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On the stability roughness of discrete dynamical systems in infinite-dimensional spaces

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ABSTRACT.
The aim of this paper is to provide new methods concerning the study of stability radius of discrete dynamical systems in infinite-dimensional spaces. We study the stability roughness of a discrete dynamical system subjected to general structured perturbations. We determine a lower bound for the stability radius in terms of the norm of the input-output operators acting between two Banach sequence spaces which are invariant under translations.

1. INTRODUCTION

Exponential stability is one of the most important properties of evolution equations which became in recent years an intensively studied subject (see [2]–[8], [14]–[18], [23]). A significant class of evolution equations with various applications in chaos, population dynamics, economics and biology is represented by the discrete dynamical systems (see [5]–[8] and the references therein).

In the last few years many research studies were focused on the asymptotic properties of discrete-time systems and to their applications in control theory (see [2]–[23]). In this context, the roughness of asymptotic properties had a central role. Roughly speaking the radius related with an asymptotic behavior estimates the size of the smallest perturbation in the presence of which the system "loses" the initial qualitative property (see [2]–[4], [14], [16], [17], [21], [23]). It is well known that the concept of stability radius was introduced by Hinrichsen, Ilchman and Pritchard in their works (see [2]–[4]) and led to various studies of exponential stability of linear systems in the presence of multi-structured feedback type perturbations (see [2]–[4], [14], [16], [17], [23]). The concept of dichotomy radius was recently studied in [19] and [21].

The aim of this paper is to present a new study concerning the roughness of the exponential stability of discrete dynamical systems. We continue the line of the study begun in [17], but we propose a distinct and more general perspective on the stability radius of discrete dynamical systems. We consider as main tool in our theory the use of input-output operators acting on Banach sequence spaces which are invariant under translations and contain at least a characteristic function of a singleton.

First, we deduce a characterization for uniform exponential stability of discrete dynamical systems in terms of Banach sequence spaces in terms of the solution of an input-output control system. After that, we associate with a discrete dynamical system (A) the perturbed system $(A + BPC)$ corresponding to a general feedback–type perturbation and introduce the stability radius $r_{stab}(A, B, C)$. We point out some new situations and we obtain various and very general lower bounds for $r_{stab}(A, B, C)$ in terms of the norm of the input-output operators between Banach sequence spaces which belong to a certain class. The main results generalize the previous estimations from the literature and also extend the applicability area to any discrete dynamical system in infinite-dimensional spaces.

2. BANACH SEQUENCE SPACES

In this section, for the sake of clarity, we will recall some basic definitions and properties of Banach sequences spaces. These spaces are often used in interpolation theory (see [1] and the references therein).

Let $\mathbb{Z}$ denote the set of the integers, let $\mathbb{N}$ denote the set of all non negative integers, let $\mathbb{R}$ denote the set of all real numbers and let $\mathcal{S}(\mathbb{N}, \mathbb{R})$ be the linear space of all sequences $s : \mathbb{N} \to \mathbb{R}$. Let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For every set $A \subset \mathbb{N}$ we denote by χ_A the characteristic function of the set A. For every $s \in \mathcal{S}(\mathbb{N}, \mathbb{R})$ we consider the sequence $s_+ : \mathbb{N} \to \mathbb{R}$ defined by $s_+(0) = 0$ and $s_+(n) = s(n-1)$, for all $n \in \mathbb{N}^*$.

Definition 2.1. A linear space $B \subset \mathcal{S}(\mathbb{N}, \mathbb{R})$ is called *normed sequence space* if there is a mapping $| \cdot |_B : B \to \mathbb{R}_+$ such that:

(i) $|s|_B = 0$ if and only if $s = 0$;
(ii) $|\alpha s|_B = |\alpha| \, |s|_B$, for all $(\alpha, s) \in \mathbb{R} \times B$;
(iii) $|s + \gamma|_B \le |s|_B + |\gamma|_B$, for all $s, \gamma \in B$;
(iv) if $|s(j)| \le |\gamma(j)|$, for all $j \in \mathbb{N}$ and $\gamma \in B$, then $s \in B$ and $|s|_B \le |\gamma|_B$.

If, moreover, $(B, | \cdot |_B)$ is complete, then B is called *Banach sequence space*.

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Definition 2.2. A Banach sequence space $(B, |\cdot|_B)$ is called *invariant under translations* if for every $s \in B$ the sequence $s_+ \in B$ and $|s_+|_B = |s|_B$.

In what follows we denote by $\mathcal{Q}(\mathbb{N})$ the class of all Banach sequence spaces B which are invariant under translations and $\chi_{\{0\}} \in B$.

Example 2.1. (*Orlicz sequence spaces*) Let $\varphi : \mathbb{R}_+ \to [0, \infty]$ be a nondecreasing left continuous function which is not identically 0 or ∞ on $(0, \infty)$. The *Young function* associated with φ is $Y_\varphi(t) = \int_0^t \varphi(s)\, ds$, for all $t \geq 0$. For every $s \in \mathcal{S}(\mathbb{N}, \mathbb{R})$, let $M_\varphi(s) := \sum\limits_{k=0}^{\infty} Y_\varphi(|s(k)|)$. Then $\ell_\varphi(\mathbb{N}, \mathbb{R}) := \{s \in \mathcal{S}(\mathbb{N}, \mathbb{R}) : \exists c > 0$ such that $M_\varphi(cs) < \infty\}$ is a Banach space with respect to the norm $|s|_\varphi := \inf\{c > 0 : M_\varphi(s/c) \leq 1\}$. The space $\ell_\varphi(\mathbb{N}, \mathbb{R})$ is called the *Orlicz sequence space* associated to φ. It is easy to see that $O_\varphi \in \mathcal{Q}(\mathbb{N})$.

Let $p \in [1, \infty)$. Immediate examples of Orlicz sequence spaces are the $\ell^p(\mathbb{N}, \mathbb{R})$-spaces with respect to the norm $||s||_p = (\sum_{k=0}^{\infty} ||s(k)||^p)^{1/p}$, which are obtained for $\varphi(t) = pt^{p-1}$.

Example 2.2. The linear space $\ell^\infty(\mathbb{N}, \mathbb{R}) = \{s \in \mathcal{S}(\mathbb{N}, \mathbb{R}) : \sup\limits_{n \in \mathbb{N}} |s(n)| < \infty\}$ is a Banach space with respect to the norm $||s||_\infty := \sup\limits_{n \in \mathbb{N}} |s(n)|$ and $\ell^\infty(\mathbb{N}, \mathbb{R}) \in \mathcal{Q}(\mathbb{N})$. If $c_0(\mathbb{N}, \mathbb{R}) = \{s \in \mathcal{S}(\mathbb{N}, \mathbb{R}) : \lim\limits_{n \to \infty} s(n) = 0\}$, then $c_0(\mathbb{N}, \mathbb{R})$ is a closed linear subspace of $\ell^\infty(\mathbb{N}, \mathbb{R})$.

Remark 2.1. If $B \in \mathcal{Q}(\mathbb{N})$, then the following properties hold:

(i) for every $A \subset \mathbb{N}$, $\chi_A \in B$;

(ii) for every $s \in B$ and every $j \in \mathbb{N}$ the sequence

$$s_j : \mathbb{N} \to \mathbb{R}, \quad s_j(n) = \begin{cases} s(n-j) & , \quad n \geq j \\ 0 & , \quad n < j \end{cases}$$

belongs to B and $|s_j|_B = |s|_B$;

(iii) $\ell^1(\mathbb{N}, \mathbb{R}) \subset B \subset \ell^\infty(\mathbb{N}, \mathbb{R})$ (see e.g. [17], Lemma 2.1).

Lemma 2.1. *Let $B \in \mathcal{Q}(\mathbb{N})$ and let $\nu > 0$. Then, for every $s \in B$, the sequence*

$$q_s : \mathbb{N} \to \mathbb{R}_+, \quad q_s(n) = \sum_{k=0}^{n} e^{-\nu(n-k)} s(k)$$

belongs to B.

Proof. Let $s \in B$. Using the notations from Remark 2.1 (ii) we have that

$$|q_s(n)| \leq \sum_{k=0}^{\infty} e^{-\nu(n-k)} |s(k)| = \sum_{j=0}^{n} e^{-\nu j} |s_j(n)| \leq \sum_{j=0}^{\infty} e^{-\nu j} |s_j(n)|, \quad \forall n \in \mathbb{N}.$$

This implies that $q_s \in B$ and $|q_s|_B \leq [1/(1 - e^{-\nu})]\, |s|_B$. □

Notation Let $(X, ||\cdot||)$ be a real or complex Banach space. For every Banach sequence space $B \in \mathcal{Q}(\mathbb{N})$ we denote by $B(\mathbb{N}, X)$ the space of all sequences $s : \mathbb{N} \to X$ with the property that the mapping $N_s : \mathbb{N} \to \mathbb{R}_+$, $N_s(m) = ||s(m)||$ belongs to B. $B(\mathbb{N}, X)$ is a Banach space with respect to the norm $||s||_{B(\mathbb{N}, X)} := |N_s|_B$.

3. Preliminary results

Let X be a real or complex Banach space and let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on X. Throughout this paper, the norm on X and on $\mathcal{L}(X)$ will be denoted by $||\cdot||$. The linear space of all sequences $s : \mathbb{N} \to X$ will be denoted by $\mathcal{S}(\mathbb{N}, X)$.

Let $(A(n))_{n \in \mathbb{N}} \subset \mathcal{L}(X)$. We consider the discrete dynamical system

$$(A) \qquad x(n+1) = A(n)x(n), \quad n \in \mathbb{N}$$

where $x \in \mathcal{S}(\mathbb{N}, X)$.

Denoting by $\Delta = \{(m, n) \in \mathbb{N} \times \mathbb{N} : m \geq n \geq 0\}$ we have that the evolution operator $\Phi = \{\Phi(m, n)\}_{(m,n) \in \Delta}$ associated with the system (A) has the expression

$$\Phi(m, n) = \begin{cases} A(m-1) \ldots A(n), & m > n \\ I_d, & m = n \end{cases}$$

where I_d denotes the identity operator on X.

Remark 3.2. $\Phi(m, k)\Phi(k, n) = \Phi(m, n)$*, for all* $(m, k), (k, n) \in \Delta$.

Definition 3.3. The system (A) is said to be *uniformly exponentially stable* if there are $K, \nu > 0$ such that

$$||\Phi(m, n)|| \leq K e^{-\nu(m-n)}, \quad \forall (m, n) \in \Delta.$$

In what follows, we associate to the system (A) the input-output control system

$$(\mathcal{S}_A) \qquad \begin{cases} x(n+1) = A(n)x(n) + s(n+1), & n \in \mathbb{N} \\ x(0) = s(0) \end{cases}$$

where $s, x \in \mathcal{S}(\mathbb{N}, X)$.

Remark 3.3. For every $s \in \mathcal{S}(\mathbb{N}, X)$ the corresponding solution of the system $(\mathcal{S}_A)$ is given by

$$x_s(n) = \sum_{k=0}^{n} \Phi(n,k)s(k), \quad \forall n \in \mathbb{N}.$$

Definition 3.4. Let $U, V \in \mathcal{Q}(\mathbb{N})$. We say that the system $(\mathcal{S}_A)$ is $(U(\mathbb{N}, X), V(\mathbb{N}, X))$-*stable* if for every $s \in U(\mathbb{N}, X)$ the corresponding solution x_s belongs to $V(\mathbb{N}, X)$.

Theorem 3.1. *Let $V \in \mathcal{Q}(\mathbb{N})$. Then, the system (A) is uniformly exponentially stable if and only if the system $(\mathcal{S}_A)$ is $(V(\mathbb{N}, X), V(\mathbb{N}, X))$-stable.*

Proof. Necessity. Let $K, \nu > 0$ be given by Definition 3.3. Let $s \in V(\mathbb{N}, X)$. Then, from Lemma 2.1 we have that the sequence

$$\alpha_s : \mathbb{N} \to \mathbb{R}_+, \quad \alpha_s(n) = \sum_{k=0}^{n} e^{-\nu(n-k)}||s(k)||$$

belongs to V. Observing that $||x_s(n)|| \leq K\alpha_s(n)$, for all $n \in \mathbb{N}$, we obtain that $x_s \in V(\mathbb{N}, X)$.

Sufficiency. If $V = \ell^1(\mathbb{N}, \mathbb{R})$ then from Theorem 2.2 in [17], we obtain that the system (A) is uniformly exponentially stable. If $\ell^1(\mathbb{N}, \mathbb{R}) \subsetneq V$, then according to Theorem 2.4 in [17], we have that the system (A) is uniformly exponentially stable. □

4. Stability radius of discrete dynamical systems

In this section we obtain a very general lower bound for the stability radius of a discrete dynamical system in terms of the norm of certain input-output operators acting on Banach sequence spaces.

If Z is a Banach space, then we denote by $\mathcal{S}(\mathbb{N}, Z)$ the linear space of all sequences $s : \mathbb{N} \to Z$. If Z, W are Banach spaces we denote by $\mathcal{L}(Z, W)$ the Banach space of all bounded linear operators $H : Z \to W$ and we set $\mathcal{L}(Z, Z) =: \mathcal{L}(Z)$.

We consider the linear space $\ell^\infty(\mathbb{N}, \mathcal{L}(Z, W)) := \{T : \mathbb{N} \to \mathcal{L}(Z, W) : \sup_{n\in\mathbb{N}} ||T(n)|| < \infty\}$, which is a Banach space with respect to the norm

$$||T||_\infty := \sup_{n\in\mathbb{N}} ||T(n)||.$$

Let X be a Banach space and let $A \in \ell^\infty(\mathbb{N}, \mathcal{L}(X))$. We consider the discrete dynamical system

$$(A) \qquad x(n+1) = A(n)x(n), \quad n \in \mathbb{N}$$

where $x \in \mathcal{S}(\mathbb{N}, X)$.

For every $D \in \ell^\infty(\mathbb{N}, \mathcal{L}(X))$ we consider the perturbed system

$$(A+D) \qquad z(n+1) = [A(n) + D(n)]z(n), \quad n \in \mathbb{N}$$

where $z \in \mathcal{S}(\mathbb{N}, X)$.

Remark 4.4. The evolution operator associated with the system $(A + D)$ has the expression

$$\Phi_D(m,n) = \Phi(m,n) + \sum_{k=n+1}^{m} \Phi(m,k)D(k-1)\Phi_D(k-1,n)$$

for every $m, n \in \mathbb{N}$ with $m > n$.

In what follows we suppose that the system (A) is uniformly exponentially stable. The main question is how large may be the norm of the perturbation $D \in \ell^\infty(\mathbb{N}, \mathcal{L}(X))$ such that the perturbed system $(A + D)$ remains uniformly exponentially stable.

In this context, it makes sense to introduce:

Definition 4.5. The number

$$r_{stab}(A) := \sup\{r > 0 : \forall D \in \ell^\infty(\mathbb{N}, \mathcal{L}(X)) \text{ with } ||D||_\infty < r \Rightarrow (A+D) \text{ is uniformly exponentially stable}\}$$

is called the stability radius of the system (A).

In what follows, we analyze a more complex situation: when the system (A) is subjected to a very general perturbation structure.

Let U, Y be two Banach spaces. Let $B \in \ell^\infty(\mathbb{N}, \mathcal{L}(U, X))$ and $C \in \ell^\infty(\mathbb{N}, \mathcal{L}(X, Y))$.

Definition 4.6. The number

$$r_{stab}(A;B,C) := \sup\{r>0 : \forall P \in \ell^\infty(\mathbb{N},\mathcal{L}(Y,U)) \text{ with } ||P||_\infty < r \Rightarrow (A+BPC) \text{ is uniformly exponentially stable }\}$$

is called the stability radius of the system (A) subjected to the perturbation structure (B,C).

Remark 4.5. In the particular case $U=Y=X$ and $B(n)=C(n)=I_d$, for all $n\in\mathbb{N}$, then

$$r_{stab}(A;B,C) = r_{stab}(A)$$

In this context, in what follows our purpose is to obtain a very general lower bound for $r_{stab}(A;B,C)$.
We consider the class $\mathcal{V}(\mathbb{N})$ of all Banach sequence spaces $B\in\mathcal{Q}(\mathbb{N})$ with the property that if $s\in\mathcal{S}(\mathbb{N},X)$ and

$$\sup_{n\in\mathbb{N}} |s\,\chi_{\{0,\dots,n\}}|_B < \infty$$

then $s\in B$.

Remark 4.6. The class of Orlicz sequence spaces is a subclass of $\mathcal{V}(\mathbb{N})$.

As in the previous section, we associate with the system (A) the input-output control system

$$(\mathcal{S}_A)\qquad \begin{cases} x(n+1) = A(n)x(n) + s(n+1), & n\in\mathbb{N} \\ x(0) = s(0) \end{cases}$$

where $s,x\in\mathcal{S}(\mathbb{N},X)$.
Let $V\in\mathcal{V}(\mathbb{N})$. Since (A) is uniformly exponentially stable, according to Theorem 3.1 we have that the system $(\mathcal{S}_A)$ is $(V(\mathbb{N},X),V(\mathbb{N},X))$-stable.
For every $u\in V(\mathbb{N},U)$ we consider the sequence

$$s_u:\mathbb{N}\to X,\quad s_u(n)=B(n)u(n).$$

From

$$||s_u(n)|| \le ||B(n)||\,||u(n)|| \le ||B||_\infty\,||u(n)||,\quad \forall n\in\mathbb{N}$$

since $u\in V(\mathbb{N},U)$ we obtain that $s_u\in V(\mathbb{N},X)$. Using the invariance to translations of $V(\mathbb{N},X)$ we have that the sequence

$$\varphi_u:\mathbb{N}\to X,\quad \varphi_u(n)=\begin{cases} B(n-1)u(n-1), & n\in\mathbb{N}^* \\ 0, & n=0 \end{cases}$$

belongs to $V(\mathbb{N},X)$. Then, from the $(V(\mathbb{N},X),V(\mathbb{N},X))$-stability of the system $(\mathcal{S}_A)$, we obtain that

$$x_u:\mathbb{N}\to X,\quad x_u(n)=\begin{cases} \sum_{k=1}^{n}\Phi(n,k)B(k-1)u(k-1), & n\in\mathbb{N}^* \\ 0, & n=0 \end{cases} \tag{4.1}$$

belongs to $V(\mathbb{N},X)$.
We consider the system

$$(S_A^{B,C})\qquad \begin{cases} x(n+1)=A(n)x(n)+B(n)u(n), & n\in\mathbb{N} \\ x(0)=0 \\ y(n)=C(n)x(n), & n\in\mathbb{N} \end{cases}$$

with $u\in V(\mathbb{N},U)$.
According to (4.1) we have that for every $u\in V(\mathbb{N},U)$ the solution $x_u\in V(\mathbb{N},X)$. From

$$||y_u(n)|| \le ||C||_\infty\,||x_u(n)||,\quad \forall n\in\mathbb{N}$$

we deduce that $y_u\in V(\mathbb{N},Y)$. So, for every input $u\in V(\mathbb{N},U)$ the corresponding solution y_u of the system $(S_A^{B,C})$ has the property that $y_u\in V(\mathbb{N},Y)$. Then, it makes sense to consider the input-output operator

$$\Gamma_V: V(\mathbb{N},U)\to V(\mathbb{N},Y),\quad \Gamma_V(u)=y_u.$$

It is easy to observe that Γ_V is a closed linear operator, so it is bounded.
In what follows we suppose that there is $c>0$ such that

$$||C(n)y|| \ge c\,||y||,\quad \forall (n,y)\in\mathbb{N}\times Y. \tag{4.2}$$

Theorem 4.2. *If $||P||_\infty < (1/||\Gamma_V||)$, then for every $(n,x)\in\mathbb{N}\times X$, the sequence*

$$s_{n,x}:\mathbb{N}\to X,\quad s_{n,x}(k)=\begin{cases} \Phi_{BPC}(k,n)x, & k\ge n \\ 0, & k<n \end{cases}$$

belongs to $V(\mathbb{N},X)$.

Proof. Let $K, \nu > 0$ be such that $||\Phi(m,n)|| \leq Ke^{-\nu(m-n)}$, for all $(m,n) \in \Delta$. Let $(n,x) \in \mathbb{N} \times X$. For every $h \in \mathbb{N}$, we consider the sequences

$$\gamma : \mathbb{N} \to X, \quad \gamma(k) = \chi_{\{0,\ldots,n+h\}}(k)C(k)s_{n,x}(k)$$

$$u : \mathbb{N} \to U, \quad u(k) = P(k)\gamma(k).$$

Since u has finite support, we have that $u \in V(\mathbb{N}, U)$. Using Remark 4.4, for every $k \in \{n+1, \ldots, n+h\}$ we have that

$$\gamma(k) = C(k)\Phi(k,n)x + C(k)\sum_{j=n+1}^{k} \Phi(k,j)(BPC)(j-1)\Phi_{BPC}(j-1,k)x =$$

$$= C(k)\Phi(k,n)x + C(k)\sum_{j=1}^{k} \Phi(k,j)B(j-1)u(j-1) =$$

$$= C(k)\Phi(k,n)x + (\Gamma_V u)(k)$$

which implies that

$$||\gamma(k)|| \leq ||C||_\infty \, Ke^{-\nu(k-n)}||x|| + ||(\Gamma_V u)(k)|| \tag{4.3}$$

for all $k \in \{n+1, \ldots, n+h\}$. Observing that $\gamma(n) = C(n)x$, we deduce that (4.3) also holds for $k = n$.

We consider the sequence

$$e_\nu : \mathbb{N} \to \mathbb{R}_+, \quad e_\nu(k) = \begin{cases} e^{-\nu(k-n)}, & k \geq n \\ 0, & k < n \end{cases}.$$

We have that $e_\nu \in \ell^1(\mathbb{N}, \mathbb{R})$, so, from Remark 2.1 we obtain that $e_\nu \in V$. Since $\gamma(k) = 0$, for $k < n$ and for $k > n+h$, from (4.3) we have that

$$||\gamma(k)|| \leq K||C||_\infty \, ||x|| \, e_\nu(k) + ||(\Gamma_V u)(k)||, \quad \forall k \in \mathbb{N}$$

which implies that

$$||\gamma||_{V(\mathbb{N},Y)} \leq K \, ||C||_\infty \, ||x|| \, |e_\nu|_V + ||\Gamma_V u||_{V(\mathbb{N},Y)}. \tag{4.4}$$

Let $m = K \, ||C||_\infty \, ||x||$. Since Γ_V is a bounded linear operator, we have that

$$||\Gamma_V u||_{V(\mathbb{N},Y)} \leq ||\Gamma_V|| \, ||u||_{V(\mathbb{N},U)}. \tag{4.5}$$

From $||u(n)|| \leq ||P(n)|| \, ||\gamma(n)||$, for all $n \in \mathbb{N}$, we have that

$$||u||_{V(\mathbb{N},U)} \leq ||P||_\infty \, ||\gamma||_{V(\mathbb{N},Y)}. \tag{4.6}$$

From relations (4.4)–(4.6) we deduce that

$$||\gamma||_{V(\mathbb{N},Y)} \leq \frac{m \, |e_\nu|_V}{1 - ||\Gamma_V|| \, ||P||_\infty}. \tag{4.7}$$

Using (4.2) we have that

$$c \, ||\chi_{\{0,\ldots,n+h\}}(k)s_{n,x}(k)|| \leq ||\gamma(k)||, \quad \forall k \in \mathbb{N}$$

which implies that

$$||\chi_{\{0,\ldots,n+h\}}s_{n,x}||_{V(\mathbb{N},X)} \leq \frac{1}{c} \, ||\gamma||_{V(\mathbb{N},Y)}. \tag{4.8}$$

From relations (4.7) and (4.8) it follows that

$$||\chi_{\{0,\ldots,n+h\}}s_{n,x}||_{V(\mathbb{N},X)} \leq \frac{m \, |e_\nu|_V}{c - c||\Gamma_V|| \, ||P||_\infty}. \tag{4.9}$$

Since $h \in \mathbb{N}$ was arbitrary, from (4.9) we deduce that

$$\sup_{p\in\mathbb{N}} ||\chi_{\{0,\ldots,p\}}s_{n,x}||_{V(\mathbb{N},X)} \leq \frac{m \, |e_\nu|_V}{c - c||\Gamma_V|| \, ||P||_\infty}. \tag{4.10}$$

Using the fact that $V \in \mathcal{V}(\mathbb{N})$ from (4.10) we conclude that $s_{n,x} \in V(\mathbb{N}, X)$, for all $(n,x) \in \mathbb{N} \times X$. □

Theorem 4.3. *The following estimation holds:*

$$r_{stab}(A; B, C) \geq \frac{1}{||\Gamma_V||}.$$

Proof. Let $P \in \ell^\infty(\mathbb{N}, \mathcal{L}(Y,U))$ with $||P||_\infty < (1/||\Gamma_V||)$. We consider the input-output control system

$$(\mathcal{S}_{BPC}) \qquad \begin{cases} z(n+1) = [A(n) + (BPC)(n)]z(n) + s(n+1), & n \in \mathbb{N} \\ z(0) = s(0) \end{cases}$$

with $s \in V(\mathbb{N}, X)$. For every $s \in V(\mathbb{N}, X)$ the corresponding solution is

$$z_s(n) = \sum_{k=0}^{n} \Phi_{BPC}(n,j)s(j), \quad \forall n \in \mathbb{N}. \tag{4.11}$$

In what follows, we prove that the system $(\mathcal{S}_{BPC})$ is $(V(\mathbb{N}, X), V(\mathbb{N}, X))$-stable.

Let $s \in V(\mathbb{N}, X)$. For every $p \in \mathbb{N}^*$ let

$$s_p : \mathbb{N} \to X, \quad s_p(k) = \chi_{\{0,\ldots,p\}}(k)s(k).$$

Denoting by

$$h_p := \sum_{j=0}^{p} \Phi_{BPC}(p,j)s(j)$$

from Theorem 4.2 we have that the sequence

$$s_{p,x_p} : \mathbb{N} \to X, \quad s_{p,x_p}(n) = \begin{cases} \Phi_{BPC}(n,p)h_p, & n \geq p \\ 0, & n < p \end{cases}$$

belongs to $V(\mathbb{N}, X)$. Observing that $z_{s_p}(n) = s_{p,x_p}(n)$, for all $n \geq p$ and setting $M := \max\{||z_s(0)||, \ldots, ||z_s(p-1)||\}$ we deduce that

$$||z_{s_p}(n)|| \leq M\, \chi_{\{0,\ldots,p-1\}}(n) + ||s_{p,x_p}(n)||, \quad \forall n \in \mathbb{N}. \tag{4.12}$$

From (4.12) it follows that $z_{s_p} \in V(\mathbb{N}, X)$. Let

$$\gamma : \mathbb{N} \to Y, \quad \gamma(n) = C(n)z_{s_p}(n).$$

From $||\gamma(n)|| \leq ||C||_\infty\, ||z_{s_p}(n)||$, for all $n \in \mathbb{N}$, we have that $\gamma \in V(\mathbb{N}, Y)$.

For every $n \in \mathbb{N}^*$, using Remark 4.4 we successively deduce that

$$\gamma(n) = C(n)\sum_{k=0}^{n} \Phi_{BPC}(n,k)s_p(k) =$$

$$= C(n)s_p(n) + C(n)\sum_{k=0}^{n-1} \Phi(n,k)s_p(k) +$$

$$+C(n)\left[\sum_{k=0}^{n-1}\sum_{j=k}^{n-1} \Phi(n,j+1)(BPC)(j)\Phi_{BPC}(j,k)s_p(k)\right] =$$

$$= C(n)\sum_{k=0}^{n} \Phi(n,k)s_p(k) + C(n)\left[\sum_{j=0}^{n-1}\sum_{k=0}^{j} \Phi(n,j+1)(BPC)(j)\Phi_{BPC}(j,k)s_p(k)\right] =$$

$$= C(n)\sum_{k=0}^{n} \Phi(n,k)s_p(k) +$$

$$+C(n)\left[\sum_{j=0}^{n-1} \Phi(n,j+1)B(j)P(j)\left(C(j)\sum_{k=0}^{j} \Phi_{BPC}(j,k)s_p(k)\right)\right] =$$

$$= C(n)\sum_{k=0}^{n} \Phi(n,k)s_p(k) + C(n)\sum_{j=1}^{n} \Phi(n,j)B(j-1)P(j-1)\gamma(j-1). \tag{4.13}$$

Let

$$x_{s_p} : \mathbb{N} \to X, \quad x_{s_p}(n) = \sum_{k=0}^{n} \Phi(n,k)s_p(k)$$

and

$$\varphi : \mathbb{N} \to U, \quad \varphi(n) = P(n)\gamma(n).$$

Since (A) is uniformly exponentially stable we have that $x_{s_p} \in V(\mathbb{N}, X)$. From $||\varphi(n)|| \leq ||P||_\infty\, ||\gamma(n)||$, for all $n \in \mathbb{N}$, we have that $\varphi \in V(\mathbb{N}, U)$ and

$$||\varphi||_{V(\mathbb{N},U)} \leq ||P||_\infty\, ||\gamma||_{V(\mathbb{N},Y)}. \tag{4.14}$$

Then, from (4.13) it follows that

$$\gamma(n) = C(n)x_{s_p}(n) + (\Gamma_V\varphi)(n), \quad \forall n \in \mathbb{N}^*$$

which implies that

$$||\gamma(n)|| \le ||C(n)x_{s_p}(n)|| + ||(\Gamma_V\varphi)(n)||, \quad \forall n \in \mathbb{N}^*.$$

Since $\gamma(0) = C(0)s(0) = C(0)x_{s_p}(0)$ we deduce that

$$||\gamma(n)|| \le ||C||_\infty \, ||x_{s_p}(n)|| + ||(\Gamma_V\varphi)(n)||, \quad \forall n \in \mathbb{N}. \tag{4.15}$$

From (4.14) and (4.15) we have that

$$||\gamma||_{V(\mathbb{N},Y)} \le ||C||_\infty \, ||x_{s_p}||_{V(\mathbb{N},X)} + ||\Gamma_V|| \, ||P||_\infty \, ||\gamma||_{V(\mathbb{N},Y)}. \tag{4.16}$$

From (4.2) we have that

$$c\, ||z_{s_p}(n)|| \le ||\gamma(n)||, \quad \forall n \in \mathbb{N}$$

which implies that

$$c\, ||z_{s_p}||_{V(\mathbb{N},X)} \le ||\gamma||_{V(\mathbb{N},Y)}. \tag{4.17}$$

From relations (4.16) and (4.17) we deduce that

$$||z_{s_p}||_{V(\mathbb{N},X)} \le \frac{||C||_\infty \, ||x_{s_p}||_{V(\mathbb{N},X)}}{c[1 - ||\Gamma_V|| \, ||P||_\infty]}. \tag{4.18}$$

We observe that $x_{s_p}(n) = x_s(n)$, for $n \in \{0, \ldots, p\}$ and $x_{s_p}(n) = \Phi(n,p)x_s(p)$, for $n \ge p$. Then, denoting by

$$e_\nu : \mathbb{N} \to \mathbb{R}, \quad e_\nu(n) = \begin{cases} e^{-\nu(n-p)}, & n \ge p \\ 0, & n < p \end{cases}$$

we obtain that

$$||x_{s_p}(n)|| \le ||x_s(n)|| + e_\nu(n)\, ||x_s(p)||, \quad \forall n \in \mathbb{N}.$$

This shows that

$$||x_{s_p}||_{V(\mathbb{N},X)} \le ||x_s||_{V(\mathbb{N},X)} + |e_\nu|_V \, ||x_s(p)||. \tag{4.19}$$

From

$$||x_s(p)|| \, \chi_{\{p\}}(k) \le ||x_s(k)||, \quad \forall k \in \mathbb{N}$$

using the invariance to translations of the space V we deduce that

$$||x_s(p)|| \, |\chi_{\{p\}}|_V = ||x_s(p)|| \, |\chi_{\{0\}}|_V \le ||x_s||_{V(\mathbb{N},X)}. \tag{4.20}$$

Setting $\lambda := 1 + (|e_\nu|_V / |\chi_{\{0\}}|_V)$, from relations (4.19) and (4.20) it follows that

$$||x_{s_p}||_{V(\mathbb{N},X)} \le \lambda \, ||x_s||_{V(\mathbb{N},X)}. \tag{4.21}$$

Observing that

$$z_{s_p}(n) = z_s(n), \quad \forall n \in \{0, \ldots, p\}$$

from relations (4.18) and (4.21) we obtain that

$$||z_s \chi_{\{0,\ldots,p\}}||_{V(\mathbb{N},X)} \le \frac{\lambda \, ||C||_\infty \, ||x_s||_{V(\mathbb{N},X)}}{c[1 - ||\Gamma_V|| \, ||P||_\infty]}$$

which implies that

$$\sup_{p \in \mathbb{N}} ||z_s \, \chi_{\{0,\ldots,p\}}||_{V(\mathbb{N},X)} < \infty.$$

Since $V \in \mathcal{V}(\mathbb{N})$ we deduce that $z_s \in V(\mathbb{N}, X)$.

In conclusion, we have that $(\mathcal{S}_{BPC})$ is $(V(\mathbb{N},X), V(\mathbb{N},X))$-stable. By applying Theorem 3.1 we obtain the conclusion. □

As a consequence of the above results we deduce

Theorem 4.4. *The following estimation holds:*

$$r_{stab}(A; B, C) \ge \sup_{V \in \mathcal{V}(\mathbb{N})} \frac{1}{||\Gamma_V||}.$$

Corollary 4.1. *For every $p \in [1, \infty]$, let $\Gamma_p := \Gamma_{\ell^p(\mathbb{N},\mathbb{R})}$. The following estimation holds:*

$$r_{stab}(A; B, C) \ge \sup_{p \in [1,\infty]} \frac{1}{||\Gamma_p||}.$$

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