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The degree of approximation by certain linear positive operators

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ABSTRACT.
We consider certain linear positive operators B_n in polynomial weighted spaces and study approximation properties of these operators, including theorems on the degree of approximation.

1. INTRODUCTION

In the paper [19] we studied approximation problems for functions $f \in C_p$ and Szasz-Mirakyan type operators

$$A_n(f;r;x) := e^{-(nx+1)^r} \sum_{k=0}^{\infty} \frac{(nx+1)^{rk}}{k!} f\left(\frac{k}{n(nx+1)^{r-1}}\right), \tag{1.1}$$

$x \in R_0 := [0,+\infty)$, $r \in R_2 := [2,+\infty)$, $n \in N := \{1,2,\dots\}$,where C_p with some fixed $p \in N_0 := \{0,1,2,\dots\}$ is a polynomial weighted space generated by the weight function

$$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1}, \quad \text{if} \quad p \geq 1, \tag{1.2}$$

i.e., C_p is the set of all real-valued functions f, continuous on R_0 and such that $w_p f$ is uniformly continuous and bounded on R_0. The norm in C_p is defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x \in R_0} w_p(x)\,|f(x)|. \tag{1.3}$$

In [19] there were proved theorems on the degree of approximation of $f \in C_p$ by the operators A_n defined by (1.1).

Theorem 1.1. *Let $p \in N_0$ and $r \in R_2$ be fixed numbers. Then there exists $M_1 \equiv M_1(p,r)$ such that for every $f \in C_p$ and $n \in N$ we have*

$$\|A_n(f;r;\cdot) - f(\cdot)\|_p \leq M_1 \omega_1\left(f; C_p; \frac{1}{n}\right), \tag{1.4}$$

where ω_1 is the modulus of continuity defined by the formula

$$\omega_1(f;C_p;t) := \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|_p, \qquad t \in R_0, \tag{1.5}$$

where $\Delta_h f(x) := f(x+h) - f(x)$, for $x, h \in R_0$.

The operators (1.1) are related to the well-known Szasz-Mirakyan operators

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1.6}$$

$x \in R_0$, $n \in N$. In [2] it was proved that if $f \in C_p$, $p \in N_0$, then for the Szasz-Mirakyan operators S_n one has the following inequality

$$w_p(x)|S_n(f;x) - f(x)| \leq M_2 \omega_2\left(f; C_p; \sqrt{\frac{x}{n}}\right), \qquad x \in R_0, \quad n \in N_0,$$

where $M_2 = const. > 0$ and ω_2 is the modulus of smoothness defined by the formula

$$\omega_2(f;C_p;t) := \sup_{0 \leq h \leq t} \|\Delta_h^2 f(\cdot)\|_p, \qquad t \in R_0,$$

where $\Delta_h^2 f(x) := f(x) - 2f(x+h) + f(x+2h)$.

Theorem 1.1 shows that the operators A_n give a better degree of approximation of functions $f \in C_p$ than the Szasz-Mirakyan operators S_n. We can observe that the degree of approximation of f by A_n is independent on $r \in R_2$.

In [15-16, 18, 20] were examined similar approximation problems for certain modified Szasz-Mirakyan operators S_n.

The degree of approximation given in Theorem 1.1 and in [15-16, 18, 20, 21] can be improved by a certain modification of formula (1.1).

In this paper we introduce certain linear positive operators and study their approximation properties.

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Recently in many papers various modifications of S_n were introduced and examined. We refer the readers to A. Ciupa [3-5], P. Gupta and V. Gupta [7], V. Gupta [8], N. Ispir and C. Atakut [1], [13], V. Gupta, V. Vasishtha and M. K. Gupta [11], S. Guo, C. Li, Y. Sun, G. Yand, S. Yue [12]. Their results improve other related results in the literature.

In this paper we shall denote by $M_k(\alpha,\beta)$, $k=1,2,\ldots$, the suitable positive constants or functions depending only on indicated parameters α,β. To this end, let C_p be the space given above and let $f\in C_p^2 := \{f\in C_p : f', f''\in C_p\}$, where f', f'' are the derivatives of f.

2. Approximation of functions of one variable

We introduce the following

Definition 2.1. Let $p\in N_0$ be a fixed number. For functions $f\in C_p$ we define the operators

$$B_n(f;x) := e^{-(nx)^3}\sum_{k=0}^{\infty}\frac{(nx)^{3k}}{k!}f\left(\frac{k}{n((nx)^2+n^{-1})}\right),\quad x\in R_0, n\in N. \tag{2.7}$$

Similarly to A_n, the operator B_n is linear and positive. We shall prove that B_n is an operator from the space C_p into C_p for every fixed $p\in N_0$. From (2.7) we derive the following formulas

$$\begin{aligned} B_n(1;x) &= 1, \\ B_n(t;x) &= \frac{x^3}{x^2+n^{-3}}, \\ B_n(t^2;x) &= (x^2+n^{-3})^{-2}\left[x^6+\frac{x^3}{n^3}\right], \\ B_n(t^3;x) &= (x^2+n^{-3})^{-3}\left[x^9+\frac{3x^6}{n^3}+\frac{x^3}{n^6}\right], \end{aligned} \tag{2.8}$$

for all $n\in N$ and $x\in R_0$.

From formulas (2.7), (2.8) and $B_n(t^k;x)$, $1\le k\le 3$, given above we obtain

Lemma 2.1. *For all $x\in R_0$ and $n\in N$ we have*

$$B_n(t-x;x) = -\frac{x}{n^3(x^2+n^{-3})},$$

$$B_n((t-x)^2;x) = \frac{x^3+n^{-3}x^2}{n^3(x^2+n^{-3})^2},$$

$$B_n((t-x)^3;x) = \frac{x^3-3x^4-n^{-3}x^3}{n^6(x^2+n^{-3})^3}.$$

Next we shall prove

Lemma 2.2. *Let $s\in N$ be a fixed number. Then there exist coefficients $\alpha_{s,j}$, depending only on s, j such that*

$$B_n(t^s;x) = (x^2+n^{-3})^{-s}\sum_{j=1}^{s}\frac{\alpha_{s,j}x^{3j}}{n^{3(s-j)}} \tag{2.9}$$

for all $n\in N$ and $x\in R_0$. Moreover, $\alpha_{s,1}=\alpha_{s,s}=1$ for $j=1,2\ldots,s$.

Proof. We shall use mathematical induction for s. The formula (2.9) for $1\le s\le 3$ is given above. Let (2.9) hold for $f(x)=x^j$, $1\le j\le s$, with fixed $s\in N$. We shall prove (2.9) for $f(x)=x^{s+1}$. From (2.7) and (2.8) it follows that

$$B_n(t^{s+1};x) = e^{-(nx)^3}\sum_{k=0}^{\infty}\frac{(nx)^{3k}}{k!}\left(\frac{k}{n((nx)^2+n^{-1})}\right)^{s+1} =$$

$$\frac{x^3}{x^2+n^{-3}}e^{-(nx)^3}\sum_{k=0}^{\infty}\frac{(nx)^{3k}}{k!}\left(\frac{k+1}{n((nx)^2+n^{-1})}\right)^{s} =$$

$$= \frac{x^3}{x^2+n^{-3}}n^{-3s}(x^2+n^{-3})^{-s}e^{-(nx)^3}\sum_{k=0}^{\infty}\frac{(nx)^{3k}}{k!}\sum_{\mu=0}^{s}\binom{s}{\mu}k^{\mu}.$$

Consequently

$$B_n(t^{s+1};x) = \frac{x^3}{x^2+n^{-3}}\sum_{\mu=0}^{s}\binom{s}{\mu}n^{3(\mu-s)}(x^2+n^{-3})^{\mu-s}B_n(t^{\mu};x).$$

By our assumption we get

$$B_n(t^{s+1};x) = \frac{x^3}{x^2+n^{-3}}\left\{n^{-3s}(x^2+n^{-3})^{-s} + \sum_{\mu=1}^{s}\binom{s}{\mu} n^{3(\mu-s)}(x^2+n^{-3})^{-s}\sum_{j=1}^{\mu}\frac{\alpha_{\mu,j}x^{3j}}{n^{3(\mu-j)}}\right\}$$

$$= (x^2+n^{-3})^{-(s+1)}\left\{n^{-3s}x^3 + \sum_{\mu=1}^{s}\binom{s}{\mu} n^{3(\mu-s)}\sum_{j=1}^{\mu}\frac{\alpha_{\mu,j}x^{3(j+1)}}{n^{3(\mu-j)}}\right\}$$

$$= (x^2+n^{-3})^{-(s+1)}\left\{n^{-3s}x^3 + \sum_{j=1}^{s}\sum_{\mu=j}^{s}\binom{s}{\mu}\frac{\alpha_{\mu,j}x^{3(j+1)}}{n^{3(s-j)}}\right\}$$

$$= (x^2+n^{-3})^{-(s+1)}\left\{n^{-3s}x^3 + \sum_{j=2}^{s+1}\frac{x^{3j}}{n^{3(s+1-j)}}\sum_{\mu=j-1}^{s}\binom{s}{\mu}\alpha_{\mu,j-1}\right\}$$

$$= (x^2+n^{-3})^{-(s+1)}\sum_{j=1}^{s+1}\frac{\alpha_{s+1,j}x^{3j}}{n^{3(s+1-j)}},$$

where $\alpha_{s+1,1} = \alpha_{s+1,s+1} = 1$, which proves (2.9) for $f(x) = x^{s+1}$. □

Lemma 2.3. *Let $p \in N_0$ be a fixed number. Then there exists a positive constant $M_3 \equiv M_3(p)$, depending only on the parameter p such that*

$$\|B_n(1/w_p(t);\cdot)\|_p \le M_3, \qquad n \in N. \tag{2.10}$$

Moreover, for every $f \in C_p$ we have

$$\|B_n(f;\cdot)\|_p \le M_3\|f\|_p, \qquad n \in N. \tag{2.11}$$

The formula (2.7) and inequality (2.11) show that B_n, $n \in N$, is a positive linear operator from the space C_p into C_p, for every $p \in N_0$.

Proof. From (2.7) we get

$$B_n(f;0) = f(0) \quad \text{for} \quad n \in N. \tag{2.12}$$

The inequality (2.10) is obvious for $p = 0$ by (1.2), (1.3) and (2.8). Let $p \in N$. By (1.2) and (2.7)-(2.9) we have

$$w_p(x)B_n(1/w_p(t);x) = w_p(x)\{1 + B_n(t^p;x)\} = \frac{1}{1+x^p} +$$

$$+\frac{1}{(1+x^p)(x^2+n^{-3})^p}\sum_{j=1}^{p}\frac{\alpha_{p,j}x^{3j}}{n^{3(p-j)}}.$$

Let $x \in [1,+\infty)$. We remark that

$$w_p(x)B_n(1/w_p(t);x) \le 1 + \frac{x^{3p}}{(1+x^p)(x^2)^p}\sum_{j=1}^{p}\frac{\alpha_{p,j}}{n^{3(p-j)}} \le M_3(p).$$

For $x \in (0,1)$ we have

$$\frac{1}{(x^2+n^{-3})^p}\sum_{j=1}^{p}\frac{\alpha_{p,j}x^{3j}}{n^{3(p-j)}} \le \sum_{j=1}^{p}\frac{\alpha_{p,j}\left(\frac{x^2}{x^2+n^{-3}}\right)^j}{(x^2+n^{-3})^{p-j}n^{3(p-j)}} \le \sum_{j=1}^{p}\frac{\alpha_{p,j}}{(n^{-3})^{p-j}n^{3(p-j)}} \le \sum_{j=1}^{p}\alpha_{p,j}.$$

Therefore the proof of inequality (2.10) is completed.

The formulas (2.7)-(2.8) and (1.2) imply

$$\|B_n(f(t);\cdot)\|_p \le \|f\|_p\|B_n(1/w_p(t);\cdot)\|_p, \qquad n \in N,$$

for every $f \in C_p$. Applying (2.10), we obtain (2.11). □

Lemma 2.4. *Let $p \in N_0$ be fixed number. Then there exists a positive function $M_4(p,x)$ which does not depend on n such that*

$$w_p(x)B_n\left(\frac{(t-x)^2}{w_p(t)};x\right) \le \frac{M_4(p,x)}{n^3} \qquad \text{for all} \quad n \in N,\ x > 0. \tag{2.13}$$

Proof. The formulas given in Lemma 2.1 and (1.2), (1.3) imply (2.13) for $p = 0$.

By (1.2) and (2.10) we have

$$B_n\left((t-x)^2/w_p(t);x\right) = B_n\left((t-x)^2;x\right) + B_n\left(t^p(t-x)^2;x\right),$$

for $p, n \in N$. If $p = 1$, then by the equality we get

$$B_n\left((t-x)^2/w_1(t);x\right) = B_n\left((t-x)^2;x\right) + B_n\left(t(t-x)^2;x\right) =$$
$$= B_n\left((t-x)^3;x\right) + (1+x)B_n\left((t-x)^2;x\right),$$

which by (1.2), (1.3) and Lemma 2.1 yields (2.13) for $p = 1$.

Let $p \geq 2$. Applying Lemma 2.2, we get

$$w_p(x)B_n\left(t^p(t-x)^2;x\right) = w_p(x)\left\{B_n\left(t^{p+2};x\right) - 2xB_n\left(t^{p+1};x\right) + x^2B_n\left(t^p;x\right)\right\} =$$
$$= w_p(x)\left\{(x^2+n^{-3})^{-(p+2)}\sum_{j=1}^{p+2}\frac{\alpha_{p+2,j}x^{3j}}{n^{3(p+2-j)}} - 2x(x^2+n^{-3})^{-(p+1)}\sum_{j=1}^{p+1}\frac{\alpha_{p+1,j}x^{3j}}{n^{3(p+1-j)}} + \right.$$
$$\left. + x^2(x^2+n^{-3})^{-p}\sum_{j=1}^{p}\frac{\alpha_{p,j}x^{3j}}{n^{3(p-j)}}\right\} =$$
$$= w_p(x)\left\{\frac{x^{3p+2}}{n^6(x^2+n^{-3})^{p+2}} + (x^2+n^{-3})^{-(p+2)}\sum_{j=1}^{p+1}\frac{\alpha_{p+2,j}x^{3j}}{n^{3(p+2-j)}}\right.$$
$$\left. -2x(x^2+n^{-3})^{-(p+1)}\sum_{j=1}^{p}\frac{\alpha_{p+1,j}x^{3j}}{n^{3(p+1-j)}} + x^2(x^2+n^{-3})^{-p}\sum_{j=1}^{p-1}\frac{\alpha_{p,j}x^{3j}}{n^{3(p-j)}}\right\},$$

which by (1.2) implies

$$w_p(x)B_n\left(t^p(t-x)^2;x\right) \leq \frac{M_4(p,x)}{n^3}, \qquad n \in N.$$

Thus the proof is completed. □

Now we shall give approximation theorems for B_n.

Theorem 2.2. *Let $p \in N_0$ be a fixed number. Then there exists a positive function $M_5(p,x)$ which does not depend on n such that for every $f \in C_p^2$ we have*

$$w_p(x)|B_n(f;x) - f(x)| \leq M_5(p,x)\frac{\|f'\|_p + \|f''\|_p}{n^3}, \qquad n \in N,\ x > 0. \tag{2.14}$$

Proof. For a fixed $x > 0$ and $f \in C_p^2$ we have

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t\int_x^s f''(u)duds, \quad t \in R_0,$$

which yields

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)f''(u)du, \quad t \in R_0.$$

From this and by (2.7) we deduce that

$$B_n(f(t);x) = f(x) + f'(x)B_n(t-x;x) + B_n\left(\int_x^t (t-u)f''(u)du;x\right) \tag{2.15}$$

for $n \in N$. By (1.2) and (1.3) we can write

$$\left|\int_x^t (t-u)f''(u)du\right| \leq \|f''\|_p\left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)}\right)(t-x)^2.$$

Applying the above inequality and Lemma 2.1 and (2.12), we derive from (2.15)

$$w_p(x)\left|B_n(f;x) - f(x)\right| \leq \|f'\|_p\frac{x}{n^3(x^2+n^{-3})} +$$
$$\|f''\|_p\left\{w_p(x)B_n\left(\frac{(t-x)^2}{w_p(t)};x\right) + B_n\left((t-x)^2;x\right)\right\} \leq$$
$$\leq M_5(p,x)\frac{\|f'\|_p + \|f''\|_p}{n^3} \quad \text{for } n \in N.$$

Thus the proof of (2.14) is completed. □

Theorem 2.3. *Let $p \in N_0$ be a fixed number. Then there exists a positive function $M_6(p,x)$ which does not depend on n such that for every $f \in C_p$ and $n \in N$ we have*

$$w_p(x)|B_n(f;x) - f(x)| \leq M_6(p,x)\left\{\frac{1}{n^{3/2}}\omega_1\left(f;C_p;\frac{1}{n^{3/2}}\right) + \omega_2\left(f;C_p;\frac{1}{n^{3/2}}\right)\right\}. \tag{2.16}$$

Proof. Let $x > 0$. Similarly as in [2] we apply the Stieklov function of $f \in C_p$:

$$f_h(x) := \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}[f(x+s+t) - f(x+2(s+t))]dsdt \tag{2.17}$$

for $x \in R_0, h > 0$. From (2.17) we get

$$f_h'(x) = \frac{1}{h^2}\int_0^{\frac{h}{2}}[8\Delta_{h/2}f(x+s) - 2\Delta_h f(x+2s)]ds,$$

$$f_h''(x) = \frac{1}{h^2}\left[8\Delta_{h/2}^2 f(x) - \Delta_h^2 f(x)\right].$$

Consequently

$$\|f_h - f\|_p \leq \omega_2\left(f, C_p; h,\right), \tag{2.18}$$

$$\|f_h'\|_p \leq 5h^{-1}\omega_1\left(f, C_p; h\right)\frac{w_p(x)}{w_p(x+h)} \tag{2.19}$$

$$\|f_h''\|_p \leq 9h^{-2}\omega_2\left(f, C_p; h\right), \tag{2.20}$$

for $h > 0$. We see that $f_h \in C_p^2$ if $f \in C_p$. Hence, for $x > 0$ and $n \in N$, we can write

$$w_p(x)\left|B_n(f;x) - f(x)\right| \leq w_p(x)\left\{\left|B_n\left(f - f_h;x\right)\right| + \right.$$

$$\left. + \left|B_n\left(f_h;x\right) - f_h(x)\right| + \left|f_h(x) - f(x)\right|\right\} := A_1 + A_2 + A_3.$$

By (2.11) and (2.18) we have

$$A_1 \leq M_3(p)\left\|f - f_h\right\|_p \leq M_3(p)\omega_2\left(f, C_p; h\right), \quad A_3 \leq \omega_2\left(f, C_p; h\right).$$

Applying Theorem 2.2 and (2.19) and (2.20), we get

$$A_2 \leq M_5(p,x)\frac{\|f_h'\|_p + \|f_h''\|_p}{n^3} \leq$$

$$\leq M_6(p,x)\left\{\frac{5}{hn^3}\omega_1\left(f, C_p; h\right)\frac{w_p(x)}{w_p(x+h)} + \frac{9}{h^2n^3}\omega_2\left(f, C_p; h\right)\right\}.$$

Combining these and setting $h = \frac{1}{n^{3/2}}$, for fixed $n \in N$, we obtain (2.16) for $x > 0$. □

From Theorem 2.2 and Theorem 2.3 we derive the following two corollaries:

Corollary 2.1. For $f \in C_p$, $p \in N_0$, we have

$$\lim_{n\to\infty} B_n(f;x) = f(x).$$

This convergence is uniform on every interval $[x_1, x_2]$, $x_1 > 0$.

Corollary 2.2. If $f \in C_p^2$, $p \in N_0$, then

$$|B_n(f;x) - f(x)| = O_x(1/n^3)$$

In a similar manner (see [21]), the Voronovskaya type theorem for B_n can be verified.

Theorem 2.4. *Let $f \in C_p^2$. Then*

$$\lim_{n\to\infty} n^3\left\{B_n\left(f;x\right) - f(x)\right\} = \frac{-f'(x) + 1/2f''(x)}{x} \tag{2.21}$$

for every $x > 0$.

3. Approximation of functions of two variables

Next, for given $p, q \in N_0$, we define the weighted function

(3.22) $$w_{p,q}(x,y) := w_p(x)w_q(y), \qquad (x,y) \in R_0^2 := R_0 \times R_0,$$

and the weighted space $C_{p,q}$ of all real-valued functions f continuous on R_0^2 for which $w_{p,q}f$ is uniformly continuous and bounded on R_0^2 and the norm is defined by the formula

(3.23) $$\|f\|_{p,q} \equiv \|f(\cdot,\cdot)\|_{p,q} := \sup_{(x,y)\in R_0^2} w_{p,q}(x,y)\,|f(x,y)|\,.$$

The modulus of continuity of $f \in C_{p,q}$ we define as usual by the formula

(3.24) $$\omega_1(f, C_{p,q}; t, s) := \sup_{0\le h\le t,\, 0\le\delta\le s} \|\Delta_{h,\delta} f(\cdot,\cdot)\|_{p,q}\,, \qquad t,s \ge 0\,,$$

where $\Delta_{h,\delta} f(x,y) := f(x+h, y+\delta) - f(x,y)$ and $(x+h, y+\delta) \in R_0^2$. Moreover let $C_{p,q}^1$ be the set of all functions $f \in C_{p,q}$ whose first partial derivatives belong also to $C_{p,q}$. From (3.24) it follows that

$$\lim_{t,s\to 0+} \omega_1(f, C_{p,q}; t, s) = 0$$

for every $f \in C_{p,q}$, $p, q \in N_0$. We introduce the following

Definition 3.2. For functions $f \in C_{p,q}$, $p, q \in N_0$, we define operators

(3.25) $$B_{m,n}(f; x, y) := e^{-((mx)^3+(ny)^3)}$$
$$\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(mx)^{3j}}{j!}\frac{(ny)^{3k}}{k!} f\left(\frac{j}{m((mx)^2+m^{-1})}, \frac{k}{n((ny)^2+n^{-1})}\right)$$

for $(x,y) \in R_0^2$, $m, n \in N$.

We deduce that $B_{m,n}(f)$ are well-defined in every space $C_{p,q}$, $p, q \in N_0$. Moreover we have

(3.26) $$B_{m,n}(1; x, y) = 1 \quad \text{for} \quad (x,y) \in R_0^2, \quad m, n \in N,$$

and if $f \in C_{p,q}$ and $f(x,y) = f_1(x)f_2(y)$ for all $(x,y) \in R_0^2$, then

(3.27) $$B_{m,n}(f; x, y) = B_m(f_1; x)B_n(f_2; y)$$

for all $(x,y) \in R_0^2$ and $m, n \in N$.

Lemma 3.5. *For fixed $p, q \in N_0$ there exists a positive constant $M_7 \equiv M_7(p,q)$ such that*

(3.28) $$\left\|B_{m,n}\left(1/w_{p,q}(t,z)\cdot,\cdot\right)\right\|_{p,q} \le M_7 \quad \text{for} \quad m, n \in N.$$

Moreover for every $f \in C_{p,q}$ we have

(3.29) $$\left\|B_{m,n}\left(f; \cdot,\cdot\right)\right\|_{p,q} \le M_7\, \|f\|_{p,q} \quad \text{for} \quad m, n \in N.$$

The formula (3.25) and the inequality (3.29) show that $B_{m,n}$, $m, n \in N$, are linear positive operators from the space $C_{p,q}$ into $C_{p,q}$.

Proof. The inequality (3.28) follows immediately from (3.22), (3.27) and (2.10).

From (3.22) and (3.25) we get for $f \in C_{p,q}$

$$\|B_{m,n}(f)\|_{p,q} \le \|f\|_{p,q}\, \|B_{m,n}(1/w_{p,q})\|_{p,q}\,, \quad m, n \in N,$$

which by (3.28) implies (3.29). □

Now we shall give two theorems on the degree of approximation of functions by $B_{m,n}$ defined by (3.25).

Theorem 3.5. *Suppose that $f \in C_{p,q}^1$ with fixed $p, q \in N_0$. Then there exists a positive function $M_8(p,q,x,y)$ which does not depend on m, n such that for all $m, n \in N$, $(x,y) \in R_+^2 := (0,+\infty) \times (0,+\infty)$*

(3.30) $$w_{p,q}(x,y)\,|B_{m,n}(f;x,y) - f(x,y)| \le M_8(p,q,x,y)\left\{\frac{1}{m^{3/2}}\|f'_x\|_{p,q} + \frac{1}{n^{3/2}}\|f'_y\|_{p,q}\right\}.$$

Proof. Let $(x,y) \in R_+^2$ be a fixed point. Then for $f \in C_{p,q}^1$

$$f(t,z) - f(x,y) = \int_x^t f'_u(u,z)du + \int_y^z f'_v(x,v)dv, \qquad (t,z) \in R_0^2.$$

Thus by (3.25)

(3.31) $$B_{m,n}(f(t,z); x, y) - f(x,y)$$
$$= B_{m,n}\left(\int_x^t f'_u(u,z)du; x, y\right) + B_{m,n}\left(\int_y^z f'_v(x,v)dv; x, y\right).$$

By (3.22)-(3.23) we have

$$\left|\int_x^t f_u'(u,z)du\right| \le \|f_x'\|_{p,q}\left|\int_x^t \frac{du}{w_{p,q}(u,z)}\right|$$
$$\le \|f_x'\|_{p,q}\left(\frac{1}{w_{p,q}(t,z)}+\frac{1}{w_{p,q}(x,z)}\right)|t-x|,$$

which by (3.22), (3.27) and (2.7)-(2.8) implies

$$w_{p,q}(x,y)\left|B_{m,n}\left(\int_x^t f_u'(u,z)du;x,y\right)\right|$$
$$\le w_{p,q}(x,y)B_{m,n}\left(\left|\int_x^t f_u'(u,z)du\right|;x,y\right)$$
$$\le \|f_x'\|_{p,q}\, w_{p,q}(x,y)\left\{B_{m,n}\left(\frac{|t-x|}{w_{p,q}(t,z)};x,y\right)+B_{m,n}\left(\frac{|t-x|}{w_{p,q}(x,z)};x,y\right)\right\}$$
$$\le \|f_x'\|_{p,q}\, w_q(y)B_n\left(\frac{1}{w_q(z)};y\right)\left\{w_p(x)B_m\left(\frac{|t-x|}{w_p(t)};x\right)+B_m\left(|t-x|;x\right)\right\}.$$

Applying the Hölder inequality and Lemma 2.1 and (2.12), we get

$$B_m\left(|t-x|;x\right) \le \left\{B_m((t-x)^2;x)B_m(1;x)\right\}^{\frac12} \le \frac{M_9(p,x)}{m^{3/2}},$$
$$w_p(x)B_m\left(\frac{|t-x|}{w_p(t)};x\right) \le \left\{w_p(x)B_m\left(\frac{(t-x)^2}{w_p(t)};x\right)\right\}^{\frac12}\left\{w_p(x)B_m\left(\frac{1}{w_p(t)};x\right)\right\}^{\frac12}$$
$$\le \frac{M_{10}(p,x)}{m^{3/2}}$$

for $x \in R_0$ and $m \in N$. Consequently

$$w_{p,q}(x,y)\left|B_{m,n}\left(\int_x^t f_u'(u,z)du;x,y\right)\right| \le \frac{M_{11}(p,q,x)}{m^{3/2}}\|f_x'\|_{p,q}, \quad m\in N.$$

Analogously we obtain

$$w_{p,q}(x,y)\left|B_{m,n}\left(\int_y^z f_v'(x,v)dv;x,y\right)\right| \le \frac{M_{12}(p,q,y)}{n^{3/2}}\|f_y'\|_{p,q}, \quad n\in N.$$

Combining these, we derive from (3.31)

$$w_{p,q}(x,y)\left|B_{m,n}(f;x,y)-f(x,y)\right| \le M_9(p,q,x,y)\left\{\frac{1}{m^{3/2}}\|f_x'\|_{p,q}+\frac{1}{n^{3/2}}\|f_y'\|_{p,q}\right\},$$

for all $m,n \in N$, where $M_9 = M_9(p,q) = const. > 0$. Thus the proof of (3.30) is completed. □

Theorem 3.6. *Suppose that $f \in C_{p,q}$, $p,q \in N_0$. Then there exists a positive function $M_{13}(p,q,x,y)$ which does not depend on m,n such that*

$$w_{p,q}(x,y)\left|B_{m,n}(f;x,y)-f(x,y)\right| \le M_{13}(p,q,x,y)\,\omega_1\left(f,C_{p,q};\frac{1}{m^{3/2}},\frac{1}{n^{3/2}}\right),\ (x,y)\in R_+^2, \tag{3.32}$$

for all $m,n \in N$.

Proof. We apply the Stieklov function $f_{h,\delta}$ for $f \in C_{p,q}$

$$f_{h,\delta}(x,y) := \frac{1}{h\delta}\int_0^h du\int_0^\delta f(x+u,y+v)dv, \quad (x,y)\in R_0^2, h,\delta>0. \tag{3.33}$$

From (3.33) it follows that

$$f_{h,\delta}(x,y)-f(x,y) = \frac{1}{h\delta}\int_0^h du\int_0^\delta \Delta_{u,v}f(x,y)dv,$$
$$(f_{h,\delta})_x'(x,y) = \frac{1}{h\delta}\int_0^\delta\left(\Delta_{h,v}f(x,y)-\Delta_{0,v}f(x,y)\right)dv,$$
$$(f_{h,\delta})_y'(x,y) = \frac{1}{h\delta}\int_0^h\left(\Delta_{u,\delta}f(x,y)-\Delta_{u,0}f(x,y)\right)du.$$

Thus

$$\left\|f_{h,\delta}-f\right\|_{p,q} \le \omega_1\left(f,C_{p,q};h,\delta\right), \tag{3.34}$$

$$\left\|(f_{h,\delta})_x'\right\|_{p,q} \le 2h^{-1}\omega_1\left(f,C_{p,q};h,\delta\right), \tag{3.35}$$

$$\left\| (f_{h,\delta})'_y \right\|_{p,q} \leq 2\delta^{-1}\omega_1 \left(f, C_{p,q}; h, \delta\right), \tag{3.36}$$

for all $h, \delta > 0$, which show that $f_{h,\delta} \in C^1_{p,q}$ if $f \in C_{p,q}$ and $h, \delta > 0$.

Now, for $B_{m,n}$ defined by (3.25), we can write

$$\begin{aligned} & w_{p,q}(x,y)\left|B_{m,n}(f;x,y) - f(x,y)\right| \\ & \leq w_{p,q}(x,y)\left\{\left|B_{m,n}\left(f(t,z) - f_{h,\delta}(t,z);x,y\right)\right| + \right. \\ & \quad + \left|B_{m,n}\left(f_{h,\delta}(t,z);x,y\right) - f_{h,\delta}(x,y)\right| \\ & \quad \left. + \left|f_{h,\delta}(x,y) - f(x,y)\right|\right\} := T_1 + T_2 + T_3. \end{aligned}$$

By (3.23), (3.29) and (3.34),

$$\begin{aligned} T_1 & \leq \left\|B_{m,n}\left(f - f_{h,\delta};\cdot,\cdot\right)\right\|_{p,q} \leq M_3(p,q)\left\|f - f_{h,\delta}\right\|_{p,q} \\ & \leq M_3(p,q)\,\omega_1\left(f, C_{p,q}; h, \delta\right), \\ T_3 & \leq \omega_1\left(f, C_{p,q}; h, \delta\right). \end{aligned}$$

Applying Theorem 3.5 and (3.35) and (3.36), we get

$$\begin{aligned} T_2 & \leq M_{14}(p,q,x,y)\left\{\frac{1}{m^{3/2}}\left\|(f_{h,\delta})'_x\right\|_{p,q} + \frac{1}{n^{3/2}}\left\|(f_{h,\delta})'_y\right\|_{p,q}\right\} \\ & \leq 2M_{14}(p,q,x,y)\omega_1\left(f, C_{p,q}; h, \delta\right)\left\{h^{-1}\frac{1}{m^{3/2}} + \delta^{-1}\frac{1}{n^{3/2}}\right\}. \end{aligned}$$

Consequently there exists $M_{15} \equiv M_{15}(p,q,x,y)$ such that

$$w_{p,q}(x,y)\left|B_{m,n}(f;x,y) - f(x,y)\right|$$

$$\leq M_{15}(p,q,x,y)\omega\left(f, C_{p,q}; h, \delta\right)\left\{1 + h^{-1}\frac{1}{m^{3/2}} + \delta^{-1}\frac{1}{n^{3/2}}\right\}, \tag{3.37}$$

for $m, n \in N$ and $h, \delta > 0$. Now, for $m, n \in N$ setting $h = \frac{1}{m^{3/2}}$ and $\delta = \frac{1}{n^{3/2}}$ to (3.37), we obtain (3.32). □

From Theorem 3.6 follows

Corollary 3.3. Let $f \in C_{p,q}$, $p, q \in N_0$. Then

$$\lim_{m,n\to\infty} B_{m,n}(f;x,y) = f(x,y). \tag{3.38}$$

Moreover (3.38) holds uniformly on every rectangle $0 < x \leq x_0$, $0 < y \leq y_0$.

Theorem 3.6 in our paper shows that operators $B_{m,n}$, $m, n \in N$, give a better degree of approximation of functions $f \in C_{p,q}$ than the classical Szasz-Mirakyan operator $S_{m,n}$ (considered in [17] for continuous and bounded functions) and some other known operators (for example, see [16, 20]).

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