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# Affine legendrians and co-legendrians

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ABSTRACT.

The aim of the paper is to define and to study legendrians and their dual objects, co-legendrians, as generalizations of lagrangians and affine hamiltonians. The structure of closed legendrians, Helmholtz conditions and some properties related to their Euler-Lagrange equations, Hamilton equations and energy form are studied. A duality between hyperregular legendrians and co-legendrians (including their Euler-Lagrange and Hamilton equations) is found.

## 1. INTRODUCTION

A lagrangian system is generally defined by a lagrangian $L : TM \to I\!R$. When the lagrangian is not globally defined, a lagrangian system can be also considered, using a closed 1–form. Hereby, for a given closed 1–form $\omega$ on the manifold $TM$, using the Poincaré Lemma, then $\omega = dL$ only locally. If the coomological class of $\omega$ is not zero, then it does not exist a global $L$ such that $\omega = dL$. In this case, one say that $\omega$ defines a lagrangian system.

The Euler-Lagrange equations of a lagrangian system defined by the lagrangian $L$ have the well-known local form $\frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = 0$.These equations comes from a variational condition imposed to the action of $L$ on curves in $M$. If the lagrangian system is defined by a closed 1–form $\omega = \omega_{(0)i}dx^i + \omega_{(1)i}dy^i$, then the Euler-Lagrange equations have the local form $\frac{d}{dt}\omega_{(1)i} - \omega_{(0)i} = 0$.

A non-lagrangian system is generally given by a lagrangian $L$ and a vertical 1-form $f = f_i(x^j, y^j)dx^i$ on $TM$ (called an *exterior force* [1, 2]). The dynamical equations of the non-lagrangian system have the local form $\frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = f_i$.

More generally, one can consider a non-lagrangian system defined by a closed form $\omega \in \mathcal{X}^*(TM)$ and a vertical 1-form $f$. The non-lagrangian system is equivalently given by a 1-form $\bar{\omega} \in \mathcal{X}^*(TM)$, that has the local form $\bar{\omega} = \omega_i^{(0)}dx^i + (\omega_i^{(1)} + f_i)dy^i$ and is generally not closed.

In the hyperregular case there is a duality between lagrangian and hamiltonian systems, and also between non-lagrangian and non-hamiltonian systems.

Therefore, a hamiltonian system is generally defined by a hamiltonian $H : T^*M \to I\!R$. More generally, a hamiltonian system can be defoned by a closed 1-form $\omega'$ on the manifold $T^*M$. In this case the existence of $H$ such that $\omega' = dH$ is only locally.

The Hamilton equations of a hamiltonian system defined by a hamiltonian $H$ have the well-known local form $\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}$.

A non-hamiltonian system is generally defined by a hamiltonian $H$ and a vertical 1-form $g = g_i(x^j, p_j)dx^i$ on $T^*M$ (see, for example, [6, 7]). The dynamical equations of the non-hamiltonian system are $\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} + g_i$.

More generally, one can consider a non-hamiltonian system defined by a closed form $\omega \in \mathcal{X}^*(T^*M)$ and a vertical 1-form $g$ as above. The non-hamiltonian system is equivalently given by a 1-form $\bar{\omega}' \in \mathcal{X}^*(T^*M)$ given locally by $\bar{\omega}' = \bar{\omega}_i^{(0)}dx^i + (\bar{\omega}^{(1)i} - g^i)dp_i$. Let us observe that, in general, $\bar{\omega}'$ is not closed.

The aim of the paper is to consider legendians as extensions of the lagrangian and non-lagrangian systems and also co-legendrians as extensions of hamiltonian and non-hamiltonian systems. The extensions are performed considering affine bundles and anchors on affine bundles. Some particular examples of legendrians are given in [4], in the case of higher order tangent spaces of a manifold.

Legendrians and co-legendrians on an affine bundle are defined in the second section. A duality between them in the hyperregular case is studied. The Helmholtz conditions and the structures of a closed affine legendrian (Proposition 2.1) and of a closed affine co-legendrian (Proposition 2.2) are given.

Legendrians and co-legendrians on affine anchored bundles are studied in the third section. The Helmholtz conditions are revised in the anchored case (Proposition 3.5). The Euler-Lagrange equations of a legendrian and the Hamilton equations of a co-legendrian are considered and some relations between these equations (Proposition 3.1) are proved. The energy form is defined and it is used to find relations between the solutions of Euler-Lagrange and Hamilton equations of two dual legendrians (Theorem 3.2).

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## 2. Legendrians and co-legendrians on affine bundles

Let $E \xrightarrow{\pi} M$ be an affine bundle, i.e. a local trivial fibration with the fiber type a real and finite dimensional affine space $A$, such that the structure functions are affine transformations. The change rules of local coordinates $(x^i, y^\alpha)$ on $E$, adapted to the affine structure, have the form

$$x^{i'} = x^{i'}(x^j), y^{\alpha'} = g_\alpha^{\alpha'}(x^j)y^{\alpha'} + v^{\alpha'}(x^j). \tag{2.1}$$

A *section* in the affine bundle $E$ is a differentiable map $M \xrightarrow{s} E$ such that $\pi \circ s = id_M$ and its local components change according to the rule $s^{\alpha'}(x^{i'}) = g_\alpha^{\alpha'}(x^j)s^\alpha(x^j) + v^{\alpha'}(x^j)$. Using a suitable partition of unity on the base $M$, it can be easily proved that always there is a (global) section $s : M \to E$.

Notice that a vector bundle is a particular affine bundle and to an affine bundle $E \xrightarrow{\pi} M$ one can associate a vector bundle $\bar{E} \xrightarrow{\bar{\pi}} M$; using local coordinates, if (2.1) are change rules of coordinates on $E$, then $x^{i'} = x^{i'}(x^j)$, $\bar{y}^{\alpha'} = g_\alpha^{\alpha'}(x^j)\bar{y}^\alpha$ are those on $\bar{E}$.

We say that a differential form on $E$, $\omega \in \mathcal{X}^*(E)$, is a *legendrian* on $E$ and a (differentiable) map $L : E \to I\!R$ is a *lagrangian* on $E$. Considering an open submanifold $\widetilde{E} \subset E$ (usually one consider that $\widetilde{E}$ is $E$ less the image of a section $s_0 : M \to E$), we can assume that $\omega$ restricts to $\omega : \widetilde{E} \to T^*\widetilde{E}$ that is differentiable on $\widetilde{E}$ and it is only continuous on $E$; similarly for a lagrangian. An example of a legendrian is the differential $dL$ of a lagrangian $L$.

A *top legendrian* $\tilde{\omega}$ on $E$ is a linear 1-form on the fibers of the vertical bundle $VE \to E$. Notice that $\tilde{\omega}$ is not a differential form on $E$. The action of a differential denoted by $d_v$ can be given on differential forms defined on the fibers of the vertical bundle $VE$. We say that $\tilde{\omega}$ is *v-closed* if $d_v\tilde{\omega} = 0$. Using local coordinates, $\tilde{\omega} = \omega_\alpha(x^j, y^\beta)dy^\alpha$ and $d_v\tilde{\omega} = \dfrac{1}{2}\left(\dfrac{\partial \omega_\alpha}{\partial y^\beta} - \dfrac{\partial \omega_\beta}{\partial y^\alpha}\right) dy^\alpha \wedge dy^\beta$. It is obviously that a *legendrian* $\omega$ defines a top legendrian $\tilde{\omega}$ (if $\omega = \omega_{(0)i}dx^i + \omega_{(0)\alpha}dy^\alpha$, then $\tilde{\omega} = \omega_{(0)\alpha}dy^\alpha$, where $dy^\alpha$ in $\omega$ and $\tilde{\omega}$ have different meanings); if $\tilde{\omega}$ is v-closed, then we say that $\omega$ is a *top closed* legendrian. If a legendrian $\omega$ has a null top legendrian, one say that $\omega$ is a *semi-basic*1*-form related to* $E$. In this case $\omega$ has a local form $\omega = \omega_i(x^j, y^\alpha)dx^i$ and the local functions $(\omega_i)$ change according to the rule $\omega_i = \dfrac{\partial x^{i'}}{\partial x^i}\omega_{i'}$.

**Proposition 2.1.** *If $\omega$ is a closed legendrian, then there is a lagrangian $L : E \to I\!R$ and a closed form $\theta' \in \mathcal{X}^*(M)$ such that $\omega = dL + \pi^*\theta'$, where $\pi : E \to M$ is the canonical projection. If $\omega$ is a top closed legendrian, then there is a lagrangian $L : E \to I\!R$ and a semi-basic*1*-form $\theta$ related to $E$ such that $\omega = dL + \theta$.*

*Proof.* If $\omega$ is closed, then according to Poincaré lemma, $\omega$ is locally exact. Thus locally, for each open set $U \subset E$ in an open cover $\mathcal{U}$ of $E$, there is a function $L_U : U \to I\!R$ (a local lagrangian) such that $\omega = dL_U$. We can take $U$ such that $U' = \pi(U) \subset M$ is open and $\{U' = \pi(U)\}$ is an open cover $\mathcal{U}'$ of $M$. Let $\{\varphi'_{U'}\}$ be a partition of unity on $M$, which is subordinated to $\mathcal{U}'$. The family $\{\varphi_U = (\pi^*\varphi'_{U'})_{|U}; U \in \mathcal{U}\} \subset \mathcal{F}(E)$ is a partition of unity on $E$, subordinate to the cover $\mathcal{U}$. Thus $L = \sum_{U \in \mathcal{U}} \varphi_U L_U \in \mathcal{F}(E)$ is a global defined lagrangian. One can prove that the form $\omega - dL$ is closed and it has the local form $\omega - dL = \theta_i dx^i$. Since $\omega$ is closed, it follows that $\theta_i = \theta_i(x^i)$, thus there is a global closed 1-form $\theta' \in \mathcal{X}^*(M)$ such that $\omega - dL = \pi^*\theta'$.

Let $\omega$ be a top closed legendrian. We can perform a similar construction as in the case of a closed lagrangian. We can take local lagrangians $\{L_U\}$ suct that $dL_U$ have the same top legendrians as $\omega$, a partition of unity that glues together all these in $L = \sum_{U \in \mathcal{U}} \varphi_U L_U \in \mathcal{F}(E)$, thus $\theta = \omega - dL$ has a null top legendrian, i.e. $\theta$ is a semi-basic1-form. □

In the case of an affine space $A$ at least two duals can be considered for $A$:

(1) The dual vector space $\bar{A}^*$, where $\bar{A}$ is the vector space that is a model for $A$ and
(2) The affine dual space $A^\dagger = \{\omega : A \to I\!R, \omega \text{ is an affine map}\}$.

Both duals $\bar{A}^*$ and $A^\dagger$ are vector spaces, but $\dim \bar{A}^* = \dim A = \dim A^\dagger - 1$. They are related by the exact sequence of vector spaces:

$$0 \to I\!R \xrightarrow{j} \mathcal{A}^\dagger \xrightarrow{\pi} \bar{A}^* \to 0, \tag{2.2}$$

where $j$ is the inclusion that associates to 1 a constant but not vanishing affine map and $\pi$ is the linear map induced on vectors.

In order to construct a duality lagrangian - hamiltonian, it is preferably to choose the affine dual.

Thus, if $(E, \pi, M)$ is an affine bundle, we denote by $(\bar{E}, \bar{\pi}, M)$ the associated vector bundle and by $(E^\dagger, \pi^\dagger, M)$ the associated affine bundle that has as fibers the affine duals of the fibers of $E$. The vector bundle map $\pi$ in (2.2) induces an epimorphism of vector bundles $\Pi : E^\dagger \to \bar{E}^*$ that can be viewed as a projection of an affine bundle with a one dimensional fiber. A section $h : \bar{E}^* \to E^\dagger$ of this affine bundle is, by definition, an *affine hamiltonian* on $E$. Let us give the local form of an affine hamiltonian. Let us consider local coordinates $(x^i, y^\alpha)$ on $E$ that change according to formulas (2.1), coordinates $(x^i, z_\alpha)$ on $\bar{E}^*$, such that $z_\alpha = g_\alpha^{\alpha'} z_{\alpha'}$, and coordinates $(x^i, z_\alpha, \omega)$ on $E^\dagger$, such that $\omega$ change according to the rule $\omega' = \omega + v^{\alpha'} z_{\alpha'}$. Thus an affine hamiltonian has the local form $(x^i, z_\alpha) \xrightarrow{h} (x^i, z_\alpha, H_0(x^i, z_\alpha))$.

Notice that the local functions $H_0$ change according to the rules

(2.3) $$H'_0 = H_0 + v^{\alpha'} z_{\alpha'}.$$

The change rules of some local coordinates $(x^i, y^\alpha, p_i, z_\alpha)$ on $T^*E$ are:

(2.4) $$x^{i'} = x^{i'}(x^i), y^{\alpha'} = g_\alpha^{\alpha'} y^\alpha + v^{\alpha'},$$

$$p_{i'} \frac{\partial x^{i'}}{\partial x^i} + \left( \frac{\partial g_\alpha^{\alpha'}}{\partial x^i} y^\alpha + \frac{\partial v^{\alpha'}}{\partial x^i} \right) z_{\alpha'} = p_i, g_\alpha^{\alpha'} z_{\alpha'} = z_\alpha,$$

It follows that there is a map $\pi' : T^*E \to \bar{E}^*$, given in local coordinates by $(x^i, y^\alpha, p_i, z_\alpha) \stackrel{\pi'}{\to} (x^i, z_\alpha)$ and this map is the canonical projection of an affine bundle.

Using relation (2.3) one can deduce that the local definition $(x^i, z_\alpha) \to \left(x^i, \frac{\partial H_0}{\partial z_\alpha}\right)$ gives a global bundle map (in general, not affine) $\mathcal{H}$:$E \to \bar{E}^*$, called the *Legendre map* of $h$. One can also verify by a straightforward computation that the local definition

(2.5) $$(x^i, z_\alpha) \to \left( x^i, \frac{\partial H_0}{\partial z_\alpha}, -\frac{\partial H_0}{\partial x^i}, z_\alpha \right).$$

gives a global map $Dh : \bar{E}^* \to T^*E$ that play the role of a differential of $h$, as well as an extension of the Legendre map of $h$. In the case when $h = H_0 : T^*M \to I\!R$ is a classical hamiltonian, then $Dh$ can be obtained as a composition of the following maps: first $\left( x^i, z_i) \stackrel{dH_0}{\to} (x^i, z_i, \frac{\partial H_0}{\partial x^i}, \frac{\partial H_0}{\partial z_i} \right)$ is the differential $d : T^*M \to T^*T^*M$ of $H_0$, then $\# : T^*T^*M \to TT^*M$, $(x^i, z_i, X^i, Z^i) \stackrel{\#}{\to} (x^i, z_i, -Z^i, X^i)$ is the canonical anchor defined by the canonical symplectic structure on $T^*M$ and $\tau : TT^*M \to T^*TM$, $(x^i, z_i, Z^i, X^i) \to (x^i, Z^i, X^i, z_i)$ is the canonical flip; finnaly $D = \tau \circ \# \circ d$. This decomposition of $D$ is not possible to be made in the general affine case.

The existence of $Dh$ suggests to define an *affine co-legendrian* on $E$ as a section $\eta : \bar{E}^* \to T^*E$ of the affine bundle defined by $\pi'$. Using local coordinates, $\eta$ has the local form

(2.6) $$(x^i, z_\alpha) \stackrel{\eta}{\to} (x^i, \eta^\alpha(x^i, z_\alpha), \eta_i(x^i, z_\alpha), z_\alpha).$$

The change rules of the local functions $(\eta^\alpha, \eta_i)$ can be deduced from the second and the third relations (2.4).

Let us consider the induced affine bundle $\bar{\pi}_0^* E \stackrel{\bar{\pi}'}{\to} \bar{E}^*$, over the base $\bar{E}^*$, where $\bar{\pi}_0 : \bar{E}^* \to M$ is the canonical projection. A *top affine co-legendrian* is a section $\bar{\eta} : \bar{E}^* \to \bar{\pi}_0^* E$ in this bundle. To give a top affine co-legendrian $\bar{\eta}$ is equivalently to give a fibered manifold map $\mathcal{L}^* : \bar{E}^* \to E$, called a *co-Legendre map*.

An affine co-legendrian $\eta : \bar{E}^* \to T^*E$ defines a top affine co-legendrian $\bar{\eta}$ with the co-Legendre map $\mathcal{L}^* = \pi'' \circ \eta$, where $\pi'' : T^*E \to E$ is the canonical projection. The *co-Legendre map* of $\eta$ is the co-Legendre map of $\bar{\eta}$. Using local coordinates, if $\eta$ has the local form (2.6), then $\bar{\eta}$ and $\mathcal{L}^*$ have the local forms $(x^i, z_\alpha) \stackrel{\bar{\eta}}{\to} (x^i, z_\alpha, \eta^\alpha(x^i, z_\alpha))$ and $(x^i, z_\alpha) \stackrel{\mathcal{L}^*}{\to} (x^i, \eta^\alpha(x^i, z_\alpha))$ respectively.

The *v-curvature* of the top affine co-legendrian $\bar{\eta} : \bar{E}^* \to \bar{\pi}_0^* E$ is the section $r : \bar{E}^* \to \wedge^2 \bar{\pi}^* \bar{E}^*$ in the vector bundle $\wedge^2 \bar{\pi}^* \bar{E}^* = \bar{\pi}^* \bar{E}^* \wedge \bar{\pi}^* \bar{E}^* \to \bar{E}^*$, defined by $r = d_v \eta^\alpha \wedge dz_\alpha = \frac{1}{2} \left( \frac{\partial \eta^\alpha}{\partial z_\beta} - \frac{\partial \eta^\beta}{\partial z_\alpha} \right) dz_\alpha \wedge dz_\beta$, where $d_v f = \frac{\partial f}{\partial z_\alpha} dz_\alpha$. It is easy to see that $r$ vanishes iff $\frac{\partial \eta^\alpha}{\partial z_\beta} = \frac{\partial \eta^\beta}{\partial z_\alpha}$, thus iff there is a local function $f : U \to I\!R$, $U \subset \bar{E}^*$, such that $\eta^\alpha = \frac{\partial f}{\partial z_\alpha}$. We say that $\bar{\eta}$ is:

: *v-closed* if it has a null curvature and

: *exact* if there is an affine hamiltonian $h$ such that $\bar{\eta}$ is the top affine co-legendrian of $Dh$.

If $\bar{\eta}$ is closed, then it has a null curvature, thus it is locally exact, as remarked above.

The *curvature* of an affine co-legendrian $\eta : \bar{E}^* \to T^*E$ is the section $R : \bar{E}^* \to \wedge^2 T^* \bar{E}^*$ in the vector bundle $\wedge^2 T^* \bar{E}^* = T^* \bar{E}^* \wedge T^* \bar{E}^* \to \bar{E}^*$, defined by $R = d\eta^\alpha \wedge dz_\alpha - d\eta_i \wedge dx^i = \frac{1}{2} \left( \frac{\partial \eta^\alpha}{\partial z_\beta} - \frac{\partial \eta^\beta}{\partial z_\alpha} \right) dz_\alpha \wedge dz_\beta - \frac{1}{2} \left( \frac{\partial \eta_i}{\partial x^j} - \frac{\partial \eta_j}{\partial x^i} \right) dx^i \wedge dx^j + \left( \frac{\partial \eta^\alpha}{\partial x^i} + \frac{\partial \eta_i}{\partial z_\alpha} \right) dz_\alpha \wedge dx^i$. It follows that $R$ vanishes iff

(2.7) $$\frac{\partial \eta_i}{\partial x^j} - \frac{\partial \eta_j}{\partial x^i} = 0, \frac{\partial \eta^\alpha}{\partial x^i} + \frac{\partial \eta_i}{\partial z_\alpha} = 0, \frac{\partial \eta^\alpha}{\partial z_\beta} - \frac{\partial \eta^\beta}{\partial z_\alpha} = 0,$$

thus iff there is a local function $f : U \to I\!R$, $U \subset \bar{E}^*$, such that $\eta^\alpha = \frac{\partial f}{\partial z_\alpha}$ and $\eta_i = -\frac{\partial f}{\partial x^i}$. We say that $\eta$ is:

: *closed* if it has a null curvature and

: *exact* if there is an affine hamiltonian such that $\eta = Dh$.

If $\eta$ is closed, then using the remark above there is a local $f$ such that $\eta^\alpha = \dfrac{\partial f}{\partial z_\alpha}$ and $\eta_i = \dfrac{\partial f}{\partial x^i}$, thus $\eta$ is locally exact.

We call relations (2.7) as *Helmholtz conditions* for an affine co-legendrian $\eta$ and the local functions $\eta_{ij} = \dfrac{\partial \eta_i}{\partial x^j} - \dfrac{\partial \eta_j}{\partial x^i}$, $\eta_i^\alpha = \dfrac{\partial \eta^\alpha}{\partial x^i} + \dfrac{\partial \eta_i}{\partial z_\alpha}$, $\eta^{\alpha\beta} = \dfrac{\partial \eta^\alpha}{\partial z_\beta} - \dfrac{\partial \eta^\beta}{\partial z_\alpha}$ as *Helmholtz coefficients.*

We say that $\eta$ is *top closed* if its associated top affine hamiltonian $\bar{h}$ is v-closed.

A *semi-basic*1*-form related to* $\bar{E}^*$ is a section $\theta' : \bar{E}^* \to \bar{\pi}_0^* TM$ in the vector bundle $\bar{\pi}_0^* TM \to \bar{E}^*$, where $\bar{\pi}_0 : \bar{E}^* \to M$ is the canonical projection. For example, if $\theta \in \mathcal{X}^*(M)$, then $\bar{\pi}_0^* \theta$ is a semi-basic1-form related to $\bar{E}^*$. If $h : \bar{E}^* \to E^\dagger$ is an affine hamiltonian and $\theta : \bar{E}^* \to \bar{\pi}_0^* TM$ is a semi-basic1-form related to $\bar{E}^*$, then one can consider the sum $\eta + \theta$, defined as follows. If $\eta$ and $\theta$ have the local forms (2.6) and $(x^i, z_\alpha) \xrightarrow{\theta'} (x^i, \theta_i(x^i, z_\alpha))$ respectively, then $(x^i, z_\alpha) \xrightarrow{\eta+\theta} (x^i, \eta^\alpha, \eta_i + \theta_i, z_\alpha)$.

**Proposition 2.2.** *If $\eta$ is a closed affine co-legendrian, then there is an affine hamiltonian $h : \bar{E}^* \to E^\dagger$ and a closed form $\theta' \in \mathcal{X}^*(M)$ such that $\eta = Dh + \bar{\pi}_0^* \theta'$. If $\eta$ is a top closed affine co-legendrian, then there is an affine hamiltonian $h : \bar{E}^* \to E^\dagger$ and a semi-basic*1*-form $\theta$ related to $\bar{E}^*$ such that $\eta = Dh + \theta$.*

*Proof.* Let $\bar{\pi} : \bar{E}^* \to M$ be the canonical projection. Since $\eta$ is closed, using equations (2.7), one can prove that $\eta$ is locally exact. Thus for each open set $U \subset \bar{E}^*$ in an open cover $\mathcal{U}$ of $\bar{E}^*$, there is a section $H_U : U \to U \times I\!R$, $u \xrightarrow{H_U} (u, H_{0U})$ (a local affine hamiltonian) such that $\eta = dH_U$. We can take $U$ such that $U' = \bar{\pi}(U) \subset M$ is open and $\{U' = \bar{\pi}(U)\}$ is an open cover $\mathcal{U}'$ of $M$. Let $\{\varphi'_{U'}\}$ be a partition of unity on $M$, which is subordinated to $\mathcal{U}'$. The family $\{\varphi_U = (\bar{\pi}^* \varphi'_{U'})_{|U}; U \in \mathcal{U}\} \subset \mathcal{F}(E)$ is a partition of unity on $E$, subordinate to the cover $\mathcal{U}$. The expression $H_0 = \sum\limits_{U \in \mathcal{U}} \varphi_U H_{0U} \in \mathcal{F}(E)$ defines a global affine hamiltonian $h$. The form $\eta - Dh$ is closed and has the local form $\omega - Dh = \theta_i dx^i$. From the vanishing curvatures of $\omega$ it follows easily that $\theta_i = \theta_i(x^i)$ comes from a global closed 1-form $\theta' \in \mathcal{X}^*(M)$, i.e. $\omega - d\eta = \pi^* \theta'$.

Let $\eta$ be a co-legendrian that is top closed. We can performe a similar construction as in the case of a closed co-legendrian. We can take local affine hamiltonians $\{H_{0U}\}$ that have the same top legendrians as $\eta$, a partition of unity that glues together all these in $h = \sum\limits_{U \in \mathcal{U}} \varphi_U H_{0U} \in \mathcal{F}(E)$, thus $\theta = \eta - Dh$ has a null top legendrian, i.e. $\theta$ is a semi-basic1-form. □

Notice that comparing with the proof of Proposition 2.1, the proof above uses that the partition of unity gives a convex hull of local sections in the affine bundle $\Pi : E^\dagger \to \bar{E}^*$, instead of real functions, and the difference $\eta - Dh$ is no longer an affine hamiltonian, but a vertical 1-form related to $\bar{E}^*$.

We say that a top affine co-legendrian $\bar{\eta}$ is *hyperregular* if its co-Legendre map $\mathcal{L}^*$ is a diffeomorphism. The inverse of $\mathcal{L}$ defines the Legendre map of a top affine legendrian $\bar{\omega}$, that we call the *inverse* of $\bar{\eta}$. We say that a co-legendrian is *hyperregular* if its associated top co-legendrian is hyperregular.

Analogous definitions can be considered for legendrians. A top affine legendrian $\bar{\omega}$ defines the *Legendre map*, that is a fibered manifold map $E \xrightarrow{\mathcal{L}} \bar{E}^*$. Then that $\bar{\omega}$ is *hyperregular* if $\mathcal{L}$ is a diffeomorphism. The inverse of $\mathcal{L}$ defines the co-Legendre map of a top affine co-legendrian $\bar{\eta}$, that we call the *inverse* of $\bar{\omega}$. A legendrian is *hyperregular* if its associated top legendrian is hyperregular.

We define below a duality between hyperregular affine legendrians and hyperregular affine co-legendrians on $E$, that in particular gives a duality between hyperregular lagrangians and hyperregular affine hamiltonians.

Let $\omega : E \to T^*E$ be a hyperregular affine legendrian and $\mathcal{L} : E \to \bar{E}^*$ be its Legendre map . We define the (hyperregular) affine co-legendrian $\eta : \bar{E}^* \to T^*E$ as the composition $\bar{E}^* \xrightarrow{\mathcal{L}^{-1}} E \xrightarrow{\omega} T^*E$. Using local coordinates $\omega = \omega_{(0)i}(x^j, y^\alpha) dx^i + \omega_{(1)\alpha}(x^j, y^\alpha) dy^\alpha$ and $\tilde{\omega} = \omega_{(1)\alpha}(x^j, y^\alpha) dy^\alpha$. Let $\tilde{\eta} = \eta^\alpha(x^j, z_\beta) dz_\alpha$ be the top co-legendrian that is inverse to the top legendrian $\tilde{\omega}$; it reads $\omega_{(1)\alpha}(x^j, \eta^\alpha(x^j, z_b)) = z_\alpha$. Then $\eta_{(0)i}(x^j, z_b) = \omega_{(0)i}(x^j, \eta^\alpha(x^j, z_b))$ and $\eta = \eta_{(0)i} dx^i + \eta^\alpha dz_\alpha$.

By duality, if $\eta : \bar{E}^* \to T^*E$ is a hyperregular co-legendrian and $\tilde{\eta}$ its associated top co-legendrian, then one can consider its dual legendrian $\omega : E \to T^*E$ as the composition $E \xrightarrow{(\mathcal{L}^*)^{-1}} \bar{E}^* \xrightarrow{\eta} T^*E$, i.e. $\omega = \eta \circ (\mathcal{L}^*)^{-1}$. Using local coordinates, $\eta = \eta_{(0)i} dx^i + \eta^\alpha dz_\alpha$ and $\tilde{\eta} = \eta^\alpha(x^j, z_b) dz_\alpha$. Let $\tilde{\omega} = \omega_{(1)\alpha}(x^j, y^\alpha) dy^\alpha$ be the top legendrian that is inverse to the top co-legendrian $\tilde{\eta}$. Then $\omega_{(0)i}(x^j, y^b) = \eta_{(0)i}(x^j, \omega_{(1)\alpha}(x^j, y^b))$ and $\omega = \omega_{(0)i} dx^i + \omega_{(1)\alpha} dy^\alpha$.

We can associate with a legendrian $\omega = \omega_{(0)i}(x^j, y^\alpha) dx^i + \omega_{(1)\alpha}(x^j, y^\alpha) dy^\alpha$ its *Helmholtz conditions:*

$$\frac{\partial \omega_{(0)i}}{\partial x^j} - \frac{\partial \omega_{(0)j}}{\partial x^i} = 0, \; \frac{\partial \omega_{(0)i}}{\partial y^\alpha} - \frac{\partial \omega_{(1)\alpha}}{\partial x^i} = 0, \; \frac{\partial \omega_{(1)\alpha}}{\partial y^\beta} - \frac{\partial \omega_{(1)\beta}}{\partial y^\alpha} = 0, \tag{2.8}$$

that come from the condition $d\omega = 0$, i.e. $\omega$ be closed. As in the case of a co-legendrian, one can call the local functions that are local components of $d\omega$ as *local Helmholtz coefficiens*.

If $L : E \to I\!R$ is a lagrangian, then it is obsiously that $\omega = dL$ is a legendrian that satisfy the Helmholtz conditions. One say that $L$ is *hyperregular* if $dL$ is a hyperegular legendrian. The validity of the following assertions are simple consequences of the definition, provided that $L$ is hyperregular:

**:** the dual co-legendrian $\eta$ of $dL$ verify the Helmholtz conditions (2.7),

**:** $\eta = Dh$ comes from an affine hamiltonian $h : \bar{E}^* \to E^\dagger$ and

**:** $\eta$ is hyperregular (i.e. its associate co-legendrian $Dh$ given by (2.5) is hyperregular).

## 3. Legendrians on anchored affine bundles

In order to have some Hamilton equations of a co-legendrian and some Lagrange equations of a legendrian, we suppose that the affine bundle has an affine anchor. It is the case of higher order legendrians of a manifold [4].

If $E \xrightarrow{\pi} M$ is an affine bundle, an affine map $\rho : E \to TM$ is called an *affine anchor* on $E$, or simply an *anchor*, when no confusion is possible. Using local coordinates, $\rho$ has the form $(x^i, y^\alpha) \xrightarrow{\rho} (x^i, \rho^j(x^i, y^\alpha))$ and the anchor has the local form:

$$\rho^i(x^i, y^\alpha) = y^\alpha D^i_\alpha(x^j) + E^i(x). \tag{3.9}$$

A special case is the higher tangent space $T^kM$, for $k \geq 2$, when $T^kM \xrightarrow{\pi_k} T^{k-1}M$ is an affine bundle and there is a inclusion map $h_k : T^kM \to TT^{k-1}M$ that is an affine bundle map, thus an (affine) anchor. We study this case in a subsequent paper.

If $E \xrightarrow{\pi} M$ is a vector bundle and $\rho : E \to TM$ is a vector bundle map, then the anchor is linear and its local form is $\rho^i(x^i, y^\alpha) = y^\alpha D^i_\alpha(x^j)$. For example, when $E = TM$, then $\rho = id_{TM}$ is an anchor. An other example is when $E \xrightarrow{\pi} M$ is an integrable distribution of constant rank on $M$, and the anchor $\rho$ is the inclusion.

Any affine anchor $\rho : E \to TM$ induces a bundle map $\rho^* : T^*M \to \bar{E}^*$, which we call a *co-anchor*. Using coordinates and (3.9), then $\rho^*(x^i, p_i) = (x^i, p_i D^i_\alpha)$.

We say that a curve $\gamma : I \to E$, $I = (a, b) \subset I\!R$, is *adapted to the anchor* if $(\pi \circ \gamma)_* = \rho \circ \gamma$, where $f_*$ denotes the differential of $f$. Using local coordinates, $\gamma$ has the form $t \xrightarrow{\gamma} (x^i(t), y^\alpha(t))$ and $\dfrac{dx^i}{dt} = \rho^i(x^i, y^\alpha)$ on $I$.

We associate the affine Hamilton equations with an affine co-legendrian $\eta$, as follows. A curve $\gamma : I \to E^*$ is a *fiber solution* of the affine Hamilton equations if the following conditions are fulfilled:

H1 The curve $\gamma_1 = \mathcal{L}^* \circ \gamma : I \to E$ is adapted to the anchor, i.e. $\rho \circ \gamma_1 = \dfrac{d(\pi \circ \gamma_1)}{dt}$, where $\pi : E \to M$ is the canonical projection. If $\gamma$ has a local form $\gamma(t) = (x^i(t), z_\alpha(t))$, then $\gamma_1(t) = (x^i(t), h^\alpha(t))$, where $h^\alpha(t) = \eta^\alpha(x^i(t), z_\beta(t))$ and $\rho^i(x^i(t), h^\alpha(t)) = \dfrac{dx^i}{dt}$.

H2 There are some local functions $f_i : I \to I\!R$, $i = \overline{1, m}$, $m = \dim M$, such that $\dfrac{df_i}{dt} = \eta_i(x^j, f_i D^i_\alpha) - f_j \dfrac{\partial \rho^j}{\partial x^i}(x^j, h^\alpha)$.

The *affine Hamilton equations* of $\eta$ are:

$$\left\{ \begin{aligned} \frac{dx^i}{dt} &= \rho^i(x^i(t), h^\alpha(t)), \\ \frac{df_i}{dt} &= \eta_i(x^j, f_i D^i_\alpha) - f_j \frac{\partial \rho^j}{\partial x^i}(x^j, h^\alpha). \end{aligned} \right. \tag{3.10}$$

Notice that the local functions $\{f_i\}_{i=\overline{1,m}}$ define a curve $\gamma^* : I \to T^*M$, $t \to f_i dx^i_{|\gamma(t)}$ that we call a *base solution* of the affine Hamilton equations.

In the case when $E = TM$ and $\rho = id_{TM}$, i.e. $\rho^i(x^j, y^j) = y^i$, the affine Hamilton equations takes the form

$$\left\{ \begin{aligned} \frac{dx^i}{dt} &= \eta^i(x^i, p_i), \\ \frac{dp_i}{dt} &= -\eta_{(0)i}(x^i, p_i), \end{aligned} \right. \tag{3.11}$$

which we call the *Hamilton equations* of $\eta$ ; it is called in [6] a *classical system*. An other case is when $E = T^kM$ and the anchor $h_k : T^kM \to TT^{k-1}M$ is the inclusion. We study this situation in a subsequent paper.

We describe in that follows the solutions of the affine Hamilton equations of an affine co-legendrian, without imposing to the affine co-legendrian the hyperregularity condition. We define below the energy of an affine co-legendrian on $T^*M$, such that the integral curves of its dual vector field are base solutions and also gives fiber solutions of the affine Hamilton equations.

We define the *energy form* of the affine co-legendrian $\eta$ as the co-legendrian $\Omega \in \mathcal{X}^*(T^*M)$, given by: $\Omega_i(x^i, z_j) = -\eta_i(x^j, p_i D^i_\alpha) + p_j \dfrac{\partial \rho^j}{\partial x^i}(x^j, h^\alpha)$, $\Omega^i = \rho^i(x^i, h^\alpha)$, where $h^\alpha = \eta^\alpha(x^i, p_i D^i_\alpha)$.

**Proposition 3.3.** *The co-legendrian $\Omega$ is well-defined.*

*Proof.* It suffices to prove that the change rule of local functions $(\Omega_i, \Omega^i)$ is $\Omega^{i'} = \dfrac{\partial x^{i'}}{\partial x^i}\Omega^i$ and $\Omega_i = \dfrac{\partial x^{i'}}{\partial x^i}\Omega_{i'} + \dfrac{\partial z_{i'}}{\partial x^i}\Omega^{i'}$. The first relation is obvious fulfilled. For the second one, one use the change rules of all local functions and coordinates; by a straightforward computation one obtain the conclusion. □

The canonical symplectic structure on the manifold $T^*M$ gives an isomorphism of vector bundles $\Phi : T^*T^*M \to TT^*M$.

**Proposition 3.4.** *The integral curves of the vector field $\Phi \circ \Omega$ are base solutions of the Hamilton equations of the co-legendrian $\eta$. The co-anchor $\rho^* : T^*M \to \bar{E}^*$ sends base solutions into fiber-solutions.*

*Proof.* The vector field $\Phi \circ \Omega$ has the local form $\Phi \circ \Omega = \Omega^i \dfrac{\partial}{\partial x^i} - \Omega_i \dfrac{\partial}{\partial p_i}$. Taking account into the definitions of $\Omega$ and of Hamilton equations of $\eta$, the conclusion follows. □

Let us consider the Helmholtz coefficients $\eta^{\alpha\beta}$, $\eta_i^\alpha$ and $\eta_{ij}$ of $\eta$. The following statement is obtained by a straightforward computation.

**Proposition 3.5.** *For $z_\alpha = D_\alpha^i p_i$, we have the following relations between Helmholtz coefficients of $\Omega$ and those of $\eta$:*

$$\left\{\begin{array}{l} \dfrac{d\Omega^i}{dp_j} - \dfrac{d\Omega^j}{dp_i} = D_\alpha^i D_\beta^j \eta^{\alpha\beta}, \\ \dfrac{d\Omega^i}{dx^j} - \dfrac{d\Omega_j}{dp_i} = D_\alpha^i \eta_j^\alpha, \\ \dfrac{d\Omega_i}{dx^j} - \dfrac{d\Omega_j}{dx^i} = \eta_{ij} + \eta_i^\alpha \dfrac{D_\alpha^k}{\partial x^j} p_k - \eta_j^\alpha \dfrac{D_\alpha^k}{\partial x^i} p_k. \end{array}\right. \tag{3.12}$$

A simple consequence is the following statement.

**Proposition 3.6.** *Let $\eta$ be top closed and the affine anchor $\rho$ be injective. Then $\Omega$ is closed iff $\eta$ is closed.*

One can obtain also the following statement, by a straightforward computation.

**Proposition 3.7.** *If $\eta$ is exact (i.e. there is an affine hamiltonian $h$ such that $\eta = Dh$), then $\Omega$ is exact (i.e. there is a hamiltonian $\mathcal{E}$ on $M$, such that $\Omega = d\mathcal{E}$); $\mathcal{E}$ is locally given by $\mathcal{E}(x^i, p_i) = p_i E^i(x^i) + H_0(x^i, p_i D_\alpha^i)$, where $(x^i, z_\alpha) \xrightarrow{h} (x^i, z_\alpha, H_0(x^i, z_\alpha))$ and $(x^i, y^\alpha) \xrightarrow{\rho} (x^i, y^\alpha D_\alpha^i + E^i)$ are the local forms of $h$ and $\rho$ respectively.*

We call $\mathcal{E}$ *the energy* of $h$. The affine Hamilton equations have the following form in this case:

$$\left\{\begin{array}{l} \dfrac{dx^i}{dt} = \rho^i(x^j, h^\alpha), \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H_0}{\partial x^i}(x^j, p_i D_\alpha^i) - p_j \dfrac{\partial \rho^j}{\partial x^i}(x^j, h^\alpha), \end{array}\right. \tag{3.13}$$

where $h^\alpha = \dfrac{\partial H_0}{\partial z_\alpha}(x^i, p_i D_\alpha^i)$.

Using Proposition 2.2, one obtain the following statement.

**Proposition 3.8.** *Let $\eta = Dh + \bar{\pi}_0^* \theta'$ be a closed affine co-legendrian and $\mathcal{E}$ be the energy of $h$. Then the energy form of $\eta$ is $\Omega = d\mathcal{E} + \theta'$, thus $\Omega$ is a closed co-legendrian. Let $\eta = Dh + \theta$ be a top closed co-legendrian and $\mathcal{E}$ be the energy of $h$. Then there is a semi-basic1-form $\theta_1$ related to $T^*M$ such that the energy form $\Omega = d\mathcal{E} + \theta_1$, thus $\Omega$ is a top closed co-legendrian.*

In the case when $E \xrightarrow{\pi} M$ is a vector bundle and $\rho : E \to TM$ is a vector bundle map that the local form (3.9), we have $E^i = 0$. Then $\mathcal{E}(x^i, z_i) = H_0(x^i, z_i D_\alpha^i)$, $\mathcal{E} : T^*M \to I\!R$ is a hamiltonian on $M$ and $H_0 : \bar{E}^* \to I\!R$ is a sub-hamiltonian on $E$.

The hamiltonian vector field $X_\mathcal{E}$ is defined according to the formula $d\mathcal{E} = i_{X_\mathcal{E}}\Omega$ and it has the local expression:

$$X_\mathcal{E} = \frac{\partial \mathcal{E}}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial \mathcal{E}}{\partial x^i}\frac{\partial}{\partial p_i}. \tag{3.14}$$

Thus an integral curve of the vector field $X_\mathcal{E}$ is a solution of the well known Hamilton equations:

$$\left\{\begin{array}{l} \dfrac{dx^i}{dt} = \dfrac{\partial \mathcal{E}}{\partial p_i}, \\ \dfrac{dp_i}{dt} = -\dfrac{\partial \mathcal{E}}{\partial x^i}. \end{array}\right. \tag{3.15}$$

A straightforward calculation shows that $\dfrac{\partial \mathcal{E}}{\partial p_i} = E^i + D_\alpha^i h^\alpha$ and

$\dfrac{\partial \mathcal{E}}{\partial x^i} = \dfrac{\partial H_0}{\partial x^i}(x^i, z_\alpha(x^i, p_i)) + z_j \dfrac{\partial E^j}{\partial x^i}(x^j, h^\alpha) + z_j \dfrac{\partial D_\alpha^j}{\partial x^i} h^\alpha$, where $z_\alpha(x^i, p_i) = p_i D_\alpha^i$ and $h^\alpha(x^i, p_i) = \dfrac{\partial H_0}{\partial \Omega_\alpha}(x^i, z_\alpha(x^i, p_i))$.

Thus we obtain the following form of the affine Hamilton equations:

$$\left\{\begin{array}{l}\dfrac{dx^i}{dt}=\rho^i(x^j,h^\alpha),\\[2mm] \dfrac{dz_i}{dt}=-\dfrac{\partial H_0}{\partial x^i}(x^j,z_\alpha)-p_j\dfrac{\partial\rho^j}{\partial x^i}(x^j,h^\alpha).\end{array}\right. \tag{3.16}$$

Notice that the hamiltonian vector field of $\mathcal{E}$ has the local form

$$X_{\mathcal{E}}=\rho^i(x^j,h^\alpha)\frac{\partial}{\partial x^i}-\left(\frac{\partial H_0}{\partial x^i}(x^j,z_\alpha)+p_j\frac{\partial\rho^j}{\partial x^i}(x^j,h^\alpha)\right)\frac{\partial}{\partial p_i}.$$

**Theorem 3.1.** *The integral curves of the hamiltonian vector field $X_{\mathcal{E}}$ define curves on $\hat{E}$ which are solutions of the affine Hamilton equations (3.10) of $H_0$.*

*Proof.* By a straightforward calculus one verify that a solution $t\to(x^i(t),f_i(t))$ of (3.16) fulfills (3.10). □

Let $\gamma:I\to T^*M$, $I=[0,1]$ be a curve, which has the local form $\gamma(t)=(x^i(t),z_i(t))$. We say that $\gamma$ is *admissible* if for every $t\in I$ one have $\dfrac{dx^i}{dt}=\rho^i(x^j,h^\alpha)$.

If one regard $\mathcal{E}:T^*M\to I\!R$ as a hamiltonian on $M$, continuous on $T^*M$ and differentiable on $\widetilde{T^*M}$, the *integral action* of $\mathcal{E}$ on a curve $\gamma:I\to T^*M$, $I=[0,1]$, which has the local form $\gamma(t)=(x^i(t),z_i(t))$ is

$$I_{\mathcal{E}}(\gamma)=\int_0^1\left[z_i\left(\frac{dx^i}{dt}-E^i\right)-H_0(x^i,z_\alpha)\right]dt,$$

where $z_\alpha=D^i_\alpha z_i$. It is well-known that the singular curves of this action are solutions of the Hamilton equations; from first equation (3.16) it follows that these integral curves are adapted to the anchor. Considering the restriction of the action $I_{\mathcal{E}}$ on curves that are adapted to the anchor, it follows that the solutions of the Hamilton equations are critical curves for this action.

We can consider, in particular, a (strict) concave hamiltonian $h$, i.e. that it has the property that the vertical hessian $\dfrac{\partial(-H_0)}{\partial\Omega_\alpha\partial\Omega_\beta}$ is (strict) positive defined. In this case the hessian is non-degenerate.

**Proposition 3.9.** *Let $h:\bar{E}^*\to E^\dagger$ be a hyperregular affine sub-hamiltonian. Then there is a hyperregular sub-lagrangian $L:E\to I\!R$ on $E$ defined by $h$.*

*Proof.* We define the sub-lagrangian $L$ by the local formula $L(x^i,y^\alpha)=L_\alpha(x^i,y^\alpha)y^\alpha-H_0(x^i,L_\gamma(x^i,y^\alpha))$,where the affine sub-hamiltonian $h$ has the local form $(z_\alpha)\xrightarrow{h}(z_\alpha,H_0(z_\alpha))$, $\mathcal{L}$ is the inverse of the co-Legendre transformation (i.e. $\mathcal{L}=\mathcal{H}^{-1}$) having the local form $\mathcal{L}(x^i,y^\alpha)=(x^i,L_\gamma(x^i,y^\alpha))$. □

**Proposition 3.10.** *Let $L:E\to I\!R$ be a hyperregular sub-lagrangian. Then there is a hyperregular affine sub-hamiltonian $h:\bar{E}^*\to E^\dagger$ defined by $L$.*

*Proof.* We define the affine sub-hamiltonian $h$ by the local formula $(z_\alpha)\xrightarrow{h}(z_\alpha,H_0(z_\alpha))$, where $H_0:\bar{E}^*\to I\!R$ is given by

$$H_0(x^i,z_\alpha)=z_\alpha H^\alpha(x^i,z_\alpha)-L(x^i,H^\gamma(x^i,z_\alpha)),\mathcal{H}$$

is the inverse of the Legendre transformation (i.e. $\mathcal{H}=\mathcal{L}^{-1}$) having the local form $\mathcal{H}(x^i,z_\alpha)=(x^i,H^\gamma(x^i,z_\alpha))$. □

An affine hamiltonian and the lagrangian corresponding by Propositions 3.9 and 3.10 are called *dual* each to the other.

In that follows we define the Euler-Lagrange equations of a legendrian defined on an anchored affine bundle. Let us consider a legendrian on the affine bundle $E\xrightarrow{\pi}M$, i.e. a differentiable form $\omega\in\mathcal{X}^*(E)$. We suppose also that the anchor $\rho:E\to TM$ is affine, thus it has the local form (3.9).

We associate with $\omega$ the affine Euler-Lagrange equations, defined as follows. A curve $\gamma:I\to E$ is a *fiber solution* of the Euler-Lagrange equations if $\gamma$ is adapted to the anchor and there are some local functions $f_i:I\to I\!R$, $i=\overline{1,m}$, $m=\dim M$, such that the *Euler-Lagrange equations* holds:

$$\left\{\begin{array}{l}\dfrac{df_i}{dt}=\omega_i(x^i,y^\alpha)-f_j\dfrac{\partial\rho^j}{\partial x^i}(x^i,y^\alpha),\\[2mm] f_iD^i_\alpha=\omega_\alpha(x^i,y^\alpha),\end{array}\right. \tag{3.17}$$

where $\gamma$ has the local form $t\xrightarrow{\gamma}(x^i(t),y^\alpha(t))$. Notice that the local functions $\{f_i\}_{i=\overline{1,m}}$ define a curve $\gamma^*:I\to T^*M$, $t\to f_i dx^i_{|\gamma(t)}$, that we call a *base solution* of the Euler-Lagrange equations.

In the case when $E=TM$ and $\rho=id_{TM}$, i.e. $\rho^i(x^j,y^j)=y^i$, then the affine Euler-Lagrange equations take the well-known form $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial y^i}\right)=\dfrac{\partial L}{\partial x^i}(x^i,y^\alpha)$ and $f_i(t)=\dfrac{\partial L}{\partial y^i}(x^i(t),y^\alpha(t))$.

We prove in the following that if $\omega$ is a hyperregular affine legendrian, then its affine Euler-Lagrange equations can be solved using the Hamilton equations of its dual affine co-legendrian.

**Theorem 3.2.** *Let $\omega \in \mathcal{X}^*(E)$ be an affine legendrian that is hyperregular, $\eta$ be its dual affine co-legendrian, $\Omega$ be its energy form and $\gamma_0$ be an integral curve of the vector field $\Phi \circ \Omega$. Then:*

(1) *The curve $\gamma_0$ is a base solution of the Hamilton equations of $\eta$ and of the Euler-Lagrange equations of $\omega$.*

(2) *The curve $\gamma = \rho^* \circ \gamma_0$ is a fiber solution of the Hamilton equations of $\eta$ and the curve $\gamma_1 = \mathcal{L}^* \circ \gamma$ is a fiber solution of the Euler-Lagrange equations of $\omega$ that corresponds to $\gamma$.*

*Proof.* Let $t \overset{\gamma_0}{\to} (x^i(t), f_i(t))$ be the local form of $\gamma_0$. Then the local forms of $\gamma$ and $\gamma_1$ are $t \overset{\gamma}{\to} D^i_\alpha(x^j) f_i$ and $t \overset{\gamma_1}{\to} y^\alpha = h^\alpha(x^i, D^i_\alpha f_i)$ respectively. Using the relations between $\omega$ and $\eta$ and Proposition 3.4, the conclusion follows. $\square$

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