |\Gamma(g)| = (|\Omega|/p)(p-1)/2 = p^{a-1}.(p-1)/2 = m.$$

Since g is an arbitrary element, all non-identity elements of G have the same movement m. Also with the same argument it can be shown that in every 2-group of order $2m$ in its regular representation all non-identity elements have the same movement. □

In what follows we will see that the regularity condition for each transitive p-group is a necessary condition. Let H be a core-free subgroup of a p-group G and consider the permutation representation by right multiplication on the right cosets of H. If $H \neq 1$ then G is not regular in this action and not all non-identity elements have the same movement . An example of such a core-free subgroup H in a non-abelian p-group G of exponent p is the cyclic group generated by any non-central element. Such elements exist provided that G is non-abelian.

Let H be cyclic of order n and $K = \langle k \rangle$ be cyclic of order m and suppose r is an integer such that $r^m \equiv 1 \pmod{n}$. For $i = 1, \ldots, m$, let $(k^i)\theta : H \longrightarrow H$ be defined by $h^{(k^i)\theta} = h^{r^i}$ for h in H. It is straightforward to verify that each $(k^i)\theta$ is an automorphism of H, and that θ is a homomorphism from K to $Aut(H)$. Hence the semidirect product $G = H : K$ (with respect to θ) exists and if $H = \langle h \rangle$, then G is given by the defining relations:

$$h^n = 1, \quad k^m = 1, \quad k^{-1}h.k = h^r, \quad with \quad r^m \equiv 1 \pmod{n}.$$

Here every element of G is uniquely expressible as $h^i k^j$, where $0 \leq i \leq n-1$, $0 \leq j \leq m-1$. Certain semi-direct products of this type (as a permutation group on a set Ω of size n) also provide examples of transitive permutation groups where every non-identity element has the same movement, and the bound in Lemma 1.1 holds (as the following lemma). We note that, if $n = p$, a prime, then by [14, Theorem 3.6.1] this group G is a subgroup of the Frobenius group $AGL(1, p) = Z_p : Z_{p-1}$.

Lemma 2.4. *Let G be a semi-direct product $Z_p : Z_{2^a}$ denote a group defined as above of order $p.2^a$ where $2^a|(p-1)$ for some $a \geq 1$. Then G acts transitively on a set Ω of size p and in this action all non-identity elements of G have the same movement equal to $(p-1)/2$.*

Proof. By the above statement of Lemma 2.4, the group G is a Frobenius group and has up to permutational isomorphism a unique transitive representation of degree p, on a set Ω, say. Let $g \in G$ such that $o(g) = p$. Then by [6, Lemma 2.1], $|\Gamma^g - \Gamma| \leq m = (p-1)/2$ for all subsets Γ, and if $\Gamma(g)$ consists of every second point of the unique cycle of g , then $|\Gamma(g)^g - \Gamma(g)|$ has size equal to m. Suppose now that $g \in G$ has order $o(g)$ a power of 2. Then g has one fixed point and $(p-1)/o(g)$ cycles of length $o(g)$ in Ω. For each $\Gamma \subseteq \Omega$, $|\Gamma^g - \Gamma|$ consists of at most $o(g)/2$ points from each cycle of g of length $o(g)$,and hence has size at most m. Since each element of G is either a 2-element or has order p , it follows that all non-identity elements of G have the same movement m. □

Lemma 2.5. *The groups A_4, and A_5 act transitively on a set of size 6 and in this action every non-identity element has the same movement equal to 2.*

Proof. By [9] the groups A_4 and A_5 have movement equal to 2. With similar argument we will show that all non-identity elements of them also have the same movement equal to 2. Let $1 \neq g \in A_4$. Then g has order 2 or 3. If g has order 2 then g has two cycles of length 2 and hence $|\Gamma(g)^g - \Gamma(g)| = 2$. Similarly, if g has order 3 then g has two cycles of length 3 and again $|\Gamma(g)^g - \Gamma(g)| = 2$. As for A_5, since every non-identity element of A_5 has order 2, 3 or 5, as above it is easy to see that every non-identity element of A_5 has movement equal to 2. □

Let F be a finite field of order 2^{s+1}, where s is a positive integer. Let K and P be the additive group and the multiplication group of F, respectively. Then standard theory of fields shows that F is a vector space over $GF(2)$ and P (acting naturally via the multiplication in F, for $v \in K$ and $x \in P$, $x : v \to xv$) acts as a group of linear transformations, and each non-identity element of P is fixed point free. This is the theory of Singer cycles, (see [8]). The following example not only proves the Theorem 1.1, but also is a positive answer to the question about more examples of the Theorem 1.2.b , we have made in [6].

Example 2.1. Let $p := 2^{s+1} - 1$ be a Mersenne prime, for some positive integer s. As above let $K := Z_2^{s+1}$, and let $P := Z_p$ be a Singer cycle of $GL(s+1, p)$. So P is fixed point free on K. Suppose that H_1 is a subgroup of order 2 of K, and $P = \langle x \rangle$.
Set $H_i = (H_1)^{x^{i-1}}$, for $1 \leq i \leq p$. Define Ω_i to be the coset space of H_i in K of size 2^s, and $\Omega = \Omega_1 \cup ... \cup \Omega_p$. Then K acts by right multiplication on Ω, K^{Ω_i} is regular with kernel H_i, where $1 \leq i \leq p$, and for each $x^j \in P$, $1 \leq j \leq p-1$, we have the linear transformation φ_{x^j} on Ω_i so that $\varphi_{x^j}(H_i k) = H_{i+j} k^{x^j}$. Now we have the permutation group G, which is the semi-direct product of K by P. It is obvious that G is transitive on Ω and each element of G is either in K or conjugate to an element of P. Let $g \in P - \{1\}$. So g is fixed point free on Ω and has 2^s cycle of length p. Suppose that $\Gamma(g)$ consists of every second point of every cycle of g. Then by definition,

$$|\Gamma(g)| = 2^s.\frac{p-1}{2} = 2^{s-1}(p-1) = m.$$

Let $k \in K - \{1\}$. So k lies in exactly one of the subgroup H_i, and then k permutes exactly $2^s(p-1)$ points and as above we have, $|\Gamma(k)| = 2^{s-1}(p-1) = m$. Hence all non-identity elements of G have the same movement m, and so all of such groups are examples for Theorem 1.1.

In continuation, we consider the intransitive case, and we will show that there certainly are families of examples of intransitive permutation groups in which every non-identity element has the same movement. The following example shows that intransitive p-groups, in which each element has the same movement do exist.

Example 2.2. Let d be a positive integer p a prime, let $G := Z_p^d$, let $t := (p^d - 1)/(p-1)$, and let $H_1, \cdots, H_t$ be an enumeration of the subgroups of index p in G. Define Ω_i to be the coset space of H_i in G and $\Omega = \Omega_1 \cup \cdots \cup \Omega_t$. If $g \in G - \{1\}$, then g lies in $(p^{d-1} - 1)/(p-1)$ of the groups H_i and therefore acts on Ω as a permutation with $p(p^{d-1} - 1)/(p-1)$ fixed points and p^{d-1} orbits of length p. Taking every second point from each of these p-cycles to form a set Γ we see that move$(g) = m \geq p^{d-1}(p-1)/2$ if p is odd or 2^{d-1} if $p = 2$, and it is not hard to prove that in fact move $(g) = m = p^{d-1}.(p-1)/2$ if p is odd or 2^{d-1} if $p = 2$. Since g is non-trivial, all non-identity elements of G have the same movement equal to m.

The last example for $p = 3$, inclined to the following example not only are examples in which every non-identity element has the same movement equal to 3^{d-1} and 2 respectively, but also gives some positive answer to the Question 1.5 in [11].

Example 2.3. Let $\Omega = \Omega_1 \cup \Omega_2$ be a set of size 7, such that $\Omega_1 = \{1, 2, 3\}$ and $\Omega_2 = \{1', 2', 3', 4'\}$. Moreover, suppose that $Z_2^2 \cong <(1'2')(3'4'), (1'3')(2'4')>$ and $Z_3 \cong <(123)(1'2'3')>$. Then the semi-direct product $G := Z_2^2 : Z_3$ with normal subgroup Z_2^2 is a permutation group on a set Ω with 2 orbits in which every non-identity element has the same movement 2, since each non-identity element of G has two cycles of length 2 or two cycles of length 3.

3. Preliminary Analysis

Throughout the rest of the paper, let G be a permutation group on a set Ω of size n in which all non-trivial elements have the same movement, and let m be a positive integer. First we note that for each permutation in G the size of its non-trivial cycles is constant.

Lemma 3.6. *Let $1 \neq g \in G$, then all non-trivial cycles of g have the same size. Moreover, g is either an odd prime or a power of* 2.

Proof. Let $g = c_1 ... c_s$ be the decomposition of g into its disjoint non-trivial cycles such that $|c_i| = l_i$ for $1 \leq i \leq s$. Then the movement of G, move(g), is the size of the subset $\Gamma(g)$ consisting of every second point of every cycle g, that is, $move(g) = \sum\limits_{i=1}^{s} \left\lfloor \frac{l_i}{2} \right\rfloor$. For each $i \leq s$, now we consider the element $h = g^{l_i}$ of G and compare the movement of h with the movement of g. As above, we have

$$move(h) \leq \sum_{j \neq i} \left\lfloor \frac{l_j}{2} \right\rfloor < \sum_{i=1}^{s} \left\lfloor \frac{l_i}{2} \right\rfloor = move(g).$$

Since all non-identity elements of G have the same movement, so $h = 1$. Hence we must have $l := l_1 = l_2 = ... = l_s$. Suppose now that l is not a power of 2, and let p be an odd prime such that $l = pk$, for some positive integer k. Then by comparing the movement of g and its power g^k we obtain

$$s \left\lfloor \frac{l}{2} \right\rfloor = move(g) = move(g^k) = sk\frac{p-1}{2}.$$

It can be easily verified that $\left\lfloor \frac{kp}{2} \right\rfloor = k(p-1)/2$ if and only if $k = 1$, and so $l = p$. The result now follows. □

Definition 3.1. A group G is called an EPPO-group if every non-identity element has prime power order. Moreover, a group G is called an EPO-group if all of its non-identity elements have prime order.

It is known that an EPPO group is a p-group, a non-abelian simple group or a Frobenius group (see [3, 7, 13). If G is solvable, then by [7, Theorem 1] $|G| = 2^a p^b$ for some non-negative integers a, b.

By Lemma 3.6, all non-trivial cycles of $g \in G$ have the same size and the number 2 and the odd prime p are the only primes that divide the order of G. Hence, by considering the above statement about the structure of EPPO-groups and compare its with all permutation groups with the same movement which have maximum degree (see the classification of all permutation groups with bounded movement in [2, 6, 9]), we have the following corollary:

Corollary 3.1. *The group G is an EPPO-group, and either $G = A_5$ or $|G| = 2^a p^b$ for some non-negative integers a, b.*

4. Transitive case

In this section we suppose that G is a transitive permutation group on a set Ω of size n in which every non-identity element has the same movement m, and p is the least odd prime dividing $|G|$. First we show that the degree n is the maximum possible.

Lemma 4.7. *If G is a 2-group then $n = 2m$, and otherwise $n = 2mp/(p-1)$.*

Proof. By Burnside's Lemma, G has a fixed point free element on a set Ω, say g. But by [5, Theorem 1], for some prime q dividing $|G|$, the fixed point free element g is a q-element of order q^c (for some positive integer c), and by Lemma 3.6 (and Corollary 3.1), either $q = 2$ or $q^c = p$ and p is the only odd prime dividing $|G|$. Let $\Gamma(g)$ consist of every second point of every cycle of g and also let o denote the number of odd length cycles. By definition, $|\Gamma(g)| = n/2$ if $q = 2$, and

$$|\Gamma(g)| = \frac{n-o}{2} = \frac{n-n/p}{2} = \frac{n(p-1)}{2p},$$

if $q = p$. On the other hand, since every non-identity element has the same movement m, so $|\Gamma(g)| = m$. Hence if G is a 2-group then $n = 2m$, and otherwise $n = 2mp/(p-1)$. By Lemma 1.1 these expressions are the maximum possible degree of G. □

We are now in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let G, Ω, m be as in Theorem 1.1 with $n = |\Omega|$. Then by Lemma 4.7, n is the maximum possible degree as in Lemma 1.1. We first suppose that G is a 2-group. By Lemma 4.7, $n = 2m$. As each $1 \neq g \in G$ has constant movement m, $|supp(g)| = 2m$, where $supp(g) = \{\alpha \in \Omega | \alpha^g \neq \alpha\}$. Thus g is a fixed point free element on Ω, that is, $G_\alpha = 1$ for each $\alpha \in \Omega$. Hence G is a regular 2-group.

Now suppose that G is not a 2-group. Then by Corollary 3.1, $G = A_5$ or $|G| = 2^a p^b$ with $b \geq 1$ and p an odd prime, and so by Lemma 4.7, $n = 2mp/(p-1)$. Then by [6, 9], one of the following holds:

(1) $|\Omega| = p$, $m = (p-1)/2$ and G is the semi-direct product $Z_p : Z_{2^a}$ where $2^a|(p-1)$ for some $a \geq 1$, as in part (1) of Theorem 1.1.

(2) G is the semi-direct product $K : P$ with K a 2-group and $P = Z_p$ is fixed point free on Ω; $|\Omega| = 2^s p$, $m = 2^{s-1}(p-1)$, and $2^s < p$, where p is a Mersenne prime, K has p-orbits of length 2^s, and each element of K moves at most $2^s(p-1)$ points of Ω. (We note that $A_4 \cong (Z_2)^2 : Z_3$ is a transitive permutation group of degree 6 which has constant movement 2, this occurs in this case where $p = 3$ and $m = 2$.)

(3) G is a p-group.

(4) $G = A_5$, $n = 6$, $m = 2$.

All groups in part (1) are examples for Theorem 1.1. In part (2) except the case when p is Mersenne prime, and the part (4) except the groups A_4 and A_5 acting on a set of size 6, we will show that the other groups have some elements whose movements are less than m, which contradicts the fact that every non-identity element of G has the same movement. In part (2), when p is not a Mersenne prime, since every non-identity element of $G = K.P$ has the same movement m, each non-identity element $k \in K$ has $(p-1)$ cycles of length 2^s. We consider the element kk^g of K. This element is fixed point free on Ω and so has movement $p.2^{s-1}$, which is a contradiction. In part (3), by Burnside's lemma, G has a fixed point free element, say g, on a set of size p^a for some positive integer a. Since every fixed point free element has order p with movement $p^a.(p-1)/2$ (see [6, Proposition 4.4), $o(g) = p$ and hence $move(g) = p^{a-1}(p-1)/2$. But by our assumption, $m = p^{a-1}(p-1)/2$. Therefore, each non-identity element g of G is a fixed point free element, so that G is a regular p-group of exponent p. This completes the proof of Theorem 1.1. □

5. Intransitive case

Suppose now that G is an intransitive permutation group on a set Ω of size n, in which every non-identity element has the same movement m. By Corollary 3.1, $|G| = 2^a p^b$ where p is an odd prime, $a \geq 0$, $b \geq 0$. Then we have the following result which is some part of the proof of Theorem 1.2.

We suppose that G is a q-group for an odd prime q. So G is either a 2-group or a p-group. First we consider the case when G is a 2-group which has maximum possible degree and also has constant movement m. Then by [4], m is a power of 2, and G is elementary abelian of order $2m$, all G-orbits have length 2, that is, G is one of the groups in Example 2.1. In this case we obtained part (1) of the theorem. Suppose now that G is a p-group. By Lemma 3.6, G has exponent p. Since G has maximum possible degree, by [1, Theorem 1] $G \cong Z_p^d$, for some positive integer d, $m = p^{d-1}(p-1)/2$, and all G-orbits have length p. Hence in this case we have part (2) of the theorem.

Assume that G is not a q-group, for an odd prime q. Then by Corollary 3.1, G is an EPPO-group and $|G| = 2^a p^b$, where a, b are positive integers. Hence by Burnside's "pq-theorem" [14, Theorem 2.10.17], G is solvable. Since by Lemma 3.6, G is a solvable EPPO-group, it has a non-trivial normal q-subgroup K for some $q \in \{2, p\}$. Suppose G also has a non-trivial normal q'-subgroup for $q' \neq p$. Then G contains an element of order $2p$ contradicting the fact that G is an EPPO-group. Now we consider two cases.

Case 1. $q = 2$. Let $K := O_2(G)$ be the largest normal 2-subgroup of G. Then by Higman's classification [7, Theorem 1] G/K is either a cyclic group whose order is a power of p or a group of order $2^s p^b$ with cyclic Sylow subgroups, p being a prime of the form $k2^t + 1$. Since every p-element of G has order p, so the cyclic p-subgroup is isomorphic to Z_p, that is, each Sylow p-subgroup of G is a cyclic group Z_p.

Now we got the following result in EPO-groups.

Lemma 5.8. *Let G be a EPO-group. Then G is a Frobenius group.*

Proof. Suppose that G is a EPO-group. Then by Lemma 3.6 and Corollary 3.1, $|G| = 2^a p$. So by Burnside's pq-theorem' (see [14, Theorem 2.10.17]), G is a solvable and non-nilpotent EPO-group. Let K be the maximal normal 2-subgroup of G and $S \in$Sylow$_2(G)$. We claim that $K = S$. If not, since G is a EPO-group with constant movement of order $2^a p$ and S is regular, G has a normal 2-complement [8, Sat 8.1 IV]. It implies that there exists in G an element of order $2p$, contradicting the hypothesis. Therefore K is normal in G and $G = K : P$ is a Frobenius group with kernel K and complement P. □

The following result is the classification of intransitive permutation EPO-groups with constant movement, which have normal 2-subgroup.

Lemma 5.9. *Let $G = K : Z_p$, denote the group defined as in Case 1, be an EPO-group where $K = O_2(G)$ of order 2^a, $a \geq 1$. Then G is the semi-direct product of $Z_2^2 : Z_3$ with normal subgroup Z_2^2, $m = 2$ and G has 2 orbits Ω_1 and Ω_2 of length 3 and 4 respectively.*

Proof. By Lemma 5.8, G is a Frobenius group on a set Ω_1, of size 2^a. Let $\Omega_2, ..., \Omega_t$ be others orbits of G such that $|\Omega_2| =, ..., = |\Omega_t| = p$. Set $\Omega = \bigcup_{i=1}^t \Omega_i$. By Burnside's lemma, $t|G| = \sum_{g \in G} |fix(g)|$. So,

$$t2^a p = |\Omega| + \sum_{1 \neq g} |fix(g)| = 2^a + (t-1)p + (2^a - 1)(t-1)p + (2^a p - 2^a)(\frac{2^a - 1}{p}).$$

This equality holds if and only if $(2^a - 1)/p = 1$.
Hence we have $2^a = p + 1$. Suppose that g is a 2-element and h is a p-element of G. Since every non-identity element of G has same movement, so

$$2^{a-1} = move(g) = move(h) = (t-1)\frac{(p-1)}{2} + \frac{2^a - 1}{p}.\frac{p-1}{2} = t\frac{(p-1)}{2}.$$

Thus $2^a = t(p-1)$. Since $t = (p+1)/(p-1)$ is an integer, so this equality holds if and only of $p = 3$, $t = 2$, and hence $G = Z_2^2 : Z_3$ as defined in Example 2.3. □

Case 2. Let $P := O_p(G)$ be the largest normal p-subgroup of G. Then again by Higman's classification [8, Theorem 1], G/P is either a cyclic 2-group, or a generalized quaternion group.

Finally, one may ask whether there exists an example in Case 2.

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Department of Mathematics
Iran University of Science and Technology
Narmak, tehrn 16844, Iran
E-mail address: alaeiyan@iust.ac.ir