Some remarks on a fixed point theorem for Ćirić-type almost contractions

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ABSTRACT.

It is pointed out that the contraction condition used in a fixed point theorem for Ćirić-type almost contractions [Berinde, V., *General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces*, Carpathian J. Math., **24** (2008), No. 2, 10-19] must be slightly modified in order to always ensure the existence of fixed points.

1. INTRODUCTION

The main result in [**?**], included there as Theorem 3.2, has the following statement

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a Ćirić almost contraction, that is, a mapping for which there exist two constants $\alpha \in [0, 1)$ and $L \geq 0$ such that*

$$(1.1) \qquad d(Tx, Ty) \leq \alpha \cdot M(x, y) + L\,d(y, Tx)\,, \quad \text{for all} \;\; x, y \in X\,,$$

where

$$M(x, y) = \max\big\{d(x, y),\, d(x, Tx),\, d(y, Ty),\, d(x, Ty),\, d(y, Tx)\big\}.$$

Then
1) $Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$;
2) For any $x_0 = x \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by

$$(1.2) \qquad x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \dots$$

converges to some $x^* \in Fix(T)$;
3) The following estimate holds

$$(1.3) \qquad d(x_n, x^*) \leq \frac{\alpha^n}{(1 - \alpha)^2}\, d(x, Tx)\,, \quad n = 1, 2, \dots$$

The Ćirić almost contraction condition (**??**) has been obtained by combining the definition of Ćirić quasi-contractions [**?**]: there exists $0 \leq h < 1$ such that for all $x, y \in X$,

$$(1.4) \qquad d(Tx, Ty) \leq h \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

on the one hand, and that of the almost contractions [**?**]: there exist the constants $\delta \in [0, 1)$ and $L \geq 0$ such that

$$(1.5) \qquad d(Tx, Ty) \leq \delta \cdot d(x, y) + L d(y, Tx)\,, \quad \text{for all} \;\; x, y \in X,$$

introduced in [**?**], on the other hand. The latter has been studied in some other papers [**?**], [**?**], [**?**], [**?**], [**?**], [**?**] etc., for the case of both single-valued and multi-valued mappings.

Condition (**??**) is known as one of the most general contractive conditions for which the unique fixed point can be approximated by means of Picard iteration. It is numbered (24) in Rhoades' classification [**?**] that include 25 classical contractive conditions.

On the other hand, condition (**??**) is a very general contractive condition that allows the operator T to have more than one unique fixed point which still could be approximated by means of Picard iteration. It includes many contractive conditions from Rhoades' classification [**?**], among which we mention: Banach's contraction condition, numbered (1), Kannan's condition [**?**], numbered (4), Chatterjea's condition [**?**], numbered (7), Zamfirescu's condition [**?**], numbered (19) etc., but not fully includes Ćirić's quasi-contraction condition (**??**), see for example [**?**] and [**?**].

Another contractive condition due to Ćirić [**?**], and numbered (21) in Rhoades' classification is the following one: there exists $0 \leq h < 1$ such that for all $x, y \in X$,

$$(1.6) \qquad d(Tx, Ty) \leq h \cdot \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}.$$

Condition (**??**) is weaker than the aforementioned ones from Rhoades' classification but it is sharper than (**??**), that is, (**??**) implies (**??**), but not vice-versa, see [**?**]. For other fixed and common fixed point theorems that use the contractive condition (**??**) see also [**?**], [**?**], [**?**], [**?**].

The main aim of this note is to show that a Ćirić type almost contraction satisfying (**??**) could fail to have a fixed point in the case

$$M(x, y) = d(x, Ty)$$

as shown by Example **??**, and to show that if we consider instead of Ćirić's type almost contractions, the class of the so called *Ćirić's strong almost contractions*, obtained by replacing $M(x, y)$ in (**??**) by

$$(1.7) \qquad M_1(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} \left[d(x, Ty) + d(y, Tx) \right] \right\},$$

then we do not need to require explicitly the existence of a fixed point.

2. MAIN RESULTS

Ćirić's strong almost contractions always have a fixed point, as shown by Theorem **??**. We thus obtain the following correct version of Theorem 3.2 in [**?**].

Theorem 2.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be a strong Ćirić almost contraction, that is, a mapping for which there exist two constants $\alpha \in [0, 1)$ and $L \geq 0$ such that*

$$(2.8) \qquad d(Tx, Ty) \leq \alpha \cdot M_1(x, y) + L\, d(y, Tx)\,, \quad \text{for all } x, y \in X\,,$$

*where $M_1(x, y)$ is given by (**??**). Then*
1) *$Fix\,(T) = \{x \in X : Tx = x\} \neq \emptyset$;*
2) *For any $x_0 = x \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (**??**)*
converges to some $x^* \in Fix\,(T)$;
3) *The following estimate holds*

$$(2.9) \qquad d(x_{n+i-1}, x^*) \leq \frac{\alpha^i}{1 - \alpha}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots; i = 1, 2, \ldots$$

Proof. Let $x \in X$ be arbitrary and let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by (**??**) with $x_0 = x$. By taking $x := x_{n-1}$, $y := x_n$ in (**??**), we obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha \cdot M_1(x_{n-1}, x_n)\,,$$

that is,

$$d(x_n, x_{n+1}) \leq \alpha \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2} \left[d(x_{n-1}, x_{n+1}) + 0 \right] \right\},$$

since $d(x_n, Tx_{n-1}) = 0$. Now, by the triangle inequality

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$$

and using the inequality $\dfrac{a + b}{2} \leq \max\{a, b\}$, we deduce that, either

$$(2.10) \qquad \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2} d(x_{n-1}, x_{n+1}) \right\} = d(x_{n-1}, x_n)$$

or

$$(2.11) \qquad \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2} d(x_{n-1}, x_{n+1}) \right\} = d(x_n, x_{n+1}).$$

The case (**??**) cannot hold because it would lead to the contradiction

$$d(x_n, x_{n+1}) \leq h\, d(x_n, x_{n+1}).$$

Hence, always (**??**) must hold, and this leads to

$$d(x_n, x_{n+1}) \leq h\, d(x_{n-1}, x_n).$$

The rest of the proof is similar to that of Theorem 3.2 in [**?**].   □

Note that the error estimate (**??**) obtained in Theorem **??** for Ćirić strong almost contractions is different from the estimate (**??**) that has been deduced in [**?**] for Ćirić strong almost contractions. While the former is the same as in Banach's contraction theorem, see for example [**?**], the latter is slightly different.

It is possible to force the uniqueness of the fixed point of a Ćirić strong almost contraction, like in the case of almost contractions [**?**], by imposing an additional contractive condition, quite similar to (**??**), as shown by the next theorem.

Theorem 2.3. *Let (X, d) be a complete metric space and let $T : X \to X$ be a Ćirić strong almost contraction for which there exist $\theta \in [0, 1)$ and some $L_1 \geq 0$ such that*

(2.12) $$d(Tx, Ty) \leq \theta \cdot d(x, y) + L_1 \cdot d(x, Tx), \quad \text{for all } x, y \in X.$$

Then

1) *T has a unique fixed point, i.e. $Fix(T) = \{x^*\}$;*
2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (??) converges to x^*, for any $x_0 \in X$;*
3) *The error estimate (??) holds.*
4) *The rate of convergence of the Picard iteration is given by*

(2.13) $$d(x_n, x^*) \leq \theta \, d(x_{n-1}, x^*), \quad n = 1, 2, \dots$$

Proof. Assume T has two distinct fixed points $x^*, y^* \in X$. Then by (??), with $x := x^*, y := y^*$ we get

$$d(x^*, y^*) \leq \theta \cdot d(x^*, y^*) \Leftrightarrow (1 - \theta) \, d(x^*, y^*) \leq 0,$$

so contradicting $d(x^*, y^*) > 0$.

Now letting $y := x_n, x := x^*$ in (??), we obtain the estimate (??).
The rest of proof follows by Theorem ??. $\qquad \square$

A stronger but simpler contractive condition that ensures the uniqueness of the fixed point has been obtained by Babu et al. [?] for almost contractions. We state the fixed point theorem corresponding to this uniqueness condition in the case of Ćirić strong almost contractions.

Theorem 2.4. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping for which there exist $\alpha \in [0, 1)$ and some $L \geq 0$ such that for all $x, y \in X$*

(2.14) $$d(Tx, Ty) \leq \alpha \cdot M_1(x, y) + L \min \{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then

1) *T has a unique fixed point, i.e., $Fix(T) = \{x^*\}$;*
2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) converges to x^*, for any $x_0 \in X$;*
3) *The error estimate (??) holds.*

To close this section we mention some important particular cases of the above fixed point theorems.

All fixed point theorems in [?] and [?] are particular cases of Theorems ??-?? in the present paper. These theorems also generalize, amongst many other fixed point theorems, the Banach's contraction principle, Kannan's fixed point theorem [?], Chatterjea's fixed point theorem [?], Reich's fixed point theorems [?], [?], Hardy and Rogers fixed point theorem [?], Zamfirescu's fixed point theorem [?], Ćirić's fixed point theorem [?].

3. EXAMPLES AND CONCLUSIONS

Example ?? illustrates a case when Theorem ?? does not ensure the existence of a fixed point, so requiring the replacement of Ćirić's almost contraction condition (??) by the Ćirić's strong almost contraction condition (??), while Example ?? shows that a Ćirić's strong almost contraction need not be neither a Ćirić quasi-contraction nor any other contraction type mappings in Rhoades' classification.

Example 3.1. Let $X = \mathbb{N} = \{0, 1, 2, \dots\}$ with the usual norm and let T be defined by $T(n) = n + 1$. Then T does satisfy (??) with $\alpha = \dfrac{1}{2}$ and $L = 2$ but T is fixed point free. Indeed, if we take $x = n, y = m, m > n$, then $d(Tx, Ty) = m - n, M(x, y) = m - n + 1, d(y, Tx) = m - n - 1$. Thus condition (??) reduces to

$$m - n \leq \alpha(m - n + 1) + 2(m - n - 1) = \frac{5}{2}(m - n) - \frac{3}{2}$$

which is true, since $m - n \geq 1$.

Example 3.2. Let $[0, 1]$ be the unit interval with usual norm and let $T : [0, 1] \to [0, 1]$ be given by $Tx = \dfrac{x}{2}$ for $x \in [0, 1)$ and $T(1) = 1$.

As T has two fixed points, that is, $Fix(T) = \{0, 1\}$, it does not satisfy neither Ćirić's quasi-contractive condition (??), nor Banach, Kannan, Chatterjea or Zamfirescu contractive conditions etc., but T satisfies the contraction condition (??), and therefore the contractive condition (??), too.

Indeed, for $x, y \in [0, 1)$, condition (??) is satisfied with $a = \dfrac{1}{2}$ and $L \geq 0$ arbitrary, while, for $x \in [0, 1)$ and $y = 1$, condition (??) reduces to

$$\left| \frac{x}{2} - 1 \right| \leq a|x - 1| + L|1 - \frac{x}{2}|,$$

which is obviously true if we take $a < 1$ arbitrary and $L \geq 1$. Therefore T satisfies (??), for all $x, y \in [0, 1]$ with $a = \dfrac{1}{2}$ and $L \geq 1$.

It is clear from their definition that any Ćirić's strong almost contraction is a Ćirić's almost contraction. As the main aim of introducing Ćirić's almost contractions has been to try to include the whole class of Ćirić's quasi contractions, the following problem still remains open: is any Ćirić quasi-contraction a Ćirić strong almost contraction ?

REFERENCES

[1] Babu, G. V. R., Vara Prasad, K. N. V. V., *Common fixed point theorems of different compatible type mappings using Ćirić's contraction type condition*, Math. Commun. **11** (2006) 87-102

[2] Babu, G. V. R., Sandhya, M. L. and Kameswari, M. V. R., *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math. **24** (2008), No. 1, 8–12

[3] Berinde, M. and Berinde V., *On a general class of multi-valued weakly Picard mappings*, J. Math. Anal. Appl. **326** (2007) 772-782

[4] Berinde, V., *General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces*, Carpathian J. Math., **24** (2008), No. 2, 10–19

[5] Berinde, V., *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math., **19** (2003), No. 1, 7–22

[6] Berinde, V., *Approximating fixed points of weak φ-contractions using the Picard iteration*, Fixed Point Theory **4** (2003), No. 2, 131–142

[7] Berinde, V., *Approximation fixed points of weak contractions using the Picard iteration*, Nonlinear Analysis Forum, **9** (2004), No. 1, 43-53

[8] Berinde, V., *Iterative Approximation of Fixed Points*, 2nd Ed., Springer Verlag, Berlin Heidelberg New York, 2007

[9] Chatterjea, S. K., *Fixed-point theorems*, C. R. Acad. Bulgare Sci. **25** (1972) 727–730

[10] Ćirić, Lj. B., *Generalized contractions and fixed-point theorems*, Publ. l'Inst. Math. (Beograd) **12** (1971) 19–26

[11] Ćirić, Lj. B., *On contraction type mappings*, Math. Balkanica **1** (1971) 52–57

[12] Ćirić, Lj. B., *A generalization of Banach's contraction principle*, Proc. Am. Math. Soc. **45** (1974) 267–273

[13] Ćirić, Lj. B., *On a family of contractive maps and fixed points*, Publ. Inst. Math. **31** (1974) 45-51 ,

[14] Ćirić, Lj. B., Razani, A. Radenović, S. and Ume, J.S., *Common fixed point theorems for families of weakly compatible maps*, Comput. Math. Appl. **55** (2008) 2533-2543

[15] Collaço, P., Carvalho e Silva, J., *A complete comparison of 25 contraction conditions*, Nonlinear Anal., TMA, **30** (1997), 741–746

[16] Hardy, G. E. and Rogers, T. D., *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. **16** (1973), No. 2, 201–206

[17] Kannan, R., *Some results on fixed points*, Bull. Calcutta Math. Soc. **10** (1968) 71–76

[18] Meszaros, J., *A comparison of various definitions of contractive type mappings*, Bull. Calcutta Math. Soc. **84** (1992), No. 2, 167–194

[19] Păcurar, Mădălina and Păcurar, R., *Approximate fixed point theorems for weak contractions on metric spaces*, Carpathian J. Math. **23** (2007), No. 1-2, 149–155

[20] Ray B.K., *On Ćirić's fixed point theorem*, Fund. Math. 94 (1977), No. 3, 221-229

[21] Reich, S., *Kannan's fixed point theorem*, Bolletino U.M.I. (4) **4** (1971), 459–465

[22] Reich, S., *Some remarks concerning contraction mappings*, Canad. Math. Bull. **14** (1971), No. 1, 121–124

[23] Rhoades, B. E., *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226** (1977) 257–290

[24] Rus, I. A., *Some fixed point theorems in metric spaces*, Rend. Ist. Mat. Univ. Trieste, **3** (1971), 169–172

[25] Rus, I. A., *Principles and Applications of the Fixed Point Theory* (in Romanian), Editura Dacia, Cluj-Napoca, 1979

[26] Rus, I. A., *Generalized contractions*, Seminar on Fixed Point Theory **3** (1983) 1–130

[27] Rus, I. A., *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001

[28] Singh S. L., Mishra S. N., *On a Ljubomir Ćirić's fixed point theorem for nonexpansive type maps with applications*, Indian J. Pure Appl. Math. **33** (2002) 531-542

[29] Zamfirescu, T., *Fix point theorems in metric spaces*, Arch. Math. (Basel) **23** (1972), 292–298

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