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Metrizable systems of autonomous second order differential equations

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ABSTRACT.
Well-known notions from tangent bundle geometry, namely nonlinear connections and semisprays, are extended to bundle-type tangent manifolds. As main subject, the metrizability of both semisprays and nonlinear connections is investigated through Obata operators.

1. INTRODUCTION

Almost tangent structures were introduced by Clark and Bruckheimer ([4]) and Eliopoulos ([10], [11], [12]) around 1960 and have been investigated by several authors, in [1], [5], [9], [20]. As is well-known, the tangent bundle of a manifold carries a canonical integrable almost tangent structure, hence the name. This almost tangent structure play an important role in the Lagrangian description of analytical mechanics ([9], [13], [17]).

The aim of present paper is to extend two natural objects, namely nonlinear connections and semisprays, from tangent bundles to tangent manifolds geometry. The first geometrical object is studied by means of vertical projectors and the second implies the existence of a global vector field of Liouville type.

The paper is structured as follows. In the second section nonlinear connections are introduced and interpreted as kernels of vertical projectors and the equivalence with other two types of vector 1-forms is proved. In the third section the notion of second order differential system (semispray on short) is defined and the relationship between semisprays and nonlinear connections is discussed in detail. As particular case, the notion of spray corresponds to a homogeneity condition. Types of curves associated in a natural manner to nonlinear connections and semisprays are studied in the next section. The significance of the notion of semispray from the point of view of differential equations is the theme of section 4.

The main part of the paper, namely the section 5, is devoted to the metrizability problem of both semisprays and nonlinear connections. Let us remark that previously, the metrizability of nonlinear connections on tangent bundles was studied by Bucataru in [3]. Therefore, our theorem 5.1 is a generalization of Theorem 2.4 of [3]. Other very interesting studies on the metrizability problem are provided by Olga Krupkova in [14] and [15] in the usual framework of tangent bundles. Thus, our notion of metric on a tangent bundle gives a natural generalization and can be put in correspondence with the main notion of paper [16].

2. NONLINEAR CONNECTIONS ON TANGENT MANIFOLDS

Let M be a smooth, m-dimensional real manifold for which we denote:
$C^\infty(M)$-the real algebra of smooth real functions on M, $\mathcal{X}(M)$-the Lie algebra of vector fields on M, $T^r_s(M)$-the $C^\infty(M)$-module of tensor fields of (r,s)-type on M. An element of $T^1_1(M)$ is usually called *vector 1-form* ([18, p. 176]).

The framework of our paper is fixed by:

Definition 2.1. $J \in T^1_1(M)$ is called *almost tangent structure* on M if:

$$imJ = \ker J. \tag{2.1}$$

The pair (M, J) is an *almost tangent manifold*.

The name is motivated by the fact that (2.1) imply the nilpotence $J^2 = 0$ exactly as the natural tangent structure of tangent bundles, [17].

Denoting $rankJ = n$ it results $m = 2n$. In addition, we suppose that J is integrable i.e.:

$$N_J(X,Y) := [JX, JY] - J[JX,Y] - J[X,JY] + J^2[X,Y] = 0 \tag{2.2}$$

and in this case J is called *tangent structure* and (M, J) is called *tangent manifold*.

In the following we shall work only on tangent manifolds. From [19] we get:

i) the distribution $imJ (= \ker J)$ defines a foliation denoted $V(M)$ and called *the vertical distribution*.

Example 2.1. $M = \mathbb{R}^2, J(x,y) = (0,x)$ is a tangent structure with $\ker J$ the Y-axis, hence the name.

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ii) there exists an atlas on M with local coordinates $(x,y)=\left(x^i,y^i\right)_{1\leq i\leq n}$ such that $J=\frac{\partial}{\partial y^i}\otimes dx^i$ i.e.:

$$J\left(\frac{\partial}{\partial x^i}\right)=\frac{\partial}{\partial y^i},\quad J\left(\frac{\partial}{\partial y^i}\right)=0. \tag{2.3}$$

We call *canonical coordinates* the above (x,y) and the change of canonical coordinates $(x,y)\to(\widetilde{x},\widetilde{y})$ is given by ([19]):

$$\begin{cases}\widetilde{x}^i=\widetilde{x}^i\left(x\right)\\ \widetilde{y}^i=\frac{\partial\widetilde{x}^i}{\partial x^a}y^a+B^i\left(x\right).\end{cases} \tag{2.4}$$

It results an alternative description in terms of G-structures. Namely, a tangent structure is a G-structure with:

$$G=\left\{C=\begin{pmatrix}A & O_n\\ B & A\end{pmatrix}\in GL(2n,\mathbb{R});A\in GL(n,\mathbb{R})\right\}$$

and G is the invariance group of the matrix $J=\begin{pmatrix}O_n & O_n\\ I_n & O_n\end{pmatrix}$ i.e. $C\in G$ if and only if $C\cdot J=J\cdot C$.

Inspired by Definition 1.1 of [2, p. 71] we give a first main notion:

Definition 2.2. A vector 1-form $v:\mathcal{X}\left(M\right)\to\mathcal{X}\left(M\right)$ satisfying:

$$\begin{cases}J\circ v=0\\ v\circ J=J\end{cases} \tag{2.5}$$

is called *vertical projector*.

From (2.5_1) $imv\subseteq\ker J=V\left(M\right)$ and from (2.5_2) $v|_{imJ}=1_{V(M)}$. In conclusion $imv=V\left(M\right)$ and $v^2=v$; these facts explain the name of v.

Another well-known notion in tangent bundles geometry extends to:

Definition 2.3. ([19]) A supplementary distribution N to the vertical distribution $V\left(M\right)$:

$$\mathcal{X}\left(M\right)=N\oplus V\left(M\right) \tag{2.6}$$

is called *normalization* or *horizontal distribution* or *nonlinear connection*. A vector field belonging to N is called *horizontal* and one belonging to $V\left(M\right)$ is called *vertical*.

Because a vertical projector v is $C^\infty\left(M\right)$-linear with $imv=V\left(M\right)$ we have a first important result:

Proposition 2.1. *A vertical projector v yields a nonlinear connection denoted $N\left(v\right)$ through relation $N\left(v\right)=\ker v$.*

This relation is a generalization of remarks from [2, p. 71] where the tangent bundles case is treated. An important remark is that the last result admits a converse. Namely, if N is a nonlinear connection let h_N,v_N the horizontal and vertical projection with respect to the decomposition (2.6).

Proposition 2.2. *v_N is a vertical projector with $N\left(v_N\right)=N$.*

Proof. From $imv_N=V\left(M\right)=\ker J$ it follows (2.5_1). v_N being projector satisfy $v_N\left(V\left(M\right)\right)=V\left(M\right)=imJ$ and then we have (2.5_2). The second fact is immediately from the definition of $N\left(v_N\right)$. □

With respect to the identification nonlinear connection=vertical projector let us point other two equivalent choices:

I) Following [13] we get:

Definition 2.4. A vector 1-form Γ is called *nonlinear connection of almost product type* if:

$$\begin{cases}\Gamma\circ J=-J\\ J\circ\Gamma=J.\end{cases} \tag{2.7}$$

Proposition 2.3. *If Γ is a nonlinear connection of almost product type then:*

(i) $v_\Gamma=\frac{1}{2}\left(1_{\mathcal{X}(M)}-\Gamma\right)$ is a vertical projector,
(ii) $V\left(M\right)$ is the (-1)-eigenspace of Γ,
(iii) $N\left(v_\Gamma\right)$ is the $(+1)$-eigenspace of Γ.

It results that every vertical projector v yields a nonlinear connection of almost product type: $\Gamma=1_{\mathcal{X}(M)}-2v$. From this last relation it results $\Gamma^2=1_{\mathcal{X}(M)}$ i.e. Γ is an almost product structure on M (hence the name).

Proof. (i) $J\circ v_\Gamma=\frac{1}{2}(J-J\circ\Gamma)\overset{(2.7_2)}{=}\frac{1}{2}(J-J)=0$ and $v_\Gamma\circ J=\frac{1}{2}(J-\Gamma\circ J)\overset{(2.7_1)}{=}\frac{1}{2}(J+J)=J$.
(ii) $V\left(M\right)=imv_\Gamma=\left\{X\in\mathcal{X}\left(M\right);\Gamma\left(X\right)=-X\right\}$.
(iii) $N\left(v_\Gamma\right)=\ker v_\Gamma=\left\{X\in\mathcal{X}\left(M\right);\Gamma\left(X\right)=X\right\}$. □

II) Inspired by [18, p. 180] we define:

Definition 2.5. A vector 1-form h is called *horizontal projector* if:

$$\left\{\begin{array}{l} h^2 = h \\ \ker h = V(M). \end{array}\right. \tag{2.8}$$

Proposition 2.4. *If h is a horizontal projector then:*
(i) $v_h = 1_{\mathcal{X}(M)} - h$ is a vertical projector,
(ii) $N(v_h)$ is the $(+1)$-eigenspace of h.
It follows that every vertical projector v yields a horizontal projector:
$h = 1_{\mathcal{X}(M)} - v$.

Proof. (i) From $h\left(1_{\mathcal{X}(M)} - h\right) = 0$ we have $im\left(1_{\mathcal{X}(M)} - h\right) \subseteq \ker h = V(M) = \ker J$ then $J \circ v_h = 0$. Also, $imJ = V(M) = \ker h$ imply $v_h \circ J = J - h \circ J = J$.
(ii) $N(v_h) = \ker v_h = \{X \in \mathcal{X}(M); h(X) = X\}$. □

In canonical coordinates a vertical projector reads:

$$v = N^i_j \frac{\partial}{\partial y^i} \otimes dx^j + \frac{\partial}{\partial y^i} \otimes dy^i = \frac{\partial}{\partial y^i} \otimes \left(N^i_j dx^j + dy^i\right) \tag{2.9}$$

and the functions $\left(N^i_j(x,y)\right)_{1 \leq i,j \leq n}$ are called *the coefficients* of v respectively $N(v)$. A basis of $\mathcal{X}(M)$ adapted to the decomposition (2.6) is $\left\{\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right\}_{1 \leq i \leq n}$ called *Berwald basis*. Then: $v = \frac{\partial}{\partial y^i} \otimes \delta y^i, h = \frac{\delta}{\delta x^i} \otimes dx^i$ where $\{dx^i, \delta y^i = dy^i + N^i_j dx^j\}$ is the dual of Berwald basis.

3. Semisprays on bundle-type tangent manifolds

In the following we suppose that $V(M)$ admits a global section $E = y^i \frac{\partial}{\partial y^i}$ called *Euler vector field* after [19] (on tangent bundles E is called *Liouville vector field*, [2, p. 70]). Again after [19] the triple (M, J, E) will be called *bundle-type tangent manifold* and in this case $\left(B^i\right)$ from (2.4_2) are zero cf. [19]. For examples of bundle-type tangent manifolds see [19].

As in the tangent bundle case ([2, p. 70]) we give a second main notion:

Definition 3.6. If (M, J, E) is a bundle-type tangent manifold then $S \in \mathcal{X}(M)$ is called *semispray* or *second order differential equation* (*sode* on short) if:

$$J(S) = E. \tag{3.10}$$

In canonical coordinates:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y) \frac{\partial}{\partial y^i} \tag{3.11}$$

and the functions $\left(G^i(x,y)\right)$ are *the coefficients* of S.

Another important result is:

Proposition 3.5. *A vertical projector v yields an unique horizontal semispray denoted $S(v)$.*

Proof. This proposition is a generalization of a similar result (without proof) from [2, p. 71]. The formula:

$$G^i = \frac{1}{2} N^i_j y^j \tag{3.12}$$

gives the conclusion. □

In other words:

$$S(v) = y^i \frac{\delta}{\delta x^i}. \tag{3.13}$$

The converse of last result is:

Proposition 3.6. *If S is a semispray then $v_S : \mathcal{X}(M) \to \mathcal{X}(M)$ given by:*

$$v_S(X) = \frac{1}{2}\left(X + [S, JX] + J[X, S]\right) \tag{3.14}$$

is a vertical projector.

Proof. Because:

$$J \circ v_S(X) = \frac{1}{2}(JX - J[JX, S]), \quad v_S \circ J(X) = \frac{1}{2}(JX + J[JX, S])$$

it must to prove that:

$$J[JX, S] = JX \tag{3.15}$$

for every $X \in \mathcal{X}(M)$.

But from (2.2) with $Y = S$:

$$[JX, E] - J[JX, S] - J[X, E] = 0 \tag{3.16}$$

and then (3.15) is equivalent with:

$$[JX, E] = J([X, E] + X). \tag{3.17}$$

Case 1) $X = \frac{\partial}{\partial x^i} \Rightarrow \left[\frac{\partial}{\partial y^i}, y^a \frac{\partial}{\partial y^a}\right] = \frac{\partial}{\partial y^i} = J\left(\frac{\partial}{\partial x^i}\right)$ i.e. (3.17) is true for this case.

Case 2) $X = \frac{\partial}{\partial y^i} \Rightarrow [0, E] = 0 = J\left(\frac{\partial}{\partial y^i} + \frac{\partial}{\partial y^i}\right)$ i.e. (3.17) is true again. □

If S is given by (3.11) then the coefficients of v_S are:

$$N^i_j = \frac{\partial G^i}{\partial y^j}. \tag{3.18}$$

A first natural question is: given the vertical projector $v\left(= N^i_j\right)$ there exists a semispray S such that $v = v_S$? Looking at (3.18) it results that $\left(N^i_j\right)$ must be a gradient with respect to $\left(y^i\right)$. Then if we define: $t^k_{ij} = \frac{\partial N^k_i}{\partial y^j} - \frac{\partial N^k_j}{\partial y^i}$ it results:

Corollary 3.1. *There exists a semispray S such that $v = v_S$ if and only if $t^k_{ij} = 0, 1 \leq i, j, k \leq n$.*

A second natural question is with respect to the sequence:

$$\begin{array}{l} S \to v_S \to S(v_S) \\ G^i \overset{(3.18)}{\to} \frac{\partial G^i}{\partial y^j} \overset{(3.12)}{\to} \frac{1}{2}\frac{\partial G^i}{\partial y^j} y^j \end{array} ;$$

when $S = S(v_S)$?

Corollary 3.2. *Let S be a semispray and v_S the associated vertical projector. Then S is exactly $S(v_S)$ given by Proposition 3.5 if and only if:*

$$[E, S] = S. \tag{3.19}$$

Proof. $v_S(S) = 0 \overset{(3.14)}{\Leftrightarrow} S + [S, E] = 0.$ □

Definition 3.7. A semispray satisfying (3.19) will be called *spray*.

Locally (3.19) means:

$$2G^i = y^j \frac{\partial G^i}{\partial y^j} \tag{3.20}$$

i.e. the functions $\left(G^i\right)$ are homogeneous of degree 2 with respect to variables $\left(y^i\right)$. In terms of the associated vertical projector $v_S = \left(N^i_j\right)$ it results, using (3.18), that $\left(N^i_j\right)$ are homogeneous of degree 1 with respect to $\left(y^i\right)$:

$$N^i_j = y^a \frac{\partial N^i_j}{\partial y^a}. \tag{3.21}$$

The above formulae can be put in a compact form using the Frölicher-Nijenhuis formalism. Recall that for a vector 1-form K and $Z \in \mathcal{X}(M)$ we have the bracket $[K, Z]_{FN} : \mathcal{X}(M) \to \mathcal{X}(M)$ given by ([18, p. 177]):

$$[K, Z]_{FN}(X) = [K(X), Z] - K[X, Z] \tag{3.22}$$

where in the R.H.S. we have the usual Lie bracket of vector fields. Then (3.14) becomes:

$$v_S = \frac{1}{2}\left(1_{\mathcal{X}(M)} - [J, S]_{FN}\right) \tag{3.23}$$

and looking to Proposition 2.3 it results that $[J, S]_{FN}$ is exactly the nonlinear connection of almost product type Γ associated to v_S.

Corollary 3.3. *A semispray S is a spray if and only if:*

$$[v_S, E]_{FN} = 0. \tag{3.24}$$

Proof. Let $X \in \mathcal{X}(M)$. The above relation means: $[v_S(X), E] = v_S([X, E])$.

I) $X = \dfrac{\partial}{\partial y^i} \Rightarrow \left[\dfrac{\partial}{\partial y^i}, E\right] = v_S\left(\left[\dfrac{\partial}{\partial y^i}, E\right]\right)$ which is true because
$\left[\dfrac{\partial}{\partial y^i}, E\right] = E \in V(M)$,

II) $X = \dfrac{\delta}{\delta x^i} \Rightarrow 0 = v_S\left(\left[\dfrac{\delta}{\delta x^i}, E\right]\right) = v_S\left(\left(y^a \dfrac{\partial N_i^j}{\partial y^a} - N_i^j\right)\dfrac{\partial}{\partial y^j}\right) =$
$\left(y^a \dfrac{\partial N_i^j}{\partial y^a} - N_i^j\right)\dfrac{\partial}{\partial y^j}$ which is equivalent with characterization (3.21). □

A third natural question is with respect to the sequence:

$$v \to S(v) \to v_{S(v)}$$
$$N_j^i \overset{(3.12)}{\to} G^i = \frac{1}{2}N_k^i y^k \overset{(3.18)}{\to} \frac{\partial G^i}{\partial y^j};$$

when $v = v_{S(v)}$? We must have: $N_j^i = \dfrac{1}{2}\dfrac{\partial}{\partial y^j}\left(N_k^i y^k\right) = \dfrac{1}{2}N_j^i + \dfrac{1}{2}y^k\dfrac{\partial N_k^i}{\partial y^j}$ and then:

Corollary 3.4. *Let $v\left(= N_j^i\right)$ be a vertical projector and $S(v)$ the associated semispray. Then v is exactly $v_{S(v)}$ given by Proposition 3.6 if and only if:*

$$N_j^i = y^k \frac{\partial N_k^i}{\partial y^j}. \tag{3.25}$$

If $v = v_{S(v)}$ then $S(v)$ is a spray and then $t_{ij}^k = 0$ and (3.21) holds.

A last question is: given the semispray $S\left(= G^i\right)$ there exists a vertical projector v such that $S = S(v)$? So, we must to solve the system $G^i = N_j^i y^j$ in the unknowns $\left(N_j^i\right)$.

We don't know the general answer, yet, but is obviously that if S is spray then the answer is positive with $v = v_S$.

4. Paths of nonlinear connections and semisprays

Let N be a nonlinear connection with associated vertical projector $v = \left(N_j^i\right)$. With respect to the Berwald basis $\left\{\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i}\right\}_{1 \le i \le n}$ we have:

$$\begin{cases} \left[\dfrac{\delta}{\delta x^i}, \dfrac{\delta}{\delta x^j}\right] = R_{ij}^a \dfrac{\partial}{\partial y^a} \\ \left[\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^j}\right] = \dfrac{\partial N_i^a}{\partial y^j}\dfrac{\partial}{\partial y^a} \\ \left[\dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial y^j}\right] = 0 \end{cases} \tag{4.26}$$

where:

$$R_{ij}^a = \frac{\delta N_i^a}{\delta x^j} - \frac{\delta N_j^a}{\delta x^i}. \tag{4.27}$$

Then the horizontal distribution N is integrable if and only if: $R_{ij}^k = 0, 1 \le i, j, k \le n$.

Let us suppose that $v = v_S$ for the semispray $S\left(= G^i\right)$. From (3.23) we get that X is symmetry for v_S if and only if: $\left[1_{\mathcal{X}(M)} - [J, S]_{FN}, X\right]_{FN} = 0$; but $\left[1_{\mathcal{X}(M)}, X\right]_{FN} = 0$ for every X and then X is symmetry for v_S if and only if:

$$\left[[J, S]_{FN}, X\right]_{FN} = 0. \tag{4.28}$$

Looking at local expressions let us note that R_{ij}^a for v_S is:

$$R_{ij}^a = \frac{\delta}{\delta x^j}\left(\frac{\partial G^a}{\partial y^i}\right) - \frac{\delta}{\delta x^i}\left(\frac{\partial G^a}{\partial y^j}\right). \tag{4.29}$$

Since we are interested in dynamics let us study curves on bundle-type tangent manifolds. Let $c = c(t)$ be a curve on M with local expression $c(t) = (x(t), y(t)) = \left(x^i(t), y^i(t)\right)$. Three cases are of importance:

I) c is an *integral curve* of the semispray S. It results from (3.11) the differential system:

$$\begin{cases} \dfrac{dx^i}{dt}(t) = y^i(t) \\ \dfrac{dy^i}{dt}(t) + 2G^i(x(t), y(t)) = 0 \end{cases} \tag{4.30}$$

which explains the name *sode* for S.
II) the tangent field of c is horizontal with respect to the vertical projector v. From (2.9):

$$v\left(\frac{dc}{dt}\right)=v\left(\frac{dx^i}{dt}\frac{\partial}{\partial x^i}+\frac{dy^i}{dt}\frac{\partial}{\partial y^i}\right)=\left(N^i_j\frac{dx^j}{dt}+\frac{dy^i}{dt}\right)\frac{\partial}{\partial y^i}. \tag{4.31}$$

Such a curve is called *h-path* of v and is solution of differential system:

$$\frac{dy^i}{dt}(t)+N^i_j(x(t),y(t))\frac{dx^j}{dt}(t)=0. \tag{4.32}$$

III) a h-path of v satisfying in addition $\frac{dx^i}{dt}=y^i$ will be called *h-integral curve* of v and is solution for:

$$\begin{cases}\frac{dx^i}{dt}(t)=y^i(t)\\ \frac{dy^i}{dt}(t)+N^i_j\left(x(t),\frac{dx}{dt}\right)\frac{dx^j}{dt}(t)=0\end{cases}. \tag{4.33}$$

With respect to Proposition 3.5 comparing (4.30) and (4.33) it results via (3.12):

Proposition 4.7. *A h-integral curve of v is an integral curve of $S(v)$.*

With respect to Proposition 3.6 there is no relation between integral curves of S and v_S in the general case. But in the homogeneous case $(3.20)-(3.21)$ we get:

Proposition 4.8. *If S is a spray then an integral curve of S is a h-integral curve of v_S.*

5. The metrizability problem

5.1. **The general problem of metric pairs.** Let us fix a semispray $S=\left(G^i\right)$ and a nonlinear connection $N=\left(N^i_j\right)$. Recall that, after (3.18), S produces a nonlinear connection $\overset{c}{N}=\left(\overset{c}{N}{}^i_j=\frac{\partial G^i}{\partial y^j}\right)$, c from canonic. Following [3] let us consider:

Definition 5.8. The *dynamical derivative* associated to the pair (S,N) is the map $\overset{SN}{\nabla}:V(M)\to V(M)$ given by:

$$\overset{SN}{\nabla}X=\overset{SN}{\nabla}\left(X^i\frac{\partial}{\partial y^i}\right):=\left(S\left(X^i\right)+N^i_jX^j\right)\frac{\partial}{\partial y^i}. \tag{5.34}$$

Properties:
I) $\overset{SN}{\nabla}\left(\frac{\partial}{\partial y^i}\right)=N^j_i\frac{\partial}{\partial y^j}$,
II) $\overset{SN}{\nabla}(X+Y)=\overset{SN}{\nabla}X+\overset{SN}{\nabla}Y$,
III) $\overset{SN}{\nabla}(fX)=S(f)X+f\overset{SN}{\nabla}X$.

It's easy to extend the action of $\overset{SN}{\nabla}$ to general vertical tensor fields by requiring to preserve the tensor product. More precisely, we will extend $\overset{SN}{\nabla}$ to a special class of tensor fields:

Definition 5.9. A *d-tensor field* (d from distinguished) on M is a tensor field whose change of components, under a change of canonical coordinates $(x,y)\to(\widetilde{x},\widetilde{y})$ on M, involves only factors of type $\frac{\partial\widetilde{x}}{\partial x}$ and (or) $\frac{\partial x}{\partial\widetilde{x}}$.

Example 5.2. i) $\left(\frac{\delta}{\delta x^i}\right)$ and $\left(\frac{\partial}{\partial y^i}\right)$ are components of d-tensor fields of $(1,0)$-type.
ii) $\left(dx^i\right)$ and $\left(\delta y^i\right)$ are components of d-tensor fields of $(0,1)$-type,
iii) $\left(G^i\right)$ are not components of a d-tensor field since a change of coordinates implies:

$$2\widetilde{G}^i=2\frac{\partial\widetilde{x}^i}{\partial x^j}G^j-\frac{\partial\widetilde{y}^i}{\partial x^j}y^j$$

but it results that given two semisprays $\overset{1}{S}$ and $\overset{2}{S}$ their difference $X=\overset{2}{S}-\overset{1}{S}$ is a vertical vector field.
iv) $\left(N^i_j\right)$ are not components of a d-tensor field since a change of coordinates implies:

$$\frac{\partial\widetilde{x}^j}{\partial x^k}N^k_i=\widetilde{N}^j_k\frac{\partial\widetilde{x}^k}{\partial x^i}+\frac{\partial\widetilde{y}^j}{\partial x^i}.$$

It follows that given two nonlinear connections $\overset{1}{N}$ and $\overset{2}{N}$ their difference $F=\overset{2}{N}-\overset{1}{N}=\left(F^i_j=\overset{2}{N}{}^i_j-\overset{1}{N}{}^i_j\right)$ is a d-tensor field of $(1,1)$-type.

Definition 5.10. A *metric* on M is a Riemannian metric g on the vertical distribution: $g = g_{ij}(x,y)\,\delta y^i \otimes \delta y^j$.

It results that (g_{ij}) are the components of a d-tensor field of $(0,2)$-type with the properties:

1) $g_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$,
2) (symmetry) $g_{ij} = g_{ji}$,
3) (nondegeneration) $\det(g_{ij}) > 0$.

From the last property we derive the existence of $g^{-1} = g^{ab}(x,y)\frac{\partial}{\partial y^a} \otimes \frac{\partial}{\partial y^b}$ which is a d-tensor field of $(2,0)$-type.

Definition 5.11. The *dynamical derivative of metric* g is $\overset{SN}{\nabla} g : V(M) \times V(M) \to V(M)$ given by:

(5.35) $$\overset{SN}{\nabla} g(X,Y) = S(g(X,Y)) - g(\overset{SN}{\nabla} X, Y) - g(X, \overset{SN}{\nabla} Y).$$

The main notion of this subsection is:

Definition 5.12. The pair (S,N) is called *metric with respect to* g if:

(5.36) $$\overset{SN}{\nabla} g = 0.$$

The aim of this subsection is to detect all nonlinear connections which together with S form a metric pair for a given g. In order to answer at this question, a look at example 5.2 iv) gives necessary a study of two operators, called *Obata* in the following, acting on the space of d-tensor fields of $(1,1)$-type:

(5.37) $$O^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k \delta^j_l - g^{ij} g_{kl}\right), \quad \overset{*}{O}{}^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k \delta^j_l + g^{ij} g_{kl}\right).$$

The Obata operators are supplementary projectors:

(5.38) $$O^{ia}_{bj} \overset{*}{O}{}^{bk}_{la} = \overset{*}{O}{}^{ia}_{bj} O^{bk}_{la} = 0, \quad O^{ia}_{bj} O^{bk}_{la} = O^{ik}_{lj}, \quad \overset{*}{O}{}^{ia}_{bj} \overset{*}{O}{}^{bk}_{la} = \overset{*}{O}{}^{ik}_{lj}$$

and tensorial equations involving these operators has solutions as follows:

Proposition 5.9. *The system of equations:*

(5.39) $$\overset{*}{O}{}^{ia}_{bj}\left(X^b_{ak}\right) = A^i_{jk}, \qquad \left(O^{ia}_{bj}(X^b_{ak}) = A^i_{jk}\right)$$

with X as unknown has solutions if and only if:

(5.40) $$O^{ia}_{bj}\left(A^b_{ak}\right) = 0, \qquad \left(\overset{*}{O}{}^{ia}_{bj}\left(A^b_{ak}\right) = 0\right)$$

and then, the general solution is:

(5.41) $$X^i_{jk} = A^i_{jk} + O^{ia}_{bj}\left(Y^b_{ak}\right), \qquad \left(X^i_{jk} = A^i_{jk} + \overset{*}{O}{}^{ia}_{bj}\left(Y^b_{ak}\right)\right)$$

with Y an arbitrary d-tensor field of $(1,1)$-type.

We are ready for one of the main results of this paper which is a natural generalization of Theorem 2.4 from [3]:

Theorem 5.1. *Set S and g. The family $\mathcal{N}(S,g)$ of all nonlinear connections $N = \left(N^i_j\right)$ such that (S,N) is metric with respect to g is given by:*

(5.42) $$N^i_j = \frac{1}{2}\overset{c}{N}{}^i_j - \frac{1}{2} g^{ia} g_{jb} \overset{c}{N}{}^b_a + \frac{1}{2} g^{ia} S(g_{aj}) + O^{ia}_{bj}\left(X^b_a\right)$$

with $X = \left(X^b_a\right)$ an arbitrary d-tensor field of $(1,1)$-type. It follows that $\mathcal{N}(S,g)$ is a $C^\infty(M)$-affine module over the $C^\infty(M)$-module of d-tensor fields of $(1,1)$-type.

Proof. We search $\left(N^i_j\right)$ of the form:

(5.43) $$N^i_j = \overset{c}{N}{}^i_j + F^i_j$$

with $\left(F^i_j\right)$ a d-tensor field of $(1,1)$-type to be determined. The local expression of equation (5.36) is:

(5.44) $$S(g_{uv}) - g_{um} N^m_v - g_{mv} N^m_u = 0$$

and inserting (5.43) in (5.44) gives:

$$S(g_{uv}) - g_{um} \overset{c}{N}{}^m_v - g_{mv} \overset{c}{N}{}^m_u = g_{um} F^m_v + g_{mv} F^m_u.$$

Multiplying the last relation with g^{ku} yields:

(5.45) $$g^{ku} S(g_{uv}) - \overset{c}{N}{}^k_v - g^{ku} g_{mv} \overset{c}{N}{}^m_u = F^k_v + g^{ku} g_{mv} F^m_u = 2\, \overset{*}{O}{}^{kb}_{av}\left(F^a_b\right).$$

Let us verify condition (5.40):

$$O^{kb}_{av}\left(g^{am}S\left(g_{mb}\right)-\overset{c}{N}{}^{a}_{b}-g^{am}g_{bl}\overset{c}{N}{}^{l}_{m}\right)=$$

$$=g^{km}S\left(g_{mv}\right)-\overset{c}{N}{}^{k}_{v}-g^{km}g_{vl}\overset{c}{N}{}^{l}_{m}-g^{km}S\left(g_{mv}\right)+g^{km}g_{vl}\overset{c}{N}{}^{l}_{m}+\overset{c}{N}{}^{k}_{v}=0.$$

It follows:

$$F^{i}_{j}=\frac{1}{2}g^{im}S\left(g_{mj}\right)-\frac{1}{2}\overset{c}{N}{}^{i}_{j}-\frac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}^{b}_{a}+O^{ib}_{aj}\left(X^{a}_{b}\right)$$

and returning to (5.43) gives the conclusion. □

In the spray case the equation (5.42) admits a simplification in writing:

Proposition 5.10. *Fix a spray S and a metric g. The family $\mathcal{N}(S,g)$ is:*

$$N^{i}_{j}=\frac{1}{2}\overset{c}{N}{}^{i}_{j}-\frac{1}{2}g^{ia}g_{jb}\overset{c}{N}{}^{b}_{a}+\frac{1}{2}g^{ia}y^{m}\frac{\delta g_{aj}}{\delta x^{m}}+O^{ia}_{bj}\left(X^{b}_{a}\right). \tag{5.46}$$

A natural problem is the variations of $\mathcal{N}(S,g)$ to various changes of S and/or g. We treat here only the well-known case of *conformal transformations*:

Corollary 5.5. *Let $f\in C^{\infty}(M), f>0$ everywhere on M. Then $\mathcal{N}(S,g)=\mathcal{N}(S,fg)$ if and only if f is a first integral of S i.e. $y^{i}\frac{\partial f}{\partial x^{i}}=2G^{i}\frac{\partial f}{\partial y^{i}}$.*

5.2. **Metrizability of nonlinear connections.** Fix a nonlinear connection $N=\left(N^{i}_{j}\right)$ and associate to N the semispray (3.12) which we will denote $S\left(N\right)$.

Definition 5.13. The nonlinear connection N is *metric with respect to* g if the pair $\left(S\left(N\right),N\right)$ is so.

Theorem 5.2. *The nonlinear connection N is metric with respect to g if and only if for all $i,j\in\{1,\ldots,n\}$:*

$$\overset{*}{O}{}^{iv}_{uj}\left(N^{u}_{v}+g^{ua}g_{vb}N^{b}_{a}+g^{um}\frac{\partial g_{mv}}{\partial y^{b}}N^{b}_{a}y^{a}\right)=g^{im}\frac{\partial g_{mj}}{\partial x^{a}}y^{a}. \tag{5.47}$$

Proof. From (5.42) it results that N is metric if:

$$\overset{*}{O}{}^{iv}_{uj}\left(N^{u}_{v}+g^{ua}g_{vb}N^{b}_{a}+g^{um}\frac{\partial g_{mv}}{\partial y^{b}}N^{b}_{a}y^{a}\right)=\overset{*}{O}{}^{iv}_{uj}\left(g^{um}\frac{\partial g_{mv}}{\partial x^{a}}y^{a}\right)$$

and a straightforward computation of the right-hand-side of last equation yields the conclusion. □

Example 5.3. **Riemannian metrics**: suppose $g=g\left(x\right)$. The last relation becomes:

$$\overset{*}{O}{}^{iv}_{uj}\left(N^{u}_{v}+g^{ua}g_{vb}N^{b}_{a}\right)=g^{im}\frac{\partial g_{mj}}{\partial x^{a}}y^{a} \tag{5.48}$$

which is equivalent with:

$$N^{i}_{j}+g^{ia}g_{jb}N^{b}_{a}=g^{im}\frac{\partial g_{mj}}{\partial x^{a}}y^{a}. \tag{5.49}$$

Let us consider that M is the tangent bundle TN and g is a Riemannian metric on N. The Levi-Civita connection of g is a linear connection on M. A symmetric linear connection with coefficients $\left(\Gamma^{i}_{jk}\right)$ yields a semispray S with:

$$G^{i}=\frac{1}{2}\Gamma^{i}_{jk}y^{j}y^{k}. \tag{5.50}$$

The canonic nonlinear connection of this semispray has the coefficients:

$$N^{i}_{j}=\Gamma^{i}_{ja}y^{a}. \tag{5.51}$$

Inserting (5.51) in (5.49) and neglecting y^{a} gives:

$$\Gamma^{i}_{ja}+g^{iu}g_{jv}\Gamma^{v}_{ua}=g^{im}\frac{\partial g_{mj}}{\partial x^{a}}. \tag{5.52}$$

But multiplying last equation with g_{si} we get:

$$g_{si}\Gamma^{i}_{ja}+g_{ji}\Gamma^{i}_{sa}=\frac{\partial g_{sj}}{\partial x^{a}} \tag{5.53}$$

which is the usual Christoffel process. So, we verified the condition (5.49) in the Riemannian setting.

5.3. **Metrizability of semisprays.**

Definition 5.14. The semispray S is called *metric with respect to g* if the pair $\left(S, \overset{c}{N}\right)$ is metric with respect to g.

Inserting $\overset{c}{N}$ in the left-hand-side of (5.42) we get:

Theorem 5.3. *The semispray S is metric with respect to g if and only if, for all $i, j \in \{1, \ldots, n\}$:*

$$\overset{*}{O}{}^{iv}_{uj}\left(\overset{c}{N}{}^{u}_{v} + g^{ua} g_{vb} \overset{c}{N}{}^{b}_{a} - g^{um} S\left(g_{mv}\right)\right) = 0. \tag{5.54}$$

Corollary 5.6. *The spray S is metric with respect to g if and only if, for all $i, j \in \{1, \ldots, n\}$:*

$$\overset{*}{O}{}^{iv}_{uj}\left(\overset{c}{N}{}^{u}_{v} + g^{ua} g_{vb} \overset{c}{N}{}^{b}_{a} - g^{um} y^{a} \frac{\delta g_{mv}}{\delta x^{a}}\right) = 0. \tag{5.55}$$

Example 5.4. Euclidean metrics. Let us consider again M as the tangent bundle TN and g is a constant metric on N i.e. g_{ij} does not depend of (x). The condition (5.54) is:

$$\overset{*}{O}{}^{iv}_{uj}\left(\overset{c}{N}{}^{u}_{v} + g^{ua} g_{vb} \overset{c}{N}{}^{b}_{a}\right) = 0 \tag{5.56}$$

which means:

$$\overset{c}{N}{}^{i}_{j} + g^{ia} g_{jb} \overset{c}{N}{}^{b}_{a} = 0. \tag{5.57}$$

If $N = \mathbb{R}^n$ and g is the usual Euclidean metric then (5.57) reads:

$$\overset{c}{N}{}^{i}_{j} + \overset{c}{N}{}^{j}_{i} = 0 \tag{5.58}$$

i.e. the matrix $\left(\overset{c}{N}{}^{i}_{j}\right)$ belongs to $o\left(n\right)$=the Lie algebra of skew-symmetric matrices of order n. So, we arrive at the well-known result that the orthogonal group $O(n)$ is the structural group of the Euclidean geometry on $\mathbb{R}^n$.

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