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Advances on affine vector fields

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ABSTRACT.

Section 1 is introductory in nature [7], [8], while Section 2 contains our results. Section 1 presents certain results on affine mappings [7] and some notions about jet bundles [10]. All of these results are necessary to develop our theory in Section 2. Here we define the energy of a d-tensor field on the first order jet bundle and we prove that this energy is positively homogeneous of degree $m-p$ with respect to the partial velocities. Then the total derivative is used to introduce affine vector fields. We prove that these fields are tangent to the spherical bundle. Our main result (Theorem 2.3) joins a symmetric tensor field, its associated energy and the affine vector fields into an original hyperbolic PDE.

1. INTRODUCTION

Let (M,g) and $(\mathbb{R}^p,h)$ be Riemannian manifolds of dimensions n and p. The local coordinates on M and $\mathbb{R}^p$ will be written $x=(x^i)$, $i=\overline{1,n}$, and $t=(t^\alpha)$, $\alpha=\overline{1,p}$, and the Christoffel symbols will be denoted by G^i_{jk}, $H^\alpha_{\beta\gamma}$ respectively. To state our results, we need some background from [7], [9]:

Definition 1.1. A mapping $\gamma\colon \mathbb{R}^p\to M$ is called affine if it moves the geodesics of the manifold $(\mathbb{R}^p,h)$ into geodesics of the manifold (M,g).

Let $t^\alpha=t^\alpha(s)$, $s\in I\subseteq\mathbb{R}$, $0\in I$, $t^\alpha(0)=0$, $\alpha=\overline{1,p}$, be a geodesic of $\mathbb{R}^p$, that is a solution of the differential system

$$\frac{d^2t^\alpha}{ds^2}+H^\alpha_{\beta\nu}\frac{dt^\beta}{ds}\frac{dt^\nu}{ds}=0,\quad \alpha=\overline{1,p}.$$

From $x^i(s)=x^i(t^\alpha(s))$, it follows $\dfrac{dx^i}{ds}=\dfrac{\partial x^i}{\partial t^\alpha}\dfrac{dt^\alpha}{ds}$, $i=\overline{1,n}$. Differentiating once again, we obtain

$$\frac{d^2x^i}{ds^2}=\frac{\partial^2x^i}{\partial t^\alpha\partial t^\beta}\frac{dt^\alpha}{ds}\frac{dt^\beta}{ds}+\frac{\partial x^i}{\partial t^\alpha}\frac{d^2t^\alpha}{ds^2}=\left(\frac{\partial^2x^i}{\partial t^\beta\partial t^\nu}-H^\alpha_{\beta\nu}x^i_\alpha\right)\frac{dt^\beta}{ds}\frac{dt^\nu}{ds},\quad i=\overline{1,n}.$$

Then

$$\frac{d^2x^i}{ds^2}+G^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=\left(\frac{\partial^2x^i}{\partial t^\beta\partial t^\nu}-H^\alpha_{\beta\nu}x^i_\alpha+G^i_{jk}x^j_\beta x^k_\nu\right)\frac{dt^\beta}{ds}\frac{dt^\nu}{ds},\quad i=\overline{1,n}.$$

Since $x^i=x^i(s)$, $i=\overline{1,n}$, is a geodesic of M, we obtain

$$\frac{d^2x^i}{ds^2}+G^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0,\quad i=\overline{1,n},$$

hence we proved the following result.

Proposition 1.1. *If the mapping $\gamma\colon\mathbb{R}^p\to M$, $\gamma(t)=(x^1(t),\ldots,x^n(t))$ is affine, then it is solution of the hyperbolic PDEs system*

$$\frac{\partial^2x^i}{\partial t^\alpha\partial t^\beta}-H^\nu_{\alpha\beta}x^i_\nu+G^i_{jk}x^j_\alpha x^k_\beta=0,\quad \alpha,\beta=\overline{1,p},\quad i=\overline{1,n}.$$

Theorem 1.1. *Suppose that the hyperbolic PDEs system*

$$\frac{\partial^2x^i}{\partial t^\alpha\partial t^\beta}-H^\nu_{\alpha\beta}x^i_\nu+G^i_{jk}x^j_\alpha x^k_\beta=0,\quad \alpha,\beta=\overline{1,n},\ i=\overline{1,n},$$

has solutions of C^3-class, where $H^\nu_{\beta\alpha\lambda}$ and $G^i_{kj\ell}$ are the Riemann tensors of the manifolds $(\mathbb{R}^p,h)$, (M,g) respectively. Then the complete integrability conditions are equivalent to

$$H^\nu_{\beta\alpha\lambda}x^i_\nu=G^i_{kj\ell}x^\ell_\lambda x^j_\alpha x^k_\beta,\quad \alpha,\beta,\lambda=\overline{1,p},\ i=\overline{1,n}.$$

For a proof, see [4], [5].

To end these preliminaries, let us point out some notions about jet bundles, [3], [10]. Let us consider the bundle $(\mathbb{R}^p\times M,\pi_1,\mathbb{R}^p)$, where π_1 is the projection $\pi_1:\mathbb{R}^p\times M\to\mathbb{R}^p$.

Definition 1.2. A mapping $\phi\colon I\subset\mathbb{R}^p\to\mathbb{R}^p\times M$ is called local section of $(\mathbb{R}^p\times M,\pi_1,\mathbb{R}^p)$ if it satisfies the condition $\pi_1\circ\phi=id_I$. If $t\in\mathbb{R}^p$, then the set of all local sections of π_1, whose domains contain the point t, will be denoted $\Gamma_t(\pi_1)$.

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If $\phi \in \Gamma_t(\pi_1)$ and (t^α, x^i) are coordinate functions around $\phi(t) \in \mathbb{R}^p \times M$, then $x^i(\phi(t)) = \phi^i(t)$.

Definition 1.3. Two local sections $\phi, \psi \in \Gamma_t(\pi_1)$ are called 1-equivalent at the point t if

$$\begin{aligned} &\phi(t) = \psi(t),\\ &\frac{\partial \phi^i}{\partial t^\alpha}(t) = \frac{\partial \psi^i}{\partial t^\alpha}(t), \quad i = \overline{1,n},\ \alpha = \overline{1,p}. \end{aligned}$$

The equivalence class containing ϕ is called the 1-*jet of* ϕ *at the point* t and is denoted by $j_t^1\phi$.

Definition 1.4. The set $J^1(\mathbb{R}^p, M) = \left\{ j_t^1\phi \,|\, t \in \mathbb{R}^p,\ \phi \in \Gamma_t(\pi_1) \right\}$ is called first order jet bundle.

Definition 1.5. Let $(U, (t^\alpha, x^i))$ be an adapted coordinate system on $\mathbb{R}^p \times M$. The induced coordinate system (U^1, u^1) on $J^1(\mathbb{R}^p, M)$ is defined by

$$U^1 = \{ j_t^1\phi \,|\, \phi(t) \in U \}, \quad u^1 = (t^\alpha, x^i, x^i_\alpha),$$

where

$$\begin{aligned} t^\alpha(j_t^1\phi) &= t^\alpha(t),\\ x^i(j_t^1\phi) &= x^i(\phi(t)),\\ x^i_\alpha(j_t^1\phi) &= \frac{\partial \phi^i}{\partial t^\alpha}(t). \end{aligned}$$

$x^i_\alpha \colon U^1 \to \mathbb{R}$ are called derivative coordinates on U^1.

Now we pass to the subject of our paper, formulating original results.

2. Main results

Here, we use an affine mapping

$$\gamma_{x,\xi_\mu} \colon \Omega \to M, \quad \Omega = \prod_{\alpha=1}^{p} [\tau_-^\alpha(x,\xi_\mu), \tau_+^\alpha(x,\xi_\mu)],$$

of C^∞-class, determined by the initial conditions $\gamma_{x,\xi_\mu}(0) = x$, $(\gamma_{x,\xi_\mu})^i_\alpha(0) = \xi^i_\alpha$, $\alpha = \overline{1,p}$, whose image is fixed by a closed border σ of dimension $p-1$ included into ∂M.

Let $(x; \xi_1, \ldots, \xi_p)$, $x \in M$, $\xi_\alpha \in \mathbb{R}^p_x M$, $\xi_\alpha \neq 0$, $\alpha = \overline{1,p}$. In a neighborhood $\mathcal{U} \subset M$ of the point $x \in M$ we consider a local coordinate system $(x^1, \ldots, x^n)$. Then $\xi_\alpha = \xi^i_\alpha \dfrac{\partial}{\partial x^i}(x)$, $\alpha = \overline{1,p}$.

Suppose ∂M is foliated by submanifolds of type σ.

Definition 2.6. The pair of metrics (h, g) is called simple if for any closed border σ of dimension $p-1$ included into ∂M, there is an unique affine mapping

$$x \colon \Omega \cup \partial\Omega \to M \cup \partial M, \quad x(\Omega \cup \partial\Omega) \cap \partial M = \sigma,$$

Ω hyperparallelepiped in $\mathbb{R}^p$, $x(\Omega \setminus \partial\Omega) \subset M$, such that x depends smoothly on σ.

Suppose that (h, g) is a pair of simple metrics.

A tensor field $F^{\alpha_1 \ldots \alpha_m}_{i_1 \ldots i_m}$ is called symmetric if it does not change under the permutation of any two indices of the same type [2].

Definition 2.7. Let $F^{\alpha_1 \ldots \alpha_m}_{i_1 \ldots i_m}$ be a symmetric tensor field of C^∞-class and $\gamma = \gamma_{x(t), x_\mu(t)}$ be the affine mapping of C^∞-class given above.

Let $\tau_-(x, x_\mu) = (\tau_-^1(x, x_\mu), \ldots, \tau_-^p(x, x_\mu))$ and $0 = (0, \ldots, 0)$. If $\Omega_{\tau_-(x,x_\mu),0}$ is the hyperparallelepiped with the diagonal opposite points $\tau_-(x, x_\mu)$ and 0, the function

$$u(t, x, x_\mu) = \int\limits_{\Omega_{\tau_-(x,x_\mu),0}} F^{\alpha_1 \ldots \alpha_m}_{i_1 \ldots i_m}(\gamma_{x,x_\mu}(s)) (\gamma_{x,x_\mu})^{i_1}_{\alpha_1}(s) \cdots (\gamma_{x,x_\mu})^{i_m}_{\alpha_m}(s) ds^1 \ldots ds^p$$

is called the energy of the tensor field F along the affine mapping γ.

Proposition 2.2. *Let* $F^{\alpha_1 \ldots \alpha_m}_{i_1 \ldots i_m}$ *be a symmetric tensor field of* C^∞*-class. Then the function* u *is positively homogeneous of degree* $m-p$ *with respect to partial velocities* x^j_μ*, that is*

$$u(t, x, \lambda x_\mu) = \lambda^{m-p} u(t^\alpha, x^i, x_\mu), \quad \lambda > 0.$$

Proof. We change the variable, by denoting $s = \tilde{s}\lambda^{-1}$.
The Jacobian of this transformation is $J = \lambda^{-p}$, while the conditions $\tau_-^\alpha(x, \lambda x_\mu) \le s^\alpha \le 0$, $\alpha = \overline{1,p}$, become $\tau_-^\alpha(x, x_\mu) \le \tilde{s}^{\,\alpha} \le 0$, $\alpha = \overline{1,p}$, that is $\tilde{s} \in \Omega_{\tau_-(x,x_\mu),0}$.

Therefore, we obtain

$$u(t, x, \lambda x_\mu) = \int\limits_{\Omega_{\tau_-(x,\lambda x_\mu),0}} F^{\alpha_1\ldots\alpha_m}_{i_1\ldots i_m}(\gamma_{x,\lambda x_\mu}(s))(\gamma_{x,\lambda x_\mu})^{i_1}_{\alpha_1}(s)\cdots(\gamma_{x,\lambda x_\mu})^{i_m}_{\alpha_m}(s)ds^1\ldots ds^p,$$

or

$$u(t, x, \lambda x_\mu) = \int\limits_{\Omega_{\tau_-(x,x_\mu),0}} F^{\alpha_1\ldots\alpha_m}_{i_1\ldots i_m}(\gamma_{x,x_\mu}(\tilde{s}))\lambda(\gamma_{x,x_\mu})^{i_1}_{\alpha_1}(\tilde{s})\cdots\lambda(\gamma_{x,x_\mu})^{i_m}_{\alpha_m}(\tilde{s})\lambda^{-p}d\tilde{s}^1\ldots d\tilde{s}^p,$$

that is

$$u(t, x, \lambda x_\mu) = \lambda^{m-p}u(t^\alpha, x^i, x_\mu).$$

□

The affine structure of the first order jet bundle involves the affine vector fields, the energy of tensor fields on affine mappings and an hyperbolic PDE. To introduce those tools, we start with the total derivative of the function u with respect to t^β, $\beta = \overline{1,p}$, that is

$$D_\beta u = \frac{\partial u}{\partial t^\beta} + \frac{\partial u}{\partial x^i}x^i_\beta + \frac{\partial u}{\partial x^i_\alpha}\frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} = \frac{\partial u}{\partial t^\beta} + \frac{\partial u}{\partial x^i}x^i_\beta$$
$$+(H^\nu_{\alpha\beta}x^i_\nu - G^i_{jk}x^j_\alpha x^k_\beta)\frac{\partial u}{\partial x^i_\alpha}, \quad \beta = \overline{1,p}.$$

Thus we obtain the vector fields

$$H_\beta = \frac{\partial}{\partial t^\beta} + x^i_\beta\frac{\partial}{\partial x^i} + (H^\nu_{\alpha\beta}x^i_\nu - G^i_{jk}x^j_\alpha x^k_\beta)\frac{\partial}{\partial x^i_\alpha}, \quad \beta = \overline{1,p},$$

which are called affine vector fields.

Theorem 2.2. *The affine vector fields H_β, $\beta = \overline{1,p}$, are tangent to the spherical bundle*

$$SM\colon\, h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta = 1.$$

Proof. The field N_{SM}, normal to SM, is

$$N_{SM} = \frac{\partial h^{\alpha\beta}}{\partial t^\nu}g_{ij}x^i_\alpha x^j_\beta\, dt^\nu + h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k}x^i_\alpha x^j_\beta dx^k + 2h^{\nu\beta}g_{\ell j}x^j_\beta x^\ell_\nu.$$

It is to notice the orthogonality

$$< N_{SM}, H_\lambda > = \frac{\partial h^{\alpha\beta}}{\partial t^\nu}g_{ij}x^i_\alpha x^j_\beta \delta^\nu_\lambda + h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k}x^i_\alpha x^j_\beta x^k_\lambda + 2h^{\nu\beta}g_{jk}x^j_\beta x^k_\varepsilon H^\varepsilon_{\nu\lambda}$$
$$-2h^{\nu\beta}g_{jk}x^j_\beta x^\ell_\nu x^q_\lambda G^k_{\ell q} = \left(\frac{\partial h^{\alpha\beta}}{\partial t^\lambda} + 2h^{\nu\beta}H^\alpha_{\nu\lambda}\right)g_{ij}x^i_\alpha x^j_\beta$$
$$+h^{\alpha\beta}\left(\frac{\partial g_{ij}}{\partial x^k} - 2g_{\ell j}G^\ell_{ik}\right)x^i_\alpha x^j_\beta x^k_\lambda = 0,$$

due to the equalities

$$G^\ell_{ik} = \frac{1}{2}g^{\ell j}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j}\right),$$
$$\frac{\partial h^{\alpha\beta}}{\partial t^\lambda} = -h^{\alpha\nu}H^\beta_{\lambda\nu} - h^{\nu\beta}H^\alpha_{\lambda\nu}\,.$$

□

Theorem 2.3. *The symmetric tensor field $F^{\alpha_1\ldots\alpha_m}_{i_1\ldots i_m}$, the energy u and the differential operators H_β, $\beta = \overline{1,p}$, are connected by the hyperbolic PDE*

$$(H_1 \circ H_2 \circ \cdots \circ H_p)(u) = F^{\alpha_1\ldots\alpha_m}_{i_1\ldots i_m}(x)x^{i_1}_{\alpha_1}\cdots x^{i_m}_{\alpha_m}.$$

Proof. Let $\gamma = \gamma_{x(t),x_\mu(t)}$ be an affine mapping, determined by its initial conditions, as above. Because of the equality $\gamma_{x(t+t_0),x_\mu(t+t_0)}(s) = \gamma(s + t_0)$, the following relation holds

$$u(t + t_0, x(t + t_0), x_\mu(t + t_0)) = \int\limits_{\Omega_{\tau_-(x,x_\mu),t_0}} F^{\alpha_1\ldots\alpha_m}_{i_1\ldots i_m}(\gamma_{x,x_\mu}(s))(\gamma_{x,x_\mu})^{i_1}_{\alpha_1}(s)\cdots$$
$$\cdots(\gamma_{x,x_\mu})^{i_m}_{\alpha_m}(s)ds^1\ldots ds^p,$$

where $\Omega_{\tau_-(x,x_\mu),t_0}$ is the hyperparallelepiped with the opposite diagonal points $\tau_-(x,x_\mu)$ and $t_0=(t_0^1,\dots,t_0^p)$.

Differentiating the previous equality with respect to t_0^p and then considering $t_0^p=0$, we obtain

$$D_p\big|_{t_0^p=0}u=\int\limits_{\tau_-^1(x,x_\mu)}^{t_0^1}\cdots\int\limits_{\tau_-^{p-1}(x,x_\mu)}^{t_0^{p-1}}F_{i_1\dots i_m}^{\alpha_1\dots\alpha_m}(\gamma_{x,x_\mu}(s^1,\dots,s^{p-1},0))$$
$$(\gamma_{x,x_\mu})_{\alpha_1}^{i_1}(s^1,\dots,s^{p-1},0)\cdots(\gamma_{x,x_\mu})_{\alpha_m}^{i_m}(s^1,\dots,s^{p-1},0)ds^1\dots ds^{p-1}.$$

This relation is differentiated with respect to t_0^{p-1} and then t_0^{p-1} is considered equal to 0, and so on, obtaining

$$D_1\big|_{t_0^1=0}\left(D_2\big|_{t_0^2=0}\cdots\left(D_p\big|_{t_0^p=0}u\right)\right)=F_{i_1\dots i_m}^{\alpha_1\dots\alpha_m}(\gamma_{x,x_\mu}(0))(\gamma_{x,x_\mu})_{\alpha_1}^{i_1}(0)\cdots(\gamma_{x,x_\mu})_{\alpha_m}^{i_m}(0).$$

Therefore, we may conclude that

$$(H_1\circ H_2\circ\cdots\circ H_p)(u)=F_{i_1\dots i_m}^{\alpha_1\dots\alpha_m}(x)x_{\alpha_1}^{i_1}\dots x_{\alpha_m}^{i_m}\qquad(*)$$

□

The relation $(*)$ is called *the hyperbolic PDE on the first order jet bundle*.

We remark that Theorem 2.3 extends a work of Prof. Sharafutdinov, [6].

Regarding related research works, we address the reader to [1], [11].

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