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Completely monotonic functions associated with gamma function and applications

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ABSTRACT.

The aim of this paper is to study the behavior of the functions defined by $\Gamma(x+1) = \sqrt{2\pi}(x/e)^x e^{s(x)}$ and $\Gamma(x+1) = \sqrt{2\pi}((x+1/2)/e)^{x+1/2} e^{w(x)}$. As applications, some inequalities involving the gamma function are obtained. Finally, an estimation of the digamma function is given.

1. INTRODUCTION

There are many situations in branches of science when we are forced to deal with big factorials. As a direct computation cannot be made even by the computer programms, approximation formulas were constructed.

One of the most known which gives good results for large values of n is the Stirling formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{1.1}$$

It was first discovered in 1733 by the French mathematician Abraham de Moivre (1667-1754) in the form

$$n! \approx \text{constant}\,\sqrt{n}\left(\frac{n}{e}\right)^n \tag{1.2}$$

(with missing constant), when he was preoccupied to estimate the sums of the form $\sum_{i\leq n} \ln n$. At that time, De Moivre used formula (1.1) in his study on the Maxwellian distribution in statistical mechanics.

In 1730, the Scottish mathematician James Stirling (1692-1770) discovered the constant $\sqrt{2\pi}$ in formula (1.2), while he was trying to give some approximations of the binomial distribution.

Although in applied statistics, formula (1.1) is satisfactory for high values of n, in pure mathematics, more precise approximations are required. One of the first improvement is of type

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\lambda_n}, \tag{1.3}$$

with

$$\frac{1}{12n+1} < \lambda_n < \frac{1}{12n}, \tag{1.4}$$

see *e.g.*, [4, 11]. The right-side bound was proved also in [3, 5, 10, 14], while succesively better values for the left-side bound $1/\left(12n+1/4\right)$, or $1/\left(12n+\dfrac{3}{4n+2}\right)$ were obtained in [3, 5].

Nanjundiah [10] obtained the following stronger result

$$\frac{1}{12n} - \frac{1}{360n^3} < \lambda_n < \frac{1}{12n},$$

if we have in mind that

$$\frac{1}{12n} - \frac{1}{360n^3} > \frac{1}{12n+1}, \quad n \geq 1,$$

$$\frac{1}{12n} - \frac{1}{360n^3} > \frac{1}{12n+\frac{3}{4n+2}}, \quad n \geq 1,$$

$$\frac{1}{12n} - \frac{1}{360n^3} > \frac{1}{12n+\frac{1}{4}}, \quad n \geq 2.$$

An even better result were given by Shi, Liu and Hu [12, Rel. 10] by

$$\frac{1}{12s} - \frac{1}{360s^3} < \lambda_s < \frac{1}{12s} - \frac{1}{360s\,(s+1)\,(s+2)}, \quad s > 0.$$

Note that $s > 0$ can be any positive number and the right-hand side bound is improved for the first time. More better results were given in [12, Theorem 6].

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The gamma function, one of the most famous functions in mathematics as in applied science,

$$\Gamma(x+1)=\int_0^\infty t^x e^{-t}dt$$

is the natural extension of the factorial function to the complex plane, excluding non-positive integers. As a direct consequence of the approximation (1.3) and its estimation (1.4), representation formulas of the form

$$\Gamma(x+1)=\sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{\gamma(x)}$$

or

$$\Gamma(x+1)=\sqrt{2\pi}\left(\frac{x}{e}\right)^x e^{\theta(x)/12x}$$

was introduced in the last decades and they are wide studied by many authors. As an example, we mention the work of Shi, Liu and Hu [12], where an asymptotic series of $\gamma(x)$ is obtained and also it is proved that $\theta(x)$ is increasing for $x>0$.

A stronger result than the monotone function is the notion of completely monotonicity. Recall that, *e.g.*, [15], a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x)\geq 0,$$

for $x\in I$ and $n\geq 0$. If these inequalities are strict, then we say that f is strictly completely monotonic.

As a corollary of the famous Hausdorff-Bernstein-Widder Theorem in [15, Theorem 12a, p. 160], a function f on $(0,\infty)$ is completely monotonic if and only if there exists a non-negative function $\alpha(t)$ such that for every $x\geq 0$,

$$f(x)=\int_0^\infty e^{-xt}\alpha(t)\,dt. \tag{1.5}$$

We prove in this paper that the derivative of the function s defined by the relation

$$\Gamma(x+1)=\sqrt{2\pi}\left(\frac{x}{e}\right)^x e^{s(x)} \tag{1.6}$$

is strictly completely monotonic. Then we use the fact that s is concave to track down the monotonicity of the defining sequence of the famous number e,

$$e_n=\left(1+\frac{1}{n}\right)^n.$$

A slightly more accurate approximation formula than the Stirling's result is the following due to W. Burnside, *e.g.*, [2],

$$\Gamma(x+1)=\sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}e^{w(x)}. \tag{1.7}$$

We prove that $-w$ is strictly completely monotonic, in particular w is concave. As a consequence of the Jensen inequality, we obtain the monotonicity of a sequence related with $(e_n)_{n\geq 1}$.

2. The Results

Shi, Liu and Hu [12] proved their results using the Euler-Maclaurin summation formula involving Bernoulli's numbers and other special functions. We prove here our results using the representation formula (1.5) and Bernstein's Theorem. Similar ideas were used in [6, 7, 8] to give estimations for gamma and polygamma functions.

The logarithmic derivative of $\Gamma(x)$, denoted by

$$\psi(x)=(\ln\Gamma(x))'=\frac{\Gamma'(x)}{\Gamma(x)}$$

is called the psi or digamma function, and their derivatives $\psi^{(i)}(x)$ for $i\geq 1$ are known as the polygamma functions. They play a basic role in the theory of special functions and have lots of extensive applications in many branches of science, as applied statistics, potential theory, numerical analysis, physics, or engineering. The polygamma functions have the following integral representations:

$$\psi^{(i)}(x)=(-1)^{i-1}\int_0^\infty \frac{t^i e^{-xt}}{1-e^{-t}}dt, \tag{2.8}$$

for $i=1,2,3,\ldots$, *e.g.*, [1, 13]. We also use the integral representations

$$\frac{1}{x^n}=\frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-xt}dt\,,\quad n\geq 1. \tag{2.9}$$

Theorem 2.1. *Let s be defined by* (1.6)*. Then s' is strictly completely monotonic.*

Proof. The function s is explicit given by

$$s(x) = \ln\Gamma(x+1) - \ln\sqrt{2\pi} - x\ln x + x.$$

By succesive derivation, we get

$$s'(x) = \psi(x) + \frac{1}{x} - \ln x, \tag{2.10}$$

since $\psi(x+1) = \psi(x) + 1/x$, and

$$s''(x) = \psi'(x) - \frac{1}{x^2} - \frac{1}{x}.$$

Using the integral representations (2.8)-(2.9), we have

$$s''(x) = \int_0^\infty \frac{te^t e^{-tx}}{e^t - 1}dt - \int_0^\infty te^{-xt}dt - \int_0^\infty e^{-xt}dt = -\int_0^\infty e^{-tx}\alpha(t)\,dt,$$

where $\alpha(t) = (e^t - t - 1)/(e^t - 1) > 0$, for $t > 0$.

Thus $s'' < 0$ and then s' is strictly decreasing. From (2.10) we get

$$0 \le s'(x) \le \ln\left(1 + \frac{1}{x}\right),$$

so $\lim_{x\to\infty} s'(x) = 0$. As s' is strictly decreasing, with limit zero at infinity, it results that $s' > 0$ and finally, s' is strictly completely monotonic. □

Corollary 2.1. *For every $x, y \ge 0$, it holds:*

$$\frac{\Gamma^2\left(\frac{x+y}{2}+1\right)}{\Gamma(x+1)\Gamma(y+1)} \ge \frac{\left(\frac{x+y}{2}\right)^{x+y}}{x^x y^y}. \tag{2.11}$$

Proof. As $s'' < 0$, it results that s is concave. Thanks to Jensen, we have

$$2s\left(\frac{x+y}{2}\right) \ge s(x) + s(y),$$

or

$$\ln\Gamma^2\left(\frac{x+y}{2}+1\right) - (x+y)\ln\frac{x+y}{2} \ge \ln\Gamma(x+1) - x\ln x + \ln\Gamma(y+1) - y\ln y,$$

which is the conclusion. □

Surpringsly, if take $x = n-1$ and $y = n+1$ in (2.11), then we obtain

$$\left(1 + \frac{1}{n}\right)^n \ge \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

Theorem 2.2. *Let w be defined by (1.7). Then $-w$ is strictly completely monotonic.*

Proof. The function w is explicit given by

$$w(x) = \ln\Gamma(x+1) - \ln\sqrt{2\pi} - \left(x + \frac{1}{2}\right)\ln\left(x + \frac{1}{2}\right) + x + \frac{1}{2}.$$

By successive derivation, we have

$$w'(x) = \psi(x) + \frac{1}{x} - \ln\left(x + \frac{1}{2}\right) \tag{2.12}$$

and

$$w''(x) = \psi'(x) - \frac{1}{x^2} - \frac{1}{x + 1/2}.$$

Using again the integral representations (2.8)-(2.9), we get

$$w''(x) = \int_0^\infty \frac{te^t e^{-tx}}{e^t - 1}dt - \int_0^\infty te^{-xt}dt - \int_0^\infty e^{-(x+1/2)t}dt = -\int_0^\infty e^{-tx}\beta(t)\,dt,$$

where $\beta(t) = \left(e^{t/2} - e^{-t/2} - t\right)/(e^t - 1) > 0$, for $t > 0$. In this sense, we have

$$\frac{d}{dt}\left(e^{t/2} - e^{-t/2} - t\right) = \frac{1}{2}\left(e^{t/2} - 1\right)^2 e^{-t/2} > 0, \quad t > 0,$$

and $\left(e^{t/2} - e^{-t/2} - t\right)\big|_{t=0} = 0$.

Thus $w'' < 0$ and then w' is strictly decreasing. Using (2.12) we get

$$\ln\frac{x}{x + \frac{1}{2}} \le w'(x) \le \ln\frac{x+1}{x + \frac{1}{2}}$$

so $\lim_{x\to\infty} w'(x) = 0$. As w' is strictly decreasing, with limit zero at infinity, it results that $w' > 0$. Further, $\lim_{x\to\infty} w(x) = 0$, and w is strictly increasing, so, $-w$ is strictly completely monotonic. □

Corollary 2.2. *For every $x, y \geq 0$, it holds:*

$$\frac{\Gamma^2\left(\frac{x+y}{2}+1\right)}{\Gamma(x+1)\Gamma(y+1)} \geq \frac{\left(\frac{x+y+1}{2}\right)^{x+y+1}}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}\left(y+\frac{1}{2}\right)^{y+\frac{1}{2}}} \tag{2.13}$$

Proof. As $-w$ is completely monotonic, it results that $w'' < 0$, so w is concave. Thanks to Jensen, we have

$$2w\left(\frac{x+y}{2}\right) \geq w(x) + w(y),$$

or

$$\ln\Gamma^2\left(\frac{x+y}{2}+1\right) - (x+y+1)\ln\frac{x+y+1}{2} \geq$$

$$\geq \ln\Gamma(x+1) - \left(x+\frac{1}{2}\right)\ln\left(x+\frac{1}{2}\right) + \ln\Gamma(y+1) - \left(y+\frac{1}{2}\right)\ln\left(y+\frac{1}{2}\right),$$

which is equivalent with (2.13). □

For $x = n-1$ and $y = n+1$ in (2.13), we obtain

$$\frac{f_n}{f_{n-1}} \geq \frac{(n+1)(2n+1)}{n(2n+3)} > 1,$$

where

$$f_n = \left(1 + \frac{1}{n+\frac{1}{2}}\right)^{n+\frac{1}{2}}.$$

In the final part, we remark that from the completely monotonicity of s', it results that $s' \geq 0$. Following (2.10), we deduce that

$$\psi(x) \geq \ln x - \frac{1}{x}. \tag{2.14}$$

Related with this result, we mention the following double inequality

$$\ln(x+1) - \frac{1}{x} \geq \psi(x) \geq \ln x - \frac{1}{x}, \tag{2.15}$$

for every $x > 0$, which is stated in [9, Cor. 2.3]. The proof of (2.15) given in [9] is not easy, it is a consequence of two lemmas, which give representations at limit for the extreme sides of (2.15).

Furthermore, from the complete monotonicity of $-w$, it results that $w' \geq 0$ and from (2.12), we get:

$$\psi(x) \geq \ln\left(x+\frac{1}{2}\right) - \frac{1}{x}. \tag{2.16}$$

By co-operation of (2.14)-(2.16), we obtain the double inequality

$$\ln(x+1) - \frac{1}{x} \geq \psi(x) \geq \ln\left(x+\frac{1}{2}\right) - \frac{1}{x},$$

which is an improvement of the estimation (2.15), stated in [9].

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