A quicker convergence toward the γ constant with the logarithm term involving the constant e

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ABSTRACT.

We introduce a new class of sequences of the form

$$\mu_n = \sum_{k=1}^{n} \frac{1}{k} + \ln(e^{a/(n+b)} - 1) - \ln a$$

which converge to the Euler-Mascheroni constant γ. Being preoccupied to accelerate the classical sequence convergent toward γ, Batir [J. Ineq. Pure Appl. Math. 6 (2005) no. 4 Art 103] and Alzer [Expo. Math. 24 (2006) 385-388] studied the case $a = b = 1$ and we show in this paper that the fastest sequence $(\mu_n)_{n\geq 1}$ is obtained for $a = 1/\sqrt{2}$, $b = (2+\sqrt{2})/4$. For these values, accurate approximations of γ can be constructed, as numerical computations made in the final part of this paper show. We also solve an open problem about the rate of convergence of some sequences defined by Batir.

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