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Some applications of CHEVIE to the theory of algebraic groups

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ABSTRACT.

The computer algebra system CHEVIE is designed to facilitate computations with various combinatorial structures arising in Lie theory, like finite Coxeter groups and Hecke algebras. We discuss some recent examples where CHEVIE has been helpful in the theory of algebraic groups, in questions related to unipotent classes, the Springer correspondence and Lusztig families.

1. INTRODUCTION

CHEVIE [21] is a computer algebra project which was initiated about 20 years ago and has been further developed ever since; general information can be found on the webpage

http://www.math.rwth-aachen.de/~CHEVIE

which also contains links to various extensions and updates of CHEVIE. The aim of CHEVIE is two-fold: firstly, it makes vast amounts of explicit data concerning Coxeter groups, Hecke algebras and groups of Lie type systematically available in electronic form; secondly, it provides tools, pre-defined functions and a programming environment (via its implementation in GAP [57] and MAPLE [9]) for performing symbolic calculations with these data. Through this combination, it has been helpful in a variety of applications; this help typically consists of:

- explicitly verifying certain properties (usually in the large groups of exceptional type) in the course of a case–by–case argument, or
- producing evidence in support of hypotheses and, conversely, searching for counter-examples, or
- performing experiments which may lead eventually to new theoretical insights (a conjecture, a theorem, a technique required in a proof, . . .),

or a combination of these. While the scope of CHEVIE is gradually expanding, the original design has been particularly suited to algorithmic questions arising from Lusztig's work [41], [47] on Hecke algebras and characters of reductive groups over finite fields.

The purpose of this article is to present selected examples of this interplay between theory and experimentation. The choice of examples is, of course, influenced by the author's own preferences. For quite some time now, algorithmic methods are well-established in various aspects of Lie theory (see, e.g., [2], [12], [33]), so another author—even another author from the CHEVIE project itself!—may easily come up with a completely different set of examples and applications.

A finite Coxeter group W can be described by a presentation with generators and defining relations, or by its action on a root system in some Euclidean space. Thus, they are particulary suitable for the application of algorithmic methods. In Section 2, we consider the conjugacy classes of W, especially questions related to elements of minimal length in the various classes—which is one of the areas where CHEVIE has been extremely helpful from its very beginnings; see [26], [25]. By recent work of Lusztig [51], this plays a role in the construction of a remarkable map from conjugacy classes in a finite Weyl group to the unipotent classes in a corresponding algebraic group; this will be explained in Section 3.

In Section 4, we shall consider certain standard operations in the character ring of W, like tensoring with the sign character and induction from parabolic subgroups—an area where one can use the full power of the highly efficient GAP functionality for character tables of finite groups. These operations are the combinatorial counter-part of a number of constructions related to unipotent classes in algebraic groups and Lusztig's families of representations.

Finally, in Section 5, we consider the problem of computing the Green functions of a finite group of Lie type. These functions provide a substantial piece of information towards the determination of the whole character table of such a group. The algorithm described by Shoji [60] and Lusztig [43, §24] is now known to work without any restriction on the characteristic, and we explain how this can be turned into an efficient GAP program. A remarkable formula combining Green functions, character values of Hecke algebras and Fourier matrices is used in Lusztig's work [51] (mentioned above) to deal with groups of exceptional type—a highlight in the applications of CHEVIE.

While most of the content of these notes is drawn from existing sources, there are a few items which are new; see, for example, the general existence result for excellent elements in the conjugacy classes of finite Coxeter groups in Section 2 and the characterisation of the a-function in Section 4. We also mention our presentation of the algorithmic questions around the computation of Green functions and Lusztig's results [51] in Section 5; in particular, we develop in somewhat more detail the fact that the $\mathbb{F}_q$-rational points in the intersections of Bruhat cells with unipotent classes

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can be counted by "polynomials in q". This, and the experimental results in [23], lead us to conjecture the existence of a natural map from the conjugacy classes of W to the Lusztig families of W; see Remark 5.14.

We assume that the reader has some familiarity with the general theory of (finite) Coxeter groups, the character theory of finite groups, and basic notions about algebraic groups; see, for example, [8], [27], [16]. The manual of the GAP part of CHEVIE (available online in GAP or on the above webpage) may actually be a good place to start to read about the algorithmic theory of Coxeter groups.

This is not meant to be a comprehensive survey about applications of CHEVIE. The interested reader may consult the bibliography for further reading; see, for example, Achar–Aubert [1], Bellamy [3], Casselman [10], Gomi [28], He [30], Himstedt–Huang [31], Lusztig [48], Reeder [56], to mention but a few from a variety of topics. Finally, Michel's development version [55] of CHEVIE contains a wealth of material around complex reflection groups and "*Spetses*" [7], a subject that we do not touch upon at all.

2. Conjugacy classes of finite Coxeter groups

Let W be a finite Coxeter group, with generating set S and corresponding length function $l\colon W \to \mathbb{Z}_{\geqslant 0}$. In CHEVIE, such a group is realised as a GAP permutation group via its action on the underlying root system; this provides highly efficient ways of performing computations with the elements of W (multiplication, length function, reduced expressions, ...); see [21, §2.2].

We shall now explain some results on conjugacy classes which have been found and established through experiments with CHEVIE.

Let $\mathrm{Cl}(W)$ be the set of all conjugacy classes of W. For $C \in \mathrm{Cl}(W)$, let

$$d_C := \min\{l(w) \mid w \in C\} \qquad \text{and} \qquad C_{\min} := \{w \in C \mid l(w) = d_C\}.$$

Thus, $C_{\min}$ is the set of elements of minimal length in C. For any subset $I \subseteq S$, let $W_I \subseteq W$ be the parabolic subgroup generated by I. We say that $C \in \mathrm{Cl}(W)$ is *cuspidal* if $C \cap W_I = \varnothing$ for all proper subsets $I \subsetneqq S$. (These classes may also be called *anisotropic* or *elliptic*.) One can show that C is cuspidal if and only if $C_{\min} \cap W_I = \varnothing$ for all proper subsets $I \subsetneqq S$; see [27, 3.1.12].

Let $w, w' \in W$. We write $w \to w'$ if there are sequences of elements $w = y_0, y_1, \ldots, y_n = w'$ in W and generators $s_1, \ldots, s_n \in S$ such that, for each $i \in \{1, \ldots, n\}$, we have $y_i = s_i y_{i-1} s_i$ and $l(y_i) \leqslant l(y_{i-1})$. This is a pre-order relation on W. Let $\leftrightarrow$ denote the associated equivalence relation, that is, we have $y \leftrightarrow w$ if and only if $y \to w$ and $w \to y$. The equivalence classes are called the *cyclic shift classes* of W; see [27, 3.2.3]. Note that all elements in a fixed cyclic shift class have the same length. Clearly, every conjugacy class of W is a union of (several, in general) cyclic shift classes.

Proposition 2.1 (See [27, 3.2.7]). *Let $C \in \mathrm{Cl}(W)$ be cuspidal. Then the elements of $C_{\min}$ form a single cyclic shift class.*

The proof of this result essentially relies on computer calculations, performed originally in [26]; see also [21, §3.2], [27, §3.3].

Using the concept of cuspidal classes, we obtain a full classification of the conjugacy classes of W. To state the following result, let us denote by $\mathcal{I}(W,S)$ the set of all pairs (I, C') where $I \subseteq S$ and $C' \in \mathrm{Cl}(W_I)$ is cuspidal (in W_I). Given two such pairs (I_1, C_1') and (I_2, C_2'), we write $(I_1, C_1') \sim (I_2, C_2')$ if there exists some $x \in W$ such that $I_2 = xI_1x^{-1}$ and $C_2' = xC_1'x^{-1}$.

Theorem 2.2 (Classification of $\mathrm{Cl}(W)$, [27, 3.2.12]). *Let $C \in \mathrm{Cl}(W)$. Then the pairs (I, C'), where $I \subseteq S$ is the set of generators involved in a reduced expression of some $w \in C_{\min}$ and C' is the conjugacy class of w in W_I, form an equivalence class in $\mathcal{I}(W,S)$. Furthermore, we obtain a bijection*

$$\mathrm{Cl}(W) \quad \xrightarrow{1-1} \quad \mathcal{I}(W,S)/\sim$$

by sending $C \in \mathrm{Cl}(W)$ to the equivalence class of pairs (I, C') as above.

(Again, the proof heavily relies on computer calculations.)

The above two results combined show that many properties about conjugacy classes of W in general can be reduced to the study of suitable elements in cuspidal classes of W. Following recent work of Lusztig [51], we will now discuss some special properties of the elements of minimal length in the classes of W. Let

$$T := \{wsw^{-1} \mid w \in W, s \in S\}$$

be the set of reflections in W.

Lemma 2.3. *Let $t \in T$. Then t can be written in the form $t = ysy^{-1}$ where $y \in W$ and $s \in S$ are such that $l(t) = 2l(y) + 1$.*

Proof. Since t has order 2, we can apply the argument in the proof of [27, 3.2.10]. This shows that there exists a subset $J \subseteq S$ and an element $y \in W$ such that $t = yw_Jy^{-1}$ where w_J is the longest element in W_J; furthermore, w_J is central in W_J and $l(t) = 2l(y) + l(w_J)$. It follows that t has $|J|$ eigenvalues equal to -1 in the standard reflection representation of W. Since t is a reflection, this forces that $|J| = 1$. So we have $w_J = s$ for some $s \in S$, as required. □

Definition 2.4 (Lusztig [51, 2.1])**.** Let $C \in \mathrm{Cl}(W)$; suppose that C corresponds to a pair (I, C') as in Theorem 2.2. An element $w \in C_{\min}$ is called *excellent* if there exist reflections $t_1, \ldots, t_r \in T$, where $r = |I|$, such that

$$w = t_1 \cdots t_r \qquad \text{and} \qquad l(w) = l(t_1) + \cdots + l(t_r).$$

Thus, using Lemma 2.3, an excellent element $w \in C_{\min}$ admits a reduced expression of the form

$$\begin{aligned} w = {} & (s_1^1 s_2^1 \cdots s_{q_1}^1 s_{q_1+1}^1 s_{q_1}^1 \cdots s_2^1 s_1)(s_1^2 s_2^2 \cdots s_{q_2}^2 s_{q_2+1}^2 s_{q_2}^2 \cdots s_2^2 s_2^2) \cdot \\ & \cdots (s_1^r s_2^r \cdots s_{q_r}^r s_{q_r+1}^r s_{q_r}^r \cdots s_2^r s_1^r), \end{aligned}$$

where $s_i^j \in S$ for all i, j and $l(w) = \sum_{1 \leqslant j \leqslant r} (2q_j + 1)$, as in [51, 2.1(a)].

Some examples are already mentioned in [51, 2.1]. In particular, these show that, for a given class $C \in \mathrm{Cl}(W)$, there can exist elements in $C_{\min}$ which are not excellent. Lusztig also establishes the existence of excellent elements in all conjugacy classes of finite Weyl groups, except when there is a component of type E_7 or E_8. Here we complete the picture by the following slightly stronger result, valid for all finite Coxeter groups.

Proposition 2.5. *Let $C \in \mathrm{Cl}(W)$; suppose that C corresponds to a pair (I, C') as in Theorem 2.2. Then, for some element $w \in C_{\min}$, there exist reflections $t_1, \ldots, t_r \in T$, where $r = |I|$, with the following properties:*

(a) *We have $w = t_1 \cdots t_r$ and $l(w) = l(t_1) + \cdots + l(t_r)$; thus, w is excellent.*

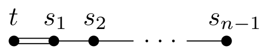
(b) *There exist subsets $\varnothing = J_0 \subseteq J_1 \subseteq \ldots \subseteq J_r \subseteq S$ such that, for $1 \leqslant i \leqslant r$, the reflection t_i lies in W_{J_i} and is a distinguished coset representative with respect to $W_{J_{i-1}}$, that is, we have $l(st_i) > l(t_i)$ for all $s \in J_{i-1}$.*

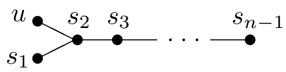
Proof. By standard reduction arguments, we can assume that (W, S) is irreducible. It will also be sufficient to deal with the case where C is a cuspidal class. Now we consider the various types of irreducible finite Coxeter groups.

First assume that W is of type $I_2(m)$ where $m \geqslant 3$. Denote the two generators of W by s_1, s_2. The cuspidal classes of W are described in [27, Exp. 3.2.8]; representatives of minimal length are given by $w_i = (s_1 s_2)^i$ where $1 \leqslant i \leqslant \lfloor m/2 \rfloor$. We see that the decomposition $w_i = (s_1)(s_2 s_1 \cdots s_1 s_2)$ (where the second factor has length $2i - 1$) satisfies the conditions (a) and (b).

If W is of type A_{n-1}, then there is only one cuspidal class C, namely, that containing the Coxeter elements. Furthermore, $C_{\min}$ consists precisely of the Coxeter elements; see [27, 3.1.16]. Clearly, a reduced expression for a Coxeter element is a decomposition as a product of reflections which satisfies (a) and (b).

Next assume that W is of type B_n or D_n, where we use the following labelling of the generators of W:

B_n 

D_n 

The cuspidal classes of W are parametrized by the partitions of n (with an even number of non-zero parts in type D_n); see [25, §2.2] or [27, §3.4]. Let $C^\alpha \in \mathrm{Cl}(W)$ be the cuspidal class corresponding to the partition α. A representative of minimal length in C^α is given as follows. For $1 \leqslant i \leqslant n-1$, we set

$$\hat{s}_i := \begin{cases} s_i s_{i-1} \ldots s_1 t s_1 \ldots s_{i-1} s_i & \text{in type } B_n, \\ s_i s_{i-1} \ldots s_2 u s_1 s_2 \ldots s_{i-1} s_i & \text{in type } D_n. \end{cases}$$

For $i = 0$ we set $\hat{s}_0 := t$ (in type B_n) and $\hat{s}_0 := 1$ (in type D_n). Given $m \geqslant 0$ and $d \geqslant 1$, we define a "negative block" of length d and starting at m by

$$b^-(m, d) := \hat{s}_m s_{m+1} s_{m+2} \cdots s_{m+d-1}.$$

Now let $1 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_h$ be the non-zero parts of α (where h is even if we are in type D_n). Let $m_i = \alpha_1 + \cdots + \alpha_{i-1}$ for $i \geqslant 1$, where $m_1 = 0$. Then we have

$$w_\alpha := b^-(m_1, \alpha_1) b^-(m_2, \alpha_2) \cdots b^-(m_h, \alpha_h) \in C^\alpha_{\min}.$$

Note that $w_\alpha = t_1 \cdots t_n$ where $t_1 = \hat{s}_0$ and $t_i \in \{s_{i-1}, \hat{s}_{i-1}\}$ for $i \geqslant 2$.

Now, in type B_n, each $\hat{s}_i$ is a reflection. It easily follows that w_α is excellent (as already noticed by Lusztig [51, 2.2(a)]) and the additional requirements in (b) are satisfied. The situation is slightly more complicated in type D_n, since $\hat{s}_i$ is not a reflection for $i \geqslant 1$. Lusztig [51, 2.3] already verified that w_α is excellent but the expression for w_α as a product of reflections described by Lusztig does not satisfy the conditions in (b). We need to somewhat modify w_α in order to make sure that (b) holds. This is done as follows. Since now h is even, we can write

$$w_\alpha = (b_1 b_2)(b_3 b_4) \cdots (b_{h-1} b_h) \quad \text{where} \quad b_i := b^-(m_i, \alpha_i) \text{ for all } i.$$

By [25, 2.2] (see also the proof of [25, Lemma 2.6(b)]), the factors $b_2, \ldots, b_h$ all commute with each other. On the other hand, note that $m_1 = 0$ and so $b_1 = b^-(m_1, \alpha_1) = s_1 s_2 \cdots s_{\alpha_1 - 1}$. In this case, we have $b_1 b_i = b_i \tilde{b}_1$ and $\tilde{b}_1 b_i = b_i b_1$ for any $i \geqslant 2$, where $\tilde{b}_1 := u s_2 \ldots s_{\alpha_i - 1}$. Since h is even, this yields

$$w_\alpha = b_1 (b_3 b_4) \cdots (b_{h-1} b_h) b_2 = (b_{h-1} b_h) \cdots (b_3 b_4)(b_1 b_2).$$

Since every element in W is conjugate to its inverse (see [27, 3.2.14]), we obtain

$$w'_\alpha := w_\alpha^{-1} = (b_1 b_2)^{-1} (b_3 b_4)^{-1} \cdots (b_{h-1} b_h)^{-1} \in C^\alpha_{\min}.$$

TABLE 1. Excellent elements in types H_3, H_4, F_4, E_6, E_7

F_4	d_C	excellent $w \in C_{\min}$
F_4	4	(4)(3)(2)(1)
B_4	6	(2)(4)(323)(1)
$F_4(a_1)$	8	(3)(4)(323)(121)
D_4	10	(2)(323)(43234)(1)
C_3+A_1	10	(1)(4)(3)(2132132)
$D_4(a_1)$	12	(3)(2)(43234)(12321)
$A_3+\tilde{A}_1$	14	(2)(323)(43234)(12321)
$A_2+\tilde{A}_2$	16	(2)(1)(4)(3213234321323)
$4A_1$	24	(2)(323)(43234)· ·(123214321324321)

E_6	d_C	excellent $w \in C_{\min}$
E_6	6	(1)(4)(2)(3)(6)(5)
$E_6(a_1)$	8	(1)(4)(3)(242)(5)(6)
$E_6(a_2)$	12	(3)(1)(5)(6)(34543)(242)
A_5+A_1	14	(1)(2)(3)(6)(5)(423454234)
$3A_2$	24	(1)(2)(3)(5)(6)· ·(4315423456542314354)

H_3	d_C	excellent $w \in C_{\min}$
6	3	(1)(2)(3)
8	5	(1)(212)(3)
9	9	(1)(212)(32123)
10	15	(1)(3)(2121321213212)

H_4	d_C	excellent $w \in C_{\min}$
11	4	(1)(2)(3)(4)
14	6	(1)(212)(3)(4)
15	8	(1)(2)(32123)(4)
17	10	(1)(212)(32123)(4)
18	12	(2)(1)(2123212)(343)
19	14	(3)(2)(12132121321)(4)
21	16	(1)(3)(2121321213212)(4)
22	16	(1)(212)(32123)(4321234)
23	18	(1)(212)(1321213)(4321234)
24	20	(1)(2)(12132121321)(4321234)
25	22	(1)(3)(2121321213212)(4321234)
26	24	(1)(2)(4)(321213212343212132123)
27	26	(2)(4)(121)(321213212343212132123)
28	28	(1)(4)(212)(32121321432121321432123)
29	30	(4)(3)(2)(123212132143212132124321213)
30	36	(3)(2)(12132121321)(43212132123432121321234)
31	38	(1)(3)(2121321213212)(43212132123432121321234)
32	40	(1)(3)(4)(2132123432121321234321213212343212132)
33	48	(1)(4)(212)(3212132123432121321234321213212343212132123)
34	60	(1)(3)(2121321213212)· ·(432121321234321213212343212132123432121321234)

E_7	d_C	excellent $w \in C_{\min}$
E_7	7	(7)(6)(5)(4)(3)(1)(2)
$E_7(a_1)$	9	(4)(7)(6)(5)(242)(3)(1)
$E_7(a_2)$	11	(5)(4)(7)(565)(242)(3)(1)
$E_7(a_3)$	13	(3)(5)(7)(6)(454)(23423)(1)
D_6+A_1	15	(2)(3)(7)(6)(5)(423454234)(1)
A_7	17	(2)(3)(6)(7)(565)(423454234)(1)
$E_7(a_4)$	21	(5)(6)(7)(45654)(2)(34543)(1234231)
$D_6(a_2)+A_1$	23	(2)(3)(7)(6)(5)(423454234)(134565431)
A_5+A_2	25	(3)(1)(2)(7)(6)(5)(4315423456542314354)
D_4+3A_1	31	(2)(3)(5)(7)(423454234)(65423456765423456)(1)
$2A_3+A_1$	33	(3)(1)(2)(5)(7)(423454234)(1654234567654231456)
$7A_1$	63	(2)(3)(5)(7)(423454234)· ·(65423456765423456)(134254316542345676542314354265431)

Finally, we verify that each product $b_i b_{i+1}$ in the above expression can be written in a suitable way as a product of reflections. First, we compute:

$$\begin{aligned} b_1 b_2 &= (s_1 s_2 \cdots s_{\alpha_1 - 1})(u_{\alpha_1} s_{\alpha_1+1} \cdots s_{\alpha_1+\alpha_2-1}) \\ &= (s_1 \cdots s_{\alpha_1-1} s_{\alpha_1} s_{\alpha_1-1} \cdots s_1) u s_2 \cdots s_{\alpha_1} s_{\alpha_1+1} \cdots s_{\alpha_1+\alpha_2-1}. \end{aligned}$$

Thus, we have $(b_1 b_2)^{-1} = t_1 \cdots t_{\alpha_1+\alpha_2}$ where

$$t_1 = s_{\alpha_1+\alpha_2-1}, \quad t_2 = s_{\alpha_1+\alpha_2-2}, \quad \ldots, \quad t_{\alpha_1+\alpha_2-2} = s_2, \quad t_{\alpha_1+\alpha_2-1} = u,$$
$$t_{\alpha_1+\alpha_2} = s_1 \cdots s_{\alpha_1-1} s_{\alpha_1} s_{\alpha_1-1} \cdots s_1;$$

note that these are all reflections and $m_3 = \alpha_1 + \alpha_2$. Note also that the generators in S which are involved in the expression for $t_{\alpha_1+\alpha_2}$ are the ones which already appeared in $t_1, \ldots, t_{\alpha_1+\alpha_2-1}$, together with s_1.

$E_8(a_6)$	24	(8)(7)(6)(5)(4)(2)(345676543)(123454231)
$E_7(a_2)+A_1$	24	(1)(2)(5)(6)(454)(314234565423143)(7)(8)
E_6+A_2	26	(3)(1)(2)(5)(6)(8)(4315423456542314354)(7)
$D_8(a_2)$	26	(2)(3)(5)(7)(6)(542345676542345)(8)(13431)
A_8	28	(1)(2)(3)(8)(7)(6)(5)(431542345676542314354)
$D_8(a_3)$	30	(1)(4)(2)(3)(7)(454)(3165423456765423145 6)(8)
D_6+2A_1	32	(2)(3)(5)(8)(7)(423454234)(65423456765423456)(1)
A_7+A_1	34	(3)(1)(2)(5)(7)(423454234)(1654234567654231456)(8)
$E_8(a_8)$	40	(3)(4)(2)(131)(454)(234565423)(13456765431)(24567876542)
$E_7(a_4)+A_1$	42	(2)(3)(4)(6)(131)(5423456542345)(1234567654231)(456787654)
$2D_4$	44	(2)(3)(5)(423454234)(1)(7)(65423456765423456)(1345678765431)
$E_6(a_2)+A_2$	44	(3)(1)(2)(5)(6)(4315423456542314354)(23456765423)(456787654)
$A_5+A_2+A_1$	46	(2)(3)(5)(423454234)(1)(8)(7)(654317654234567876542314354 6576)
$D_5(a_1)+A_3$	46	(3)(1)(2)(5)(7)(6)(34254316542345676542314354 26543)(456787654)
$2A_4$	48	(1)(2)(3)(5)(6)(7)(8)(435426543176542345678765423143542654 37654)
$2D_4(a_1)$	60	(4)(2)(454)(3)(8)(7)(6542345678765423456)(1342543165423456765423143542 65431)
D_4+4A_1	64	(2)(3)(5)(7)(423454234)(65423456765423456)(1342543165423456765423143542 65431)(8)
$2A_3+2A_1$	66	(2)(3)(5)(7)(8)(423454234)(65423456765423456)(134254316542345678765423143542 65431)
$4A_2$	80	(3)(1)(2)(6)(5)(8)(4315423456542314354)(765423143542654317654234567876542314354265431 7654234567)
$8A_1$	120	(2)(3)(5)(7)(423454234)(65423456765423456)(1342543165423456765423143542 65431)· ·(876542314354265431765423456787654231435426543176542345678)

Similarly, for $i \geqslant 3$, we find:

$$\begin{aligned} b_i b_{i+1} &= (u_{m_i} s_{m_i+1} \cdots s_{m_i+\alpha_i-1})(u_{m_i+\alpha_i} s_{m_i+\alpha_i+1} \cdots s_{m_i+\alpha_i+\alpha_{i+1}-1}) \\ &= (u_{m_i} s_{m_i+1} \cdots s_{m_i+\alpha_i-1} s_{m_i+\alpha_i} s_{m_i+\alpha_i-1} \cdots s_{m_i+1} u_{m_i}) \cdot \\ &\qquad \cdot s_{m_i+1} s_{m_i+2} \cdots s_{m_i+\alpha_i+\alpha_{i+1}-1}. \end{aligned}$$

Thus, we have $(b_i b_{i+1})^{-1} = t_{m_i+1} \cdots t_{m_i+\alpha_i+\alpha_{i+1}}$ where

$$\begin{gathered} t_{m_i+1} = s_{m_i+\alpha_i+\alpha_{i+1}-1}, \quad t_{m_i+2} = s_{m_i+\alpha_i+\alpha_{i+1}-2}, \\ \ldots, \quad t_{m_i+\alpha_i+\alpha_{i+1}-1} = s_{m_i+1}, \\ t_{m_i+\alpha_i+\alpha_{i+1}} = u_{m_i} s_{m_i+1} \cdots s_{m_i+\alpha_i-1} s_{m_i+\alpha_i} s_{m_i+\alpha_i-1} \cdots s_{m_i+1} u_{m_i}; \end{gathered}$$

note that these are all reflections and $m_{i+2} = m_i + \alpha_i + \alpha_{i+1}$. Note also that the generators in S which are involved in the expression for $t_{m_i+\alpha_i+\alpha_{i+1}}$ are the ones which already appeared in $t_1, \ldots, t_{m_i+\alpha_i+\alpha_{i+1}-1}$, together with s_{m_i}.

Combining these formulae, we obtain an expression $w'_\alpha = t_1 \cdots t_n$ such that condition (a) holds by construction. It is now also straightforward to verify that (b) holds. (This uses the above-mentioned information concerning the generators in S which are involved in the expressions for the t_i; we omit further details.) Thus, the assertion is proved for W of type B_n and D_n.

Finally, in order to deal with the remaining groups of exceptional type, we use algorithmic methods and computer programs written in CHEVIE. This involves the following steps. Let $C \in \mathrm{Cl}(W)$. An element $w \in C_{\min}$ is explicitly specified in the tables in [27, App. B]. First we compute the whole set $C_{\min}$. By Proposition 2.1, this set is the cyclic shift class containing w, and so it can be effectively computed using Algorithm G in [27, §3.2]. To procede, it will be convenient to introduce the following notation. Given any element $w \in W$, we let $J(w)$ be the set of all $s \in S$ which appear in a reduced expression for w. (It is well-known that this does not depend on the choice of the reduced expression.) Then we say that w is *pre-excellent* if there exists a reflection $t \in T$ such that $l(wt) = l(w) - l(t)$ and $J(wt) \subsetneqq J(w)$. These conditions can be effectively verified using the standard programs available in CHEVIE. Given any subset $X \subseteq W$, we define

$$\begin{aligned} X' &:= \{w \in X \mid w \text{ pre-excellent}\}, \\ \hat{X} &:= \{wt \mid w \in X', t \in T \text{ such that } l(wt) = l(w) - l(t) \text{ and } J(wt) \subsetneqq J(w)\}. \end{aligned}$$

Now we set $\mathcal{C}_0 := C_{\min}$ and then define recursively $\mathcal{C}_i := \hat{\mathcal{C}}_{i-1}$ for $i = 1, 2, \ldots, |S|$. If the set $\hat{\mathcal{C}}_{|S|}$ is non-empty and just contains the identity element then, clearly, the recursive procedure for reaching that set determines an element in $C_{\min}$ together with a decomposition $w = t_1 \cdots t_r$ as required in (a); furthermore, it yields subsets $\varnothing = J_0 \subsetneqq J_1 \subsetneqq \ldots \subsetneqq J_r \subseteq W$ such that $t_i \in W_{J_i} \setminus W_{J_{i-1}}$ for $1 \leqslant i \leqslant r$. Given such a decomposition, it is then also straightforward to verify if the remaining conditions in (b) hold.

It turns out that this procedure is successful for all W of exceptional type. The results are given in Tables 1 and 2 (where we use the notation of [27, App. B]). □

We remark that condition (b) in Proposition 2.5 was essential in turning the question of the existence of excellent elements for the large exceptional types into a feasible problem. In fact, the formulation of that condition itself was found by experiments with CHEVIE in small rank examples.

3. Bruhat decomposition and unipotent classes

Following Lusztig [50], [51], the results and concepts discussed in the previous section can be seen to have a geometric significance. Let k be an algebraic closure of the finite field $\mathbb{F}_p$ where p is a prime. Let G be a connected reductive algebraic group over k. Let $B \subseteq G$ be a Borel subgroup and $T \subseteq G$ be a maximal torus contained in B. Let $W = N_G(T)/T$ be the Weyl group of G, a finite Coxeter group. We have the Bruhat decomposition

$$G = \coprod_{w \in W} B\dot{w}B$$

where $\dot{w}$ denotes a representative of $w \in W$ in $N_G(T)$. Let G_{uni} be the unipotent variety of G. It is known [35] that G_{uni} is the union of finitely many conjugacy classes of G which are called the *unipotent classes* of G. We can now state:

Theorem 3.1 (Lusztig [51, 0.4])**.** *Assume that p is good for G. Let $C \in \mathrm{Cl}(W)$. Then there exists a unique unipotent class in G, denoted by $\mathcal{O}_C$, with the following properties:*

(a) *We have $\mathcal{O}_C \cap B\dot{w}B \neq \varnothing$ for some $w \in C_{\min}$.*
(b) *Given any $w' \in C_{\min}$ and any unipotent class $\mathcal{O}'$ we have $\mathcal{O}' \cap B\dot{w}'B = \varnothing$, unless $\mathcal{O}_C$ is contained in the Zariski closure of $\mathcal{O}'$.*

Furthermore, the assignment $C \mapsto \mathcal{O}_C$ defines a surjective map from $\mathrm{Cl}(W)$ to the set of unipotent classes of G.

Recall that p is "good" for G if p is good for each simple factor involved in G; the conditions for the various simple types are as follows.

$$\begin{aligned} A_n: &\quad \text{no condition,}\\ B_n, C_n, D_n: &\quad p \neq 2,\\ G_2, F_4, E_6, E_7: &\quad p \neq 2, 3,\\ E_8: &\quad p \neq 2, 3, 5. \end{aligned}$$

Remark 3.2. Let $C \in \mathrm{Cl}(W)$ and $\mathcal{O}$ be a unipotent class in G. Let $w, w' \in C_{\min}$. As pointed out in [51, 0.2], we have the equivalence:

$$\mathcal{O} \cap B\dot{w}B \neq \varnothing \quad \Leftrightarrow \quad \mathcal{O} \cap B\dot{w}'B \neq \varnothing.$$

(This follows from Remark 3.5 and Corollary 3.7 below.) Hence, in condition (a) of the theorem we have in fact $\mathcal{O}_C \cap B\dot{w}B \neq \varnothing$ for *all* $w \in C_{\min}$.

The *excellent* elements in the conjugacy classes of W (see Definition 2.4) play a role in the proof of Theorem 3.1 for G of classical type. More generally, they enter the picture via the following conjecture which would provide an alternative and more direct description of the map $C \mapsto \mathcal{O}_C$.

Conjecture 3.3 (Lusztig [51, 4.7])**.** *Let $C \in \mathrm{Cl}(W)$ and $w \in C_{\min}$ be excellent, with a decomposition $w = t_1 \cdots t_r$ as in Definition 2.4. Define a corresponding unipotent element $u_w \in G$ as in* [51, 2.4]. *Then $u_w \in \mathcal{O}_C$.*

Example 3.4. Let Φ be the root system of G with respect to T and $\{\alpha_s \mid s \in S\} \subseteq \Phi$ be the system of simple roots determined by B. Let $X_\alpha = \{x_\alpha(\xi) \mid \xi \in k\} \subseteq G$ be the root subgroup corresponding to $\alpha \in \Phi$. Now let $s \in S$ and $C \in \mathrm{Cl}(W)$ be the conjugacy class containing s. Clearly, s is excellent. By the procedure in [51, 2.4], we obtain the unipotent element $u_s = x_{-\alpha_s}(1) \in G$; note that $u_s \in B\dot{s}B$. Then $\mathcal{O}_C$ is the unipotent class containing u_s. (This immediately follows from the reduction arguments in [51, 1.1], which show that we can assume without loss of generality that $W = \langle s \rangle$ and, hence, G is a group of type A_1.)

Remark 3.5. Let q be a power of p and $F\colon G \to G$ be the Frobenius map with respect to a split $\mathbb{F}_q$-rational structure on G, such that $F(t) = t^q$ for all $t \in T$. Then B and all unipotent classes of G are F-stable; furthermore, F acts as the identity on W. For each $w \in W$, we can choose $\dot{w} \in N_G(T)$ such that $F(\dot{w}) = \dot{w}$. Given an F-stable subset $M \subseteq G$, we write $M^F := \{m \in M \mid F(m) = m\}$. Then, for any $w \in W$ and any unipotent class $\mathcal{O}$ of G, we have the equivalence:

(a) $$\mathcal{O} \cap B\dot{w}B \neq \varnothing \quad \Leftrightarrow \quad |(\mathcal{O} \cap B\dot{w}B)^F| \neq 0 \quad \text{for } q \text{ sufficiently large.}$$

Hence, the conditions in Theorem 3.1 can be verified by working in the finite groups G^F. (This remark already appeared in [51, 1.2].)

Remark 3.6. The cardinalities on the right hand side of the equivalence in Remark 3.5 can be computed using the representation theory of the finite group G^F. Namely, consider the permutation module $\mathbb{C}[G^F/B^F]$ for G^F and let

$$\mathcal{H}_q = \mathrm{End}_{\mathbb{C}G^F}\big(\mathbb{C}[G^F/B^F]\big)^{\mathrm{opp}}$$

be the corresponding *Hecke algebra*. (Here, "opp" denotes the opposite algebra; thus, $\mathcal{H}_q$ acts on the right on $\mathbb{C}[G^F/B^F]$.) For $w \in W$, the linear map

$$T_w\colon \mathbb{C}[G^F/B^F] \to \mathbb{C}[G^F/B^F], \qquad xB^F \mapsto \sum_{\substack{yB^F \in G^F/B^F\\ x^{-1}y \in B^F\dot{w}B^F}} yB^F,$$

is contained in $\mathcal{H}_q$. Furthermore, $\{T_w \mid w \in W\}$ is a basis of $\mathcal{H}_q$ and the multiplication is given as follows, where $s \in S$ and $w \in W$:

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w),\\ qT_{sw} + (q-1)T_w & \text{if } l(sw) < l(w); \end{cases}$$

see, for example, [11, §67A], [27, §8.4]. Now $\mathbb{C}[G^F/B^F]$ is a $(\mathbb{C}G^F, \mathcal{H}_q)$-bimodule. For any $g \in G^F$ and $w \in W$, one easily finds using the defining formulae:

$$\mathrm{trace}\big((g, T_w), \mathbb{C}[G^F/B^F]\big) = \frac{|C_{G^F}(g)|}{|B^F|}\,|O_g \cap B^F\dot{w}B^F|$$

where O_g denotes the conjugacy class of g in G^F. Now, for any irreducible representation $V \in \mathrm{Irr}(\mathcal{H}_q)$ there is a corresponding irreducible representation $\rho_V \in \mathrm{Irr}_{\mathbb{C}}(G^F)$, and this gives rise to a direct sum decomposition

(a) $$\mathbb{C}[G^F/B^F] \cong \sum_{V \in \mathrm{Irr}(\mathcal{H}_q)} \rho_V \otimes V$$

as $(\mathbb{C}G^F, \mathcal{H}_q)$-bimodules; see, for example, [11, §68B], [27, 8.4.4]. In combination with the previous discussion, this yields the formula

(b) $$|O_g \cap B^F\dot{w}B^F| = \frac{|B^F|}{|C_{G^F}(g)|} \sum_{V \in \mathrm{Irr}(\mathcal{H}_q)} \mathrm{trace}(g, \rho_V)\,\mathrm{trace}(T_w, V),$$

which already appeared in [50, 1.5(a)]. We shall illustrate the use of this formula in a small rank example below. Some more sophisticated techniques for the evaluation of the right hand side of (b) will be discussed in Section 5.

Corollary 3.7 (Lusztig [50, 1.5], [51, 1.2]). *Let $\mathcal{O}$ be a unipotent class in G.*

(a) *For a fixed $g \in \mathcal{O}^F$, the linear map $\mathcal{H}_q \to \mathbb{C}$, $T_w \mapsto |O_g \cap B^F \dot{w} B^F|$, is a trace function on $\mathcal{H}_q$.*
(b) *The linear map $\mathcal{H}_q \to \mathbb{C}$, $T_w \mapsto |(\mathcal{O} \cap B\dot{w}B)^F|$, is a trace function on $\mathcal{H}_q$.*
(c) *Let $C \in \mathrm{Cl}(W)$ and $w, w' \in C_{\min}$. Then $|(\mathcal{O} \cap B\dot{w}B)^F| = |(\mathcal{O} \cap B\dot{w}'B)^F|$.*

Proof. (a) The formula in Remark 3.6(b) shows that the map $T_w \mapsto |O_g \cap B^F\dot{w}B^F|$ is a $\mathbb{C}$-linear combination of characters of $\mathcal{H}_q$ and, hence, a trace function.

(b) First note that $(B\dot{w}B)^F = B^F\dot{w}B^F$. (This follows from the sharp form of the Bruhat decomposition; see [8, 2.5.13], [16, 1.7.2].) Now let $u_1, \ldots, u_d \in G^F$ be representatives of the G^F-conjugacy classes contained in $\mathcal{O}^F$. Then

$$\begin{aligned} |(\mathcal{O} \cap B\dot{w}B)^F| &= |\mathcal{O}^F \cap B^F\dot{w}B^F| = \sum_{1 \leqslant i \leqslant d} |O_{u_i} \cap B^F\dot{w}B^F| \\ &= |B^F| \sum_{1 \leqslant i \leqslant d} |C_{G^F}(u_i)|^{-1} \mathrm{trace}\big((u_i, T_w), \mathbb{C}[G^F/B^F]\big). \end{aligned}$$

So the assertion follows from (a).

(c) This is a general property of trace functions on $\mathcal{H}_q$; see [27, 8.2.6]. □

Remark 3.8. Lusztig's formulation [51, 0.4] of Theorem 3.1 looks somewhat different: Instead of using the intersections $\mathcal{O} \cap B\dot{w}B$, he uses certain sub-varieties $\mathcal{B}_w^\gamma \subseteq G \times G/B$ (where γ denotes $\mathcal{O}$). However, we have

$$|(\mathcal{B}_w^\gamma)^F| = \sum_{g \in \gamma^F} \mathrm{trace}\big((g, T_w), \mathbb{C}[G^F/B^F]\big) = |G^F/B^F|\,|(\mathcal{O} \cap B\dot{w}B)^F|$$

where the first equality holds by [51, 1.2] and the second by Remark 3.6 (see the proof of Corollary 3.7(b)). In combination with Remark 3.5 we see that, indeed, the formulation of Theorem 3.1 is equivalent to Lusztig's version [51].

Example 3.9. Let $G = \mathrm{Sp}_4(k)$ where W is of type B_2, with generators $S = \{s, t\}$. The algebra $\mathcal{H}_q$ has 5 irreducible representations; their traces on basis elements T_w ($w \in C_{\min}$) are given as follows; see [27, Tab. 8.1, p. 270]:

	T_1	T_t	T_{stst}	T_s	T_{st}
ind	1	q	q^4	q	q^2
σ	2	$q-1$	$-2q^2$	$q-1$	0
sgn_1	1	-1	q^2	q	$-q$
sgn_2	1	q	q^2	-1	$-q$
sgn	1	-1	1	-1	1

Now assume that $\mathrm{char}(k) \neq 2$. (Recall that 2 is a bad prime for type B_2.) There are four unipotent classes in G which we denote by $\mathcal{O}_\mu$ where the subscript μ specifies the Jordan type of the elements in the class. For example, the class $\mathcal{O}_{(211)}$ consists of unipotent matrices with one Jordan block of size 2 and two blocks of size 1. The set $\mathcal{O}_{(22)}^F$ splits into two classes in G^F which we denote by $O_{(22)}$ and $O'_{(22)}$; each of the remaining classes $\mathcal{O}_\mu$ gives rise to exactly one class in G^F which we denote by O_μ. The values of the irreducible characters of G^F corresponding to $\mathrm{Irr}(\mathcal{H}_q)$ can be extracted from Srinivasan's table [66]:

	$O_{(1111)}$	$O_{(211)}$	$O_{(22)}$	$O'_{(22)}$	$O_{(4)}$
$\lvert C_{G^F}(u)\rvert$	$\lvert G^F\rvert$	$q^4(q^2-1)$	$2q^3(q-1)$	$2q^3(q+1)$	q^2
ρ_{ind}	1	1	1	1	1
ρ_σ	$\frac{1}{2}q(q+1)^2$	$\frac{1}{2}q(q+1)$	q	0	0
ρ_{sgn_1}	$\frac{1}{2}q(q^2+1)$	$-\frac{1}{2}q(q-1)$	q	0	0
ρ_{sgn_2}	$\frac{1}{2}q(q^2+1)$	$\frac{1}{2}q(q+1)$	0	q	0
ρ_{sgn}	q^4	0	0	0	0

We now multiply the transpose of the character table of $\mathcal{H}_q$ with the above piece of Srinivasan's matrix. By the formula in Remark 3.6(b), this yields (up to a factor $|C_{G^F}(u)|/|B^F|$) the matrix of cardinalities $|O_u \cap B^F\dot{w}B^F|$ where

$u \in G^F$ is unipotent and $w \in C_{\min}$ for some $C \in \mathrm{Cl}(W)$:

	$O_{(1111)}$	$O_{(211)}$	$O_{(22)}$	$O'_{(22)}$	$O_{(4)}$
1	$\lvert G^F/B^F\rvert$	q^2+2q+1	$3q+1$	$q+1$	1
t	0	q^3+q^2	q^2-q	q^2+q	q
$stst$	0	0	q^4-q^3	q^4+q^3	q^4
s	0	0	$2q^2$	0	q
st	0	0	0	0	q^2

The closure relation among the unipotent classes is a linear order, in the sense that $\mathcal{O} \subsetneq \overline{\mathcal{O}}'$ if and only if $\dim \mathcal{O} < \dim \mathcal{O}'$. Thus, Theorem 3.1 yields the map

$$C_1 \mapsto \mathcal{O}_{(1111)}, \quad C_s \mapsto \mathcal{O}_{(22)}, \quad C_t \mapsto \mathcal{O}_{(211)}, \quad C_{st} \mapsto \mathcal{O}_{(4)}, \quad C_{stst} \mapsto \mathcal{O}_{(22)}$$

where C_w denotes the conjugacy class of W containing w.

Now assume that $\mathrm{char}(k) = 2$. We verify that, in this "bad" characteristic case, the assertions of Theorem 3.1 still hold. We use a similar convention for denoting unipotent classes as above; just note that, now, there are two unipotent classes in G with elements of Jordan type (22), which we denote by $\mathcal{O}_{(22)}$ and $\mathcal{O}^*_{(22)}$. The values of the irreducible characters of G^F corresponding to $\mathrm{Irr}(\mathcal{H}_q)$ have been determined by Enomoto [13] (with some corrections due to Lübeck):

	$O_{(1111)}$	$O_{(211)}$	$O^*_{(22)}$	$O_{(22)}$	$O_{(4)}$	$O'_{(4)}$
$\lvert C_{G^F}(u)\rvert$	$\lvert G^F\rvert$	$q^4(q^2-1)$	$q^4(q^2-1)$	q^4	$2q^2$	$2q^2$
ρ_{ind}	1	1	1	1	1	1
ρ_σ	$\frac{1}{2}q(q+1)^2$	$\frac{1}{2}q(q+1)$	$\frac{1}{2}q(q+1)$	$\frac{q}{2}$	$\frac{q}{2}$	$-\frac{q}{2}$
ρ_{sgn_1}	$\frac{1}{2}q(q^2+1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q+1)$	$\frac{q}{2}$	$-\frac{q}{2}$	$\frac{q}{2}$
ρ_{sgn_2}	$\frac{1}{2}q(q^2+1)^2$	$\frac{1}{2}q(q+1)$	$-\frac{1}{2}q(q-1)$	$\frac{q}{2}$	$-\frac{q}{2}$	$\frac{q}{2}$
ρ_{sgn}	q^4	0	0	0	0	0

As before, this yields (up to a factor $\lvert C_{G^F}(u)\rvert/\lvert B^F\rvert$) the matrix of cardinalities $\lvert O_u \cap B^F\dot{w}B^F\rvert$ where $u \in G^F$ is unipotent and $w \in C_{\min}$ for some $C \in \mathrm{Cl}(W)$:

	$O_{(1111)}$	$O_{(211)}$	$O^*_{(22)}$	$O_{(22)}$	$O_{(4)}$	$O'_{(4)}$
1	$\lvert G^F/B^F\rvert$	q^2+2q+1	q^2+2q+1	$2q+1$	1	1
t	0	q^3+q^2	0	q^2	q	q
$stst$	0	0	0	q^4	q^4-2q^3	q^4+2q^3
s	0	0	q^3+q^2	q^2	q	q
st	0	0	0	0	$2q^2$	0

We conclude that the conditions in Theorem 3.1 hold for the map

$$C_1 \mapsto \mathcal{O}_{(1111)}, \quad C_s \mapsto \mathcal{O}^*_{(22)}, \quad C_t \mapsto \mathcal{O}_{(211)}, \quad C_{st} \mapsto \mathcal{O}_{(4)}, \quad C_{stst} \mapsto \mathcal{O}_{(22)}.$$

As pointed out by Lusztig [52, 4.8], there is considerable evidence that, in general, Theorem 3.1 will continue to hold in bad characteristic.

4. Characters of finite Coxeter groups

All the general GAP functionality for working with character tables of finite groups is available for finite Coxeter groups: For example, we can form tensor products of characters, induce characters from subgroups, and decompose the characters so obtained into irreducibles. For a finite Coxeter group W, the following versions of the above operations are particularly relevant:

- tensoring with the sign character (usually denoted here by "sgn");
- inducing characters from parabolic subgroups (or reflection subgroups).

Beginning with [37], Lusztig developed the idea that various data which are important in the representation theory of reductive algebraic groups can be recovered purely in terms of the above operations together with certain numerical functions on the irreducible characters of W. (See Lusztig [49] for more recent work in this direction.) Quite often this leads to explicit recursive descriptions of these data, which can be effectively implemented in programs written in the GAP language. We discuss some examples in this section.

Probably the most subtle of the numerical functions on the irreducible characters of W is given by the so-called "$\boldsymbol{a}$-invariants". These are originally defined in [37] by using the "generic degrees" of the corresponding generic Iwahori–Hecke algebra; see [11, §68C], [27, 9.3.6]. Developing an idea in [27, §6.5], we begin by showing that these

"$\boldsymbol{a}$-invariants" can be characterised purely in terms of the characters of W, without reference to the generic Iwahori–Hecke algebra.

We shall work in the general "multi-parameter" setting of [40]. To describe this, let Γ be an abelian group (written additively). Following Lusztig [47], we say that a function $L\colon W \to \Gamma$ is a *weight function* if we have

$$L(ww') = L(w) + L(w') \quad \text{for all } w, w' \in W \text{ such that } l(ww') = l(w) + l(w').$$

Note that such a function L is uniquely determined by the values $\{L(s) \mid s \in S\}$. Furthermore, if $\{c_s \mid s \in S\}$ is a collection of elements in Γ such that $c_s = c_t$ whenever $s, t \in S$ are conjugate in W, then there is (unique) weight function $L\colon W \to \Gamma$ such that $L(s) = c_s$ for all $s \in S$. (This follows from Matsumoto's Lemma; see [27, §1.2].) We will further assume that Γ admits a total ordering $\leqslant$ which is compatible with the group structure, that is, whenever $g, g' \in \Gamma$ are such that $g \leqslant g'$, we have $g + h \leqslant g' + h$ for all $h \in \Gamma$. Then we will require that

$$L(s) \geqslant 0 \qquad \text{for all } s \in S.$$

(The standard and most important example of this whole setting is $\Gamma = \mathbb{Z}$ with its natural ordering; if, moreover, we have $L(s) = 1$ for all $s \in S$, then we say that we are in the "equal parameter case".)

Let $\mathrm{Irr}(W)$ be the set of (complex) irreducible representations of W (up to isomorphism). Having fixed $L, \Gamma, \leqslant$ as above, we wish to define a function

$$\mathrm{Irr}(W) \to \Gamma_{\geqslant 0}, \qquad E \mapsto \tilde{\boldsymbol{a}}_E.$$

We need one further piece of notation. Recall that $T = \{wsw^{-1} \mid w \in W, s \in S\}$ is the set of all reflections in W. Let $S' \subseteq S$ be a set of representatives of the conjugacy classes of W which are contained in T. For $s \in S'$, let N_s be the cardinality of the conjugacy class of s; thus, $|T| = \sum_{s \in S'} N_s$. Now let $E \in \mathrm{Irr}(W)$ and $s \in S'$. Since s has order 2, it is clear that $\mathrm{trace}(s, E) \in \mathbb{Z}$. Hence, by a well-known result in the character theory of finite groups, the quantity $N_s \mathrm{trace}(s, E)/\dim E$ is an integer. Thus, we can define

$$\omega_L(E) := \sum_{s \in S'} \frac{N_s \,\mathrm{trace}(s, E)}{\dim E}\, L(s) \in \Gamma.$$

(Note that this does not depend on the choice of the set of representatives $S' \subseteq S$.)

Definition 4.1. We define a function $\mathrm{Irr}(W) \to \Gamma$, $E \mapsto \tilde{\boldsymbol{a}}_E$, inductively as follows. If $W = \{1\}$, then $\mathrm{Irr}(W)$ only consists of the unit representation (denoted 1_W) and we set $\tilde{\boldsymbol{a}}_{1_W} := 0$. Now assume that $W \neq \{1\}$ and that the function $E \mapsto \tilde{\boldsymbol{a}}_E$ has already been defined for all proper parabolic subgroups of W. Then, for any $E \in \mathrm{Irr}(W)$, we can define

$$\tilde{\boldsymbol{a}}'_E := \max\{\tilde{\boldsymbol{a}}_M \mid M \in \mathrm{Irr}(W_J) \text{ where } J \subsetneqq S \text{ and } M \uparrow E\}.$$

Here, we write $M \uparrow E$ if E is an irreducible constituent of the representation obtained by inducing M from W_J to W. Finally, we set

$$\tilde{\boldsymbol{a}}_E := \begin{cases} \tilde{\boldsymbol{a}}'_E & \text{if } \tilde{\boldsymbol{a}}'_{E\otimes\mathrm{sgn}} - \tilde{\boldsymbol{a}}'_E \leqslant \omega_L(E), \\ \tilde{\boldsymbol{a}}'_{E\otimes\mathrm{sgn}} - \omega_L(E) & \text{otherwise.} \end{cases}$$

One immediately checks that this function satisfies the following conditions:

$$\tilde{\boldsymbol{a}}_E \geqslant \tilde{\boldsymbol{a}}'_E \geqslant 0 \quad \text{and} \quad \tilde{\boldsymbol{a}}_{E\otimes\mathrm{sgn}} - \tilde{\boldsymbol{a}}_E = \omega_L(E) \quad \text{for all } E \in \mathrm{Irr}(W).$$

This also shows that $\tilde{\boldsymbol{a}}_E \geqslant \tilde{\boldsymbol{a}}_M$ if $M \uparrow E$ where $M \in \mathrm{Irr}(W_J)$ and $J \subsetneqq S$.

Example 4.2. (a) If $L(s) = 0$ for all $s \in S$, then $\tilde{\boldsymbol{a}}_E = 0$ for any $E \in \mathrm{Irr}(W)$.

(b) Assume that we are in type A_{n-1}, where $W \cong \mathfrak{S}_n$ and there is a natural labelling $\mathrm{Irr}(W) = \{E^\lambda \mid \lambda \vdash n\}$. All generators in S are conjugate and so any non-zero weight function L takes a constant value $a > 0$ on S. Then we have:

$$\tilde{\boldsymbol{a}}_{E^\lambda} = \sum_{1 \leqslant i \leqslant r} (i-1)\lambda_i\, a \quad \text{where} \quad \lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r \geqslant 0);$$

This can be shown by a direct argument, as indicated in [27, Example 6.5.8].

Remark 4.3. As already mentioned, Lusztig originally defined an "$\boldsymbol{a}$-function" $\mathrm{Irr}(W) \to \Gamma$, $E \mapsto \boldsymbol{a}_E$, using the "generic degrees" of the generic Iwahori–Hecke algebra associated with W and the weight function L. It is known that this function has the following properties:

- (A0) *We have* $\boldsymbol{a}_{1_W} = 0$.
- (A1) *Let* $J \subsetneqq S$, $M \in \mathrm{Irr}(W_J)$ *and* $E \in \mathrm{Irr}(W)$ *be such that* $M \uparrow E$. *Then* $\boldsymbol{a}_M \leqslant \boldsymbol{a}_E$.
- (A2) *Let* $J \subsetneqq S$ *and* $M \in \mathrm{Irr}(W_J)$. *Then there exists some* $E \in \mathrm{Irr}(W)$ *such that* $M \uparrow E$ *and* $\boldsymbol{a}_M = \boldsymbol{a}_E$. *In this case, we write* $M \rightsquigarrow_L E$.
- (A3) *Let* $E \in \mathrm{Irr}(W)$. *Then there exists some* $J \subsetneqq S$ *and some* $M \in \mathrm{Irr}(W_J)$ *such that* $M \rightsquigarrow_L E$ *or* $M \rightsquigarrow_L E \otimes \mathrm{sgn}$.
- (A4) *Let* $E \in \mathrm{Irr}(W)$. *Then* $\boldsymbol{a}_{E\otimes\mathrm{sgn}} - \boldsymbol{a}_E = \omega_L(E)$.

(For the original definition of $\boldsymbol{a}_E$ in the equal parameter case, see Lusztig [37]; in that article, one can also find (A1) and (A2). Analogous definitions and arguments work for a general weight function L; see [15, §3], [47, Chap. 20] for details. (A0) is clear by the definition of $\boldsymbol{a}_E$. A version of (A3) for "special" representations in the equal parameter case already appeared in [37, §6]; the general case follows from [47, Prop. 22.3]. Note that there does not seem to be a notion of "special" representations for the general multi-parameter case; see [17, Rem. 4.11]. (A4) follows from [27, Prop. 9.4.3].)

Following the argument in [27, 6.5.6], let us now prove that $\boldsymbol{a}_E = \tilde{\boldsymbol{a}}_E$ for all $E \in \mathrm{Irr}(W)$. We proceed by induction on the order of W. If $W = \{1\}$, then $\mathrm{Irr}(W)$ only consists of 1_W and we have $\boldsymbol{a}_{1_W} = \tilde{\boldsymbol{a}}_{1_W} = 0$; see (A0). Now assume that $W \neq \{1\}$ and that the assertion is already proved for all proper parabolic subgroups of W. Consequently, using (A1), we have

$$(*) \qquad \boldsymbol{a}_E \geqslant \tilde{\boldsymbol{a}}'_E \qquad \text{for all } E \in \mathrm{Irr}(W).$$

Now fix $E \in \mathrm{Irr}(W)$. Using (A3), we distinguish two cases. Assume first that $M \rightsquigarrow_L E$ for some $M \in \mathrm{Irr}(W_J)$ where $J \subsetneqq S$. By $(*)$, we have $\boldsymbol{a}_E \geqslant \tilde{\boldsymbol{a}}'_E \geqslant \boldsymbol{a}_M$. Since $\boldsymbol{a}_E = \boldsymbol{a}_M$, we deduce that $\boldsymbol{a}_E = \tilde{\boldsymbol{a}}'_E$. Now, by $(*)$ applied to $E \otimes \mathrm{sgn}$, we also have $\boldsymbol{a}_{E\otimes\mathrm{sgn}} \geqslant \tilde{\boldsymbol{a}}'_{E\otimes\mathrm{sgn}}$ and so, using (A4), $\tilde{\boldsymbol{a}}'_{E\otimes\mathrm{sgn}} - \tilde{\boldsymbol{a}}'_E \leqslant \boldsymbol{a}_{E\otimes\mathrm{sgn}} - \boldsymbol{a}_E = \omega_L(E)$. Hence, we are in the first case of Definition 4.1 and so $\tilde{\boldsymbol{a}}_E = \tilde{\boldsymbol{a}}'_E = \boldsymbol{a}_E$, as required.

Now assume that $M \rightsquigarrow_L E \otimes \mathrm{sgn}$ for some $M \in \mathrm{Irr}(W_J)$ where $J \subsetneqq S$. Arguing as before, we have $\boldsymbol{a}_{E\otimes\mathrm{sgn}} = \tilde{\boldsymbol{a}}'_{E\otimes\mathrm{sgn}}$. Using $(*)$ and (A3), we obtain

$$\tilde{\boldsymbol{a}}'_{E\otimes\mathrm{sgn}} - \tilde{\boldsymbol{a}}'_E = \boldsymbol{a}_{E\otimes\mathrm{sgn}} - \tilde{\boldsymbol{a}}'_E \geqslant \boldsymbol{a}_{E\otimes\mathrm{sgn}} - \boldsymbol{a}_E = \omega_L(E).$$

If this inequality is an equality, then $\tilde{\boldsymbol{a}}'_E = \boldsymbol{a}_E$; furthermore, we are in the first case of Definition 4.1 and so $\tilde{\boldsymbol{a}}_E = \tilde{\boldsymbol{a}}'_E = \boldsymbol{a}_E$, as required. If the above inequality is strict, then we are in the second case of Definition 4.1 and, using (A4), we obtain $\tilde{\boldsymbol{a}}_E = \tilde{\boldsymbol{a}}'_{E\otimes\mathrm{sgn}} - \omega_L(E) = \boldsymbol{a}_{E\otimes\mathrm{sgn}} - \omega_L(E) = \boldsymbol{a}_E$, as required.

Remark 4.4. Out of the five properties (A0)–(A4), it seems that (A3) is the most subtle one. In fact, (A0), (A1), (A2) and (A4) are proved by general arguments while the proof of (A3) relies on an explicit case–by–case verification. Consider the following related statement:

(A3′) *Let $E \in \mathrm{Irr}(W)$ be such that $\omega_L(E) \geqslant 0$. Then there exists some proper subset $J \subsetneqq S$ and some $M \in \mathrm{Irr}(W_J)$ such that $M \uparrow E$ and $\boldsymbol{a}_M = \boldsymbol{a}_E$.*

Note that $\omega_L(E \otimes \mathrm{sgn}) = -\omega_L(E)$, so (A3′) certainly implies (A3). The above property has first been formulated and checked (in the equal parameter case) by Spaltenstein [63, §5] (see also [20, Lemma 4.9]). As far as groups of exceptional type are concerned, Spaltenstein just says that "we can use tables". So here is a place where CHEVIE can provide more systematic algorithmic verifications. It would certainly be interesting to find a general argument for proving (A3′).

The following definition is inspired by Lusztig [41, 4.2] and Spaltenstein [63].

Definition 4.5 (See [20, 2.10])**.** We define a relation $\preceq_L$ on $\mathrm{Irr}(W)$ inductively as follows. If $W = \{1\}$, then $\mathrm{Irr}(W)$ only consists of the unit representation and this is related to itself. Now assume that $W \neq \{1\}$ and that $\preceq_L$ has already been defined for all proper parabolic subgroups of W. Let $E, E' \in \mathrm{Irr}(W)$. Then we write $E \preceq_L E'$ if there is a sequence $E = E_0, E_1, \ldots, E_m = E'$ in $\mathrm{Irr}(W)$ such that, for each $i \in \{1, 2, \ldots, m\}$, the following condition is satisfied. There exists a subset $I_i \subsetneqq S$ and $M'_i, M''_i \in \mathrm{Irr}(W_{I_i})$, where $M'_i \preceq_L M''_i$ within $\mathrm{Irr}(W_{I_i})$, such that either

$$M'_i \uparrow E_{i-1} \quad \text{and} \quad M''_i \uparrow E_i \quad \text{where} \quad \tilde{\boldsymbol{a}}_{E_i} = \tilde{\boldsymbol{a}}_{M''_i}$$

or

$$M'_i \uparrow E_i \otimes \mathrm{sgn} \quad \text{and} \quad M''_i \uparrow E_{i-1} \otimes \mathrm{sgn} \quad \text{where} \quad \tilde{\boldsymbol{a}}_{E_{i-1}\otimes\mathrm{sgn}} = \tilde{\boldsymbol{a}}_{M''_i}.$$

Let $\sim_L$ be the equivalence relation associated with $\preceq_L$, that is, we have $E \sim_L E'$ if and only if $E \preceq_L E'$ and $E' \preceq_L E$. Then we have an induced partial order on the set of equivalence classes of $\mathrm{Irr}(W)$ which we denote by the same symbol $\preceq_L$.

Example 4.6. (a) If $L(s) = 0$ for all $s \in S$, then $E \preceq_L E'$ for any $E, E' \in \mathrm{Irr}(W)$.

(b) Assume that $W \cong \mathfrak{S}_n$ and $L(s) = a > 0$ for $s \in S$, as in Example 4.2. Let λ, μ be partitions of n. Then we have $E^\lambda \preceq_L E^\mu$ if and only if $\lambda \trianglelefteq \mu$, where $\trianglelefteq$ denotes the *dominance order* on partitions; see [22, Exp. 3.5].

Remark 4.7. One can show that the following "monotony" property holds:

(a) *If $E, E' \in \mathrm{Irr}(W)$ are such that $E \preceq_L E'$, then $\tilde{\boldsymbol{a}}_{E'} \leqslant \tilde{\boldsymbol{a}}_E$;*

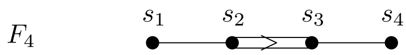
see [20, Prop. 4.4], [22, §6]. Consequently, the equivalence classes of $\mathrm{Irr}(W)$ under $\sim_L$ are precisely the "families" as defined by Lusztig [41, 4.2], [47, 23.1]. (This immediately follows from the definitions, see the argument in [20, Prop. 4.4].) In particular, the following holds:

(b) *The function $E \mapsto \tilde{\boldsymbol{a}}_E$ is constant on the "families" of $\mathrm{Irr}(W)$.*

In the equal parameter case, this appeared originally in [41, 4.14.1]; see also [39].

It is straightforward to implement the recursion in Definition 4.5 in the GAP programming language. In this way, one can for example systematically re-compute the families of $\mathrm{Irr}(W)$ for the exceptional types (in the equal parameter case), which are listed in [41, Chap. 4]. Similar computations can be performed for a general weight function L.

Example 4.8. Let W be of type F_4 with generators labelled as follows:



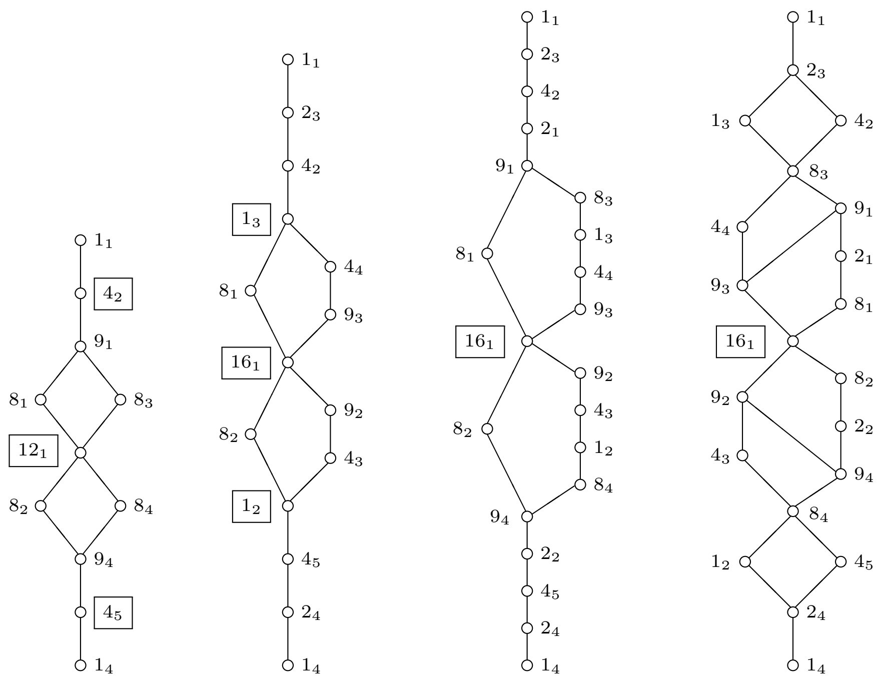
Assume that $a := L(s_1) = L(s_2) > 0$ and $b := L(s_3) = L(s_4) > 0$. By the symmetry of the Dynkin diagram, we may also assume without loss of generality that $b \geqslant a$. The results of the computation of $\preceq_L$ and $\sim_L$ are presented in Table 3. The notation for $\mathrm{Irr}(W)$ follows [27, App. C]; for example, $1_1 = 1_W$, $1_4 = \mathrm{sgn}$ and 4_2 is the standard reflection representation.

Quite remarkably, it turns out that there are only 4 essentially different cases. Note that, a priori, one has to deal with infinitely many values of a, b; a reduction to a finite set of values is achieved by using similar techniques as in [17]; in any case, the final result is the same as that given in the table in [17, p. 362].

The partition of $\mathrm{Irr}(W)$ into *families* follows from the earlier results of Lusztig [47, 22.17]. (Note that there is an error for $b = 2a$ in [47, 22.17]; this has been corrected in [17, 4.10], based on the explicit computations using CHEVIE.)

This example, and Guilhot's results [29] on affine Weyl groups of rank 2 (which also rely on explicit computations using GAP), provide considerable evidence in support of Bonnafé's "*semicontinuity conjectures*" [5].

TABLE 3. Partial order $\preceq_L$ on families in type F_4



$a = b$ $\qquad$ $b = 2a$ $\qquad$ $2a > b > a$ $\qquad$ $b > 2a$

A box indicates a family containing several irreducible representations:

$\boxed{4_2} = \{2_1, 2_3, 4_2\}$, $\boxed{4_5} = \{2_2, 2_4, 4_5\}$, $\boxed{1_3} = \{1_3, 2_1, 8_3, 9_1\}$, $\boxed{1_2} = \{1_2, 2_2, 8_4, 9_4\}$,

$\boxed{12_1} = \{1_2, 1_3, 4_1, 4_3, 4_4, 6_1, 6_2, 9_2, 9_3, 12_1, 16_1\}$, $\boxed{16_1} = \{4_1, 6_1, 6_2, 12_1, 16_1\}$.

Otherwise, the family contains just one irreducible respresentation.

Remark 4.9. The idea of partitioning $\mathrm{Irr}(W)$ into "families" originally arose from the representation theory of finite groups of Lie type, see Lusztig [36, §8]. A completely new interpretation appeared in the theory of *Kazhdan–Lusztig cells*; see [32], [40]. Among others, this gives rise not only to a partition but to a natural pre-order relation $\leqslant_{\mathcal{LR}}$ on $\mathrm{Irr}(W)$; see [41, 5.15], [20, Def. 2.2]. The relation $\leqslant_{\mathcal{LR}}$ is an essential ingredient, for example, in the construction of a "cellular structure" in the generic Iwahori–Hecke algebra associated with W, L; see [18], [19]. One can show by a general argument that

$$E \preceq_L E' \qquad \Rightarrow \qquad E \leqslant_{\mathcal{LR}} E' \qquad\quad (E, E' \in \mathrm{Irr}(W));$$

see [20, Prop. 3.4]. In the equal parameter case, it is known that the reverse implication also holds; see [20, Theorem 4.11]. The computations involved in Example 4.8 provide considerable evidence that this will also hold for general weight functions L.—Thus, $\preceq_L$ may be regarded as a purely combinatorial (and computable!) characterisation of $\leqslant_{\mathcal{LR}}$.

Finally, let us assume that W is the Weyl group of a connected reductive algebraic group G over $\overline{\mathbb{F}}_p$ where p is a good prime. Let $\mathcal{N}_G$ be the set of all pairs $(\mathcal{O}, \mathcal{L})$ where $\mathcal{O}$ is a unipotent class in G and $\mathcal{L}$ is a G-equivariant irreducible $\overline{\mathbb{Q}}_\ell$-local system on $\mathcal{O}$ (up to isomorphism); here, ℓ is a prime different from p. By the Springer correspondence (see [65], [42]), we obtain a natural injective map

$$\mathrm{Irr}(W) \hookrightarrow \mathcal{N}_G, \qquad E \mapsto \iota_E = (\mathcal{O}_E, \mathcal{L}_E).$$

It is known that, for any unipotent class $\mathcal{O}$, the pair $(\mathcal{O}, \overline{\mathbb{Q}}_\ell) \in \mathcal{N}_G$ (where $\overline{\mathbb{Q}}_\ell$ stands for the trivial local system) is in the image of this map. Hence, the map

$$\mathrm{Irr}(W) \rightarrow \{\text{unipotent classes of } G\}, \qquad E \mapsto \mathcal{O}_E,$$

is surjective.

Remark 4.10. The Springer correspondence is explicitly known in all cases. In good characteristic, the results are systematically presented in Section 13.3 of Carter [8]; for bad characteristic, see [53], [64]. It turns out that $\mathcal{N}_G$ and the map $E \mapsto \iota_E$ are independent of p (in a suitable sense) as long as p is good; some compatibility properties of the Springer correspondence in good and bad characteristic are established in [24, §2].

Remark 4.11. Let $\Gamma = \mathbb{Z}$ and consider the "equal parameter" weight function L_0 such that $L_0(s) = 1$ for all $s \in S$. Let $\mathcal{F} \subseteq \mathrm{Irr}(W)$ be a family with respect to L_0 (see Remark 4.7) and consider the following collection of unipotent classes in G:

$$\mathcal{C}(\mathcal{F}) := \{\mathcal{O}_E \mid E \in \mathrm{Irr}(W) \text{ such that } E \in \mathcal{F}\}.$$

Then it is known that there exists a unique unipotent class in $\mathcal{C}(\mathcal{F})$, which we denote by $\mathcal{O}_\mathcal{F}$, such that $\mathcal{O} \subseteq \overline{\mathcal{O}}_\mathcal{F}$ for all $\mathcal{O} \in \mathcal{C}(\mathcal{F})$; see [24, Prop. 2.2]. (Here, and below, $\overline{X}$ denotes the Zariski closure in G for any subset $X \subseteq G$.) Thus, $\mathcal{O}_\mathcal{F}$ is the maximum of the elements in $\mathcal{C}(\mathcal{F})$ with respect to the partial order given by the Zariski closure. A unipotent class of the form $\mathcal{O}_\mathcal{F}$ will be called a "*special*" unipotent class. Thus, we have a bijection

$$\{\text{families of } \mathrm{Irr}(W)\} \xrightarrow{1-1} \{\text{special unipotent classes of } G\}, \quad \mathcal{F} \mapsto \mathcal{O}_\mathcal{F}.$$

(Special unipotent classes were originally defined by Lusztig [37, §9]. The above equivalent characterisation appeared in [20, 5.2]; see also Remark 4.13 below.)

Now we can formulate the following geometric interpretation of the pre-order relation $\preceq_{L_0}$ in Definition 4.5.

Theorem 4.12 (Spaltenstein [63])**.** *Let $\mathcal{F}, \mathcal{F}'$ be families in* $\mathrm{Irr}(W)$ *(with respect to the equal parameter weight function L_0). Then we have*

$$\mathcal{F} \preceq_{L_0} \mathcal{F}' \qquad \Leftrightarrow \qquad \mathcal{O}_\mathcal{F} \subseteq \overline{\mathcal{O}}_{\mathcal{F}'}.$$

Spaltenstein uses a slightly different definition of $\preceq_{L_0}$; the equivalence with the one in Definition 4.5 is shown in [20, Cor. 5.6]. The proofs rely on some explicit verifications for exceptional types; Spaltenstein just says that "we can then use tables" [63, p. 215]. So here again, CHEVIE provides a more systematic algorithmic way of verifying such statements.

Remark 4.13. Let $\mathcal{S}_W$ be the set of all $E \in \mathrm{Irr}(W)$ such that $\mathcal{L}_E \cong \overline{\mathbb{Q}}_\ell$ and $\mathcal{O}_E$ is a special unipotent class; see Remark 4.11. Then every family of $\mathrm{Irr}(W)$ as above contains a unique representation in $\mathcal{S}_W$. It is known that $\mathcal{S}_W$ is the set of "special" representations of W as defined by Lusztig [36], [37]. (This follows from [24, Prop. 2.2].) Following Lusztig [46], we define the "*special piece*" corresponding to $E \in \mathcal{S}_W$ to be the set of all elements in $\overline{\mathcal{O}}_E$ which are not contained in $\overline{\mathcal{O}}_{E'}$ where $E' \in \mathcal{S}_W$ is such that $\mathcal{O}_{E'} \subsetneqq \overline{\mathcal{O}}_E$. By Spaltenstein [62] and Lusztig [46], the various special pieces form a partition of G_{uni}. Note that every special piece is a union of a special unipotent class (which is open dense in the special piece) and of a certain number (possibly zero) of non-special unipotent classes.—We will encounter the special pieces of G_{uni} again in Conjecture 5.3 below.

5. Green functions

We begin by describing a basic algorithm which is inspired by the computation of Green functions and [23]. It can be formulated without any reference to algebraic groups; in fact, it will work for any finite Coxeter group W (including the dihedral groups and groups of type H_3, H_4). Let u be an indeterminate over $\mathbb{Q}$. We define a matrix

$$\Omega = \big(\omega_{E,E'}\big)_{E,E' \in \mathrm{Irr}(W)},$$

as follows. Let $D_W := u^{l(w_0)}(u-1)^{|S|} \sum_{w \in W} u^{l(w)}$ where $w_0 \in W$ is the longest element. Then, for any $E, E' \in \mathrm{Irr}(W)$, we set

$$\omega_{E,E'} := \frac{D_W}{|W|} \sum_{w \in W} \frac{\mathrm{trace}(w,E)\,\mathrm{trace}(w,E')}{\det(u\,\mathrm{id}_V - w)} \in \mathbb{Q}(u);$$

here, W is regarded as a subgroup of $\mathrm{GL}(V)$ via the natural reflection representation on a vector space V of dimension $|S|$. It is known that $\omega_{E,E'} \in \mathbb{Z}[u]$ for all $E, E' \in \mathrm{Irr}(W)$; see [8, 11.1.1].

Lemma 5.1 (Cf. [23, §2]). *Let us fix a partition* $\mathrm{Irr}(W) = \mathcal{I}_1 \sqcup \mathcal{I}_2 \sqcup \ldots \sqcup \mathcal{I}_r$ *and a sequence of integers* $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_r$. *Correspondingly, we write* Ω *in block form:*

$$\Omega = \begin{bmatrix} \Omega_{1,1} & \Omega_{1,2} & \cdots & \Omega_{1,r} \\ \Omega_{2,1} & & & \vdots \\ \vdots & & & \Omega_{r-1,r} \\ \Omega_{r,1} & \cdots & \Omega_{r,r-1} & \Omega_{r,r} \end{bmatrix}$$

where $\Omega_{i,j}$ *has entries* $\omega_{E,E'}$ *for* $E \in \mathcal{I}_i$ *and* $E' \in \mathcal{I}_j$. *Then there is a unique factorisation*

$$\Omega = P^{\mathrm{tr}} \cdot \Lambda \cdot P, \qquad P = \big(p_{E,E'}\big)_{E,E' \in \mathrm{Irr}(W)}, \qquad \Lambda = \big(\lambda_{E,E'}\big)_{E,E' \in \mathrm{Irr}(W)},$$

such that P *and* Λ *have corresponding block shapes as follows:*

$$P = \begin{bmatrix} u^{b_1} I_{n_1} & P_{1,2} & \cdots & P_{1,r} \\ 0 & u^{b_2} I_{n_2} & & \vdots \\ \vdots & & \ddots & P_{r-1,r} \\ 0 & \cdots & 0 & u^{b_r} I_{n_r} \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \Lambda_r \end{bmatrix};$$

here, $n_i = |\mathcal{I}_i|$ *and* I_{n_i} *denotes the identity matrix of size* n_i. *Furthermore, the block* $P_{i,j}$ *has entries* $p_{E,E'} \in \mathbb{Q}(u)$ *for* $E \in \mathcal{I}_i$ *and* $E' \in \mathcal{I}_j$; *similarly, the block* Λ_i *has entries* $\lambda_{E,E'} \in \mathbb{Q}(u)$ *for* $E, E' \in \mathcal{I}_i$.

Proof. This relies on the following remark due to Lusztig (see [23, Lemma 2.1]):

($*$) All the principal minors of Ω are non-zero.

Now P and Λ are constructed inductively by the following well-known procedure (see for example [58, Chap. 8] and note that Ω is symmetric). We begin with the first block column. We have $u^{2b_1}\Lambda_1 = \Omega_{1,1}$, which determines Λ_1. For $i > 1$ we have $u^{b_1} P_{1,i}^{\mathrm{tr}} \Lambda_1 = \Omega_{i,1}$. By ($*$), we know that $\det \Omega_{1,1} \neq 0$. Hence Λ_1 is invertible, and we can determine $P_{1,i}$. Now consider the j-th block column, where $j > 1$. Assume that the first $j-1$ block columns of P and the first $j-1$ diagonal blocks of Λ have already been determined. We have an equation

$$u^{2b_j}\Lambda_j + P_{j-1,j}^{\mathrm{tr}} \Lambda_{j-1} P_{j-1,j} + \cdots + P_{1,j}^{\mathrm{tr}} \Lambda_1 P_{1,j} = \Omega_{j,j},$$

which can be solved uniquely for Λ_j. In particular, we have now determined all coefficients in P and Λ which belong to the first j blocks. We consider the subsystem of equations made up of these blocks; this subsystem looks like the original system written in matrix form above, with r replaced by j. By ($*$), the right hand side has a non-zero determinant. Hence so have the blocks $\Lambda_1, \ldots, \Lambda_j$. Now we can determine the coefficients of P in the i-th row: for $i > j$, we have

$$u^{b_j} P_{j,i}^{\mathrm{tr}} \Lambda_j + P_{j-1,i}^{\mathrm{tr}} \Lambda_{j-1} P_{j-1,j} + \cdots + P_{1,i}^{\mathrm{tr}} \Lambda_1 P_{1,j} = \Omega_{i,j}.$$

Since Λ_j is invertible, $P_{j,i}$ is determined. Continuing in this way, the above system of equations is solved. □

Example 5.2. Let W be of type B_2, with generators $S = \{s, t\}$. We write $\mathrm{Irr}(W) = \{\mathrm{sgn}, \mathrm{sgn}_2, \mathrm{sgn}_1, \sigma, 1_W\}$ (and use this ordering for the rows and columns of the matrices below). The values of the corresponding characters are obtained by formally setting $q = 1$ in the table in Example 3.9. We have

$$\det(u\,\mathrm{id}_V - 1) = (u-1)^2, \quad \det(u\,\mathrm{id}_V - s) = \det(u\,\mathrm{id}_V - t) = u^2 - 1,$$
$$\det(u\,\mathrm{id}_V - st) = u^2 + 1, \qquad \det(u\,\mathrm{id}_V - stst) = (u+1)^2;$$

furthermore, $D_W = u^4(u^2-1)(u^4-1)$. Using this information, we obtain:

$$\Omega = \begin{bmatrix} u^8 & u^6 & u^6 & u^7{+}u^5 & u^4 \\ u^6 & u^8 & u^4 & u^7{+}u^5 & u^6 \\ u^6 & u^4 & u^8 & u^7{+}u^5 & u^6 \\ u^7{+}u^5 & u^7{+}u^5 & u^7{+}u^5 & u^8{+}2u^6{+}u^4 & u^7{+}u^5 \\ u^4 & u^6 & u^6 & u^7{+}u^5 & u^8 \end{bmatrix}$$

We shall now determine three factorisations of Ω.

(a) Consider the partition $\mathrm{Irr}(W) = \{\mathrm{sgn}\} \sqcup \{\mathrm{sgn}_2\} \sqcup \{\mathrm{sgn}_1, \sigma\} \sqcup \{1_W\}$, together with the sequence of integers 4, 2, 1, 0. We obtain the matrices:

$$P = \begin{bmatrix} u^4 & u^2 & u^2 & u^3{+}u & 1 \\ 0 & u^2 & 0 & u & 1 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & u^4{-}1 & 0 & 0 & 0 \\ 0 & 0 & u^6{-}u^2 & u^5{-}u & 0 \\ 0 & 0 & u^5{-}u & u^6{-}u^2 & 0 \\ 0 & 0 & 0 & 0 & u^8{-}u^6{-}u^4{+}u^2 \end{bmatrix}.$$

(We will see below that this yields the Green functions of $\mathrm{Sp}_4(\mathbb{F}_q)$, q odd.)

(b) Consider the partition $\mathrm{Irr}(W) = \{\mathrm{sgn}\} \sqcup \{\mathrm{sgn}_2\} \sqcup \{\mathrm{sgn}_1\} \sqcup \{\sigma\} \sqcup \{1_W\}$, together with the sequence of integers $4, 2, 2, 1, 0$. We obtain the matrices:

$$P = \begin{bmatrix} u^4 & u^2 & u^2 & u^3+u & 1 \\ 0 & u^2 & 0 & u & 1 \\ 0 & 0 & u^2 & u & 1 \\ 0 & 0 & 0 & u & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & u^4-1 & 0 & 0 & \\ 0 & 0 & u^4-1 & 0 & 0 \\ 0 & 0 & 0 & u^6-u^4-u^2+1 & 0 \\ 0 & 0 & 0 & 0 & u^8-u^6-u^4+u^2 \end{bmatrix}$$

(We will see below that this yields the Green functions of $\mathrm{Sp}_4(\mathbb{F}_q)$, q even.)

(c) As in [23, 2.9], consider the partition $\mathrm{Irr}(W) = \{\mathrm{sgn}\} \sqcup \{\mathrm{sgn}_2, \mathrm{sgn}_1, \sigma\} \sqcup \{1_W\}$, together with the sequence of integers $4, 1, 0$. We obtain the matrices:

$$P = \begin{bmatrix} u^4 & u^2 & u^2 & u^3+u & 1 \\ 0 & u & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & u^6-u^2 & 0 & u^5-u & 0 \\ 0 & 0 & u^6-u^2 & u^5-u & 0 \\ 0 & u^5-u & u^5-u & u^6+u^4-u^2-1 & 0 \\ 0 & 0 & 0 & 0 & u^8-u^6-u^4+u^2 \end{bmatrix}$$

Quite remarkably, in all three cases the solutions are in $\mathbb{Z}[u]$. (One easily finds partitions of $\mathrm{Irr}(W)$ for which this does not hold, for example, $\mathrm{Irr}(W) = \{\mathrm{sgn}, \mathrm{sgn}_2\} \sqcup \{\mathrm{sgn}_1\} \sqcup \{\sigma\} \sqcup \{1_W\}$.)

It is straightforward to implement the algorithm in the proof of Lemma 5.1 in the GAP programming language. In those cases where one expects that polynomial solutions exist, it is most efficient to first specialise u to a large number of integer values, then solve the resulting systems of equations over $\mathbb{Q}$, and finally interpolate to obtain polynomial solutions. (In order to avoid working with large rational numbers, one can further reduce the specialised systems of equations modulo various prime numbers, then solve the resulting systems over finite fields, and finally use "chinese remainder" techniques to recover the solutions over $\mathbb{Q}$; similar methods have been used in the proof of [27, Prop. 11.5.13] where it was necessary to invert certain matrices with polynomial entries.) All this works well for W of rank up to 8, including all exceptional types.

Although this turns the actual chronological development of things upside down, the discussion in the previous section leads us to consider the partition of $\mathrm{Irr}(W)$ into families with respect to the "equal parameter" weight function $L_0 \colon W \to \mathbb{Z}$ such that $L_0(s) = 1$ for all $s \in S$. The following conjecture has been found through extensive experimentation with CHEVIE. It is verified for all W of exceptional type; the answer for W of classical type is open.

Conjecture 5.3 (Geck–Malle [23, §2]). *Consider the partition $\mathrm{Irr}(W) = \mathcal{F}_1 \sqcup \ldots \sqcup \mathcal{F}_r$ where $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are the families with respect to L_0. Let b_i be the constant value of the function $E \mapsto \tilde{\boldsymbol{a}}_E$ on $\mathcal{F}_i$; see Remark 4.7. Assume that $b_1 \geqslant \ldots \geqslant b_r$. Let P and Λ be the matrices obtained by Lemma 5.1. Then the following hold.*

(a) *All the entries of P and Λ are polynomials in $\mathbb{Z}[u]$; furthermore, the polynomials in P have non-negative coefficients.*

(b) *Assume that W is the Weyl group of a connected reductive algebraic group G over $k = \overline{\mathbb{F}}_p$, with a split $\mathbb{F}_q$-rational structure where q is a power of p (as in Remark 3.5). Let $E_i \in \mathcal{S}_W$. Then $\lambda_{E_i,E_i}(q)$ is the number of $\mathbb{F}_q$-rational points in the "special piece" corresponding to E_i; see Remark 4.13.*

We now turn to the discussion of Green functions. Let G be a connected reductive algebraic group over $k = \overline{\mathbb{F}}_p$. Let $B \subseteq G$ be a Borel subgroup and $T \subseteq G$ be a maximal torus contained in B. Let $W = N_G(T)/T$ be the Weyl group of G. Let q be a power of p and $F \colon G \to G$ be a Frobenius map with respect to a split $\mathbb{F}_q$-rational structure on G, as in Remark 3.5. Recall that then B and all unipotent classes of G are F-stable; furthermore, F acts as the identity on W.

Let $w \in W$ and $T_w \subseteq G$ be an F-stable maximal torus obtained from T by twisting with w. Let $\theta \in \mathrm{Irr}(T_w^F)$ and $R_{T_w}^\theta$ be the character of the corresponding virtual representation of G^F defined by Deligne and Lusztig; see Carter [8, §7.2]. Then the restriction of $R_{T_w}^\theta$ to G_{uni}^F only depends on w but not on θ (see [8, 7.2.9]). This restriction is called the *Green function* corresponding to $w \in W$; it will be denoted by Q_w. There is a character formula which reduces the computation of the values of $R_{T_w}^\theta$ to the computation of the values of various Green functions (see [8, 7.2.8]). It is known that the values of Q_w are integers (see [8, §7.6]), but it is a very hard problem to compute these values explicitly.

Let $E \in \mathrm{Irr}(W)$. Following Lusztig [41, §3.7], we define

$$R_E := \frac{1}{|W|} \sum_{w \in W} \mathrm{trace}(w, E)\, R_{T_w}^1 \qquad \text{and} \qquad Q_E := R_E|_{G_{\mathrm{uni}}^F},$$

where the superscript 1 stands for the unit representation of T_w^F. Note that $Q_w = \sum_{E \in \mathrm{Irr}(W)} \mathrm{trace}(w, E)\, Q_E$, so Q_E and Q_w determine each other. Now the entries of the matrix Ω introduced above have the following interpretation:

$$\omega_{E,E'}(q) = \sum_{u \in G_{\mathrm{uni}}^F} Q_E(u) Q_{E'}(u) \qquad (E, E' \in \mathrm{Irr}(W)).$$

(This follows from the orthogonality relations for Green functions; see [8, 7.6.2]. It also uses the formulae for $|T_w^F|$ and $|(N_G(T_w)/T_w)^F|$ in [8, §3.3].)

As in the previous section, let $\mathcal{N}_G$ be the set of all pairs $(\mathcal{O},\mathcal{L})$ where $\mathcal{O}$ is a unipotent class in G and $\mathcal{L}$ is a G-equivariant irreducible $\overline{\mathbb{Q}}_\ell$-local system on $\mathcal{O}$ (up to isomorphism). Recall that the *Springer correspondence* defines an injection

$$\mathrm{Irr}(W) \hookrightarrow \mathcal{N}_G, \qquad E \mapsto \iota_E = (\mathcal{O}_E, \mathcal{L}_E).$$

The Frobenius map F acts naturally on $\mathcal{N}_G$. Given $\iota = (\mathcal{O},\mathcal{L}) \in \mathcal{N}_G^F$, we obtain a class function $Y_\iota \colon G^F \to \mathbb{C}$ as in [43, 24.2.3]. We have $Y_\iota(g) = 0$ unless $g \in \mathcal{O}^F$. Furthermore, the matrix $\big(Y_\iota(g)\big)$ (with rows labelled by all $\iota = (\mathcal{O},\mathcal{L}) \in \mathcal{N}_G^F$ where $\mathcal{O}$ is fixed, and columns labelled by a set of representatives of the G^F-classes contained in $\mathcal{O}^F$) is, up to multiplication of the rows by roots of unity, the "F-twisted" character table of the finite group $A(u) = C_G(u)/C_G(u)^\circ$ ($u \in \mathcal{O}$); see [43, 24.2.4, 24.2.5]. In particular, the following hold:

Remark 5.4. (a) The functions $\{Y_\iota \mid \iota \in \mathcal{N}_G^F\}$ form a basis of the space of class functions on G^F_{uni}.

(b) Let $\iota = (\mathcal{O},\mathcal{L}) \in \mathcal{N}_G^F$ where $\mathcal{L} \cong \overline{\mathbb{Q}}_\ell$. Then there is a root of unity η such that $Y_\iota(g) = \eta$ for all $g \in \mathcal{O}^F$. (It will turn out that $\eta = \pm 1$; see Remark 5.6 below.)

Let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ be the unipotent classes of G, where the labelling is chosen such that $\dim \mathcal{O}_1 \leqslant \dim \mathcal{O}_2 \leqslant \ldots \leqslant \dim \mathcal{O}_r$. For $i \in \{1,\ldots,r\}$, we set

$$\mathcal{I}_i^* := \{E \in \mathrm{Irr}(W) \mid \mathcal{O}_E = \mathcal{O}_i\}, \text{ and}$$
$$b_i^* := \frac{1}{2}(\dim G - \dim T - \dim \mathcal{O}_i).$$

(It is known that $\dim G - \dim T - \dim \mathcal{O}_i$ always is an even number; see [8, 5.10.2].) Recall that all pairs $(\mathcal{O}_i, \overline{\mathbb{Q}}_\ell) \in \mathcal{N}_G$ belong to the image of the Springer correspondence. Thus, we obtain a partition $\mathrm{Irr}(W) = \mathcal{I}_1^* \sqcup \ldots \sqcup \mathcal{I}_r^*$, and a decreasing sequence of integers $b_1^* \geqslant \ldots \geqslant b_r^*$. Hence, Lemma 5.1 yields a factorisation

$$\Omega = (P^*)^{\mathrm{tr}} \cdot \Lambda^* \cdot P^*.$$

Recall that the entries of P^*, Λ^* are in $\mathbb{Q}(u)$; we denote these entries by $p^*_{E,E'}$ and $\lambda^*_{E,E'}$. With this notation, we can now state the following fundamental result.

Theorem 5.5 (Springer [65]; Shoji [60], [61, §5]; Lusztig [43, §24], [44]; see also [14, §3]). *In the above setting, the entries of P^* and Λ^* are polynomials in u. We have*

$$Q_E = \sum_{E' \in \mathrm{Irr}(W)} p^*_{E',E}(q)\, Y_{\iota_{E'}} \quad \textit{and}$$
$$\lambda^*_{E,E'}(q) = \sum_{u \in G^F_{\mathrm{uni}}} Y_{\iota_E}(u)\, Y_{\iota_{E'}}(u)$$

for all $E, E' \in \mathrm{Irr}(W)$. Furthermore, the polynomial $p^*_{E',E}$ is 0 unless $\mathcal{O}_{E'} \subseteq \overline{\mathcal{O}}_E$.

Remark 5.6. The above result shows that, for any $E \in \mathrm{Irr}(W)$, we have

$$Q_E = q^{d_E}\, Y_{\iota_E} + \sum_{\substack{E' \in \mathrm{Irr}(W) \\ \mathcal{O}_{E'} \subsetneqq \overline{\mathcal{O}}_E}} p^*_{E',E}(q)\, Y_{\iota_{E'}}.$$

These equations can be inverted and, hence, every function Y_{ι_E} can be expressed as a $\mathbb{Q}$-linear combination of the Green functions Q_w ($w \in W$). Since the values of the Green functions are integers (see [8, §7.6]), we deduce that the values of Y_{ι_E} are rational numbers. Since they are also algebraic integers, they must be integers. In particular, the root of unity η in Remark 5.4 must be ± 1.

Remark 5.7. (a) Note that, in order to run the algorithm in Lemma 5.1, we only need to know the map $E \mapsto \mathcal{O}_E$ and the dimensions $\dim \mathcal{O}_E$. The finer information on the local systems $\mathcal{L}_E$ only comes in at a later stage.

(b) As formulated above, Theorem 5.5 does not say anything about the tricky question of determining the values of the functions Y_{ι_E}. This relies on the careful choice of a representative in $\mathcal{O}^F$, where the situation is optimal when a so-called "split" element can be found; see the discussion by Beynon–Spaltenstein [4, §3]. Such split elements exist for G of classical type in good characteristic; see Shoji [59]. On the other hand, in type E_8 where $q \equiv -1 \bmod 3$, there is one unipotent class which does not contain any split element; see [4, Case V, p. 591].—For our purposes here, the information in Remark 5.4(b) will be sufficient.

Example 5.8. Let us re-interprete the computations in Example 5.2 in the light of Theorem 5.5. By Carter [8, p. 424] and Lusztig–Spaltenstein [53, 6.1], the Springer correspondence for $G = \mathrm{Sp}_4(k)$ is given by the following tables:

char(k) $\neq 2$			b_E^*
sgn	$\mapsto$	$\mathcal{O}_{(1111)}$	4
sgn_2	$\mapsto$	$\mathcal{O}_{(211)}$	2
sgn_1	$\mapsto$	$\mathcal{O}_{(22)}$	1 ($\mathcal{L}_E \ncong \overline{\mathbb{Q}}_\ell$)
σ	$\mapsto$	$\mathcal{O}_{(22)}$	1
1_W	$\mapsto$	$\mathcal{O}_{(4)}$	0

char(k) $= 2$			b_E^*
sgn	$\mapsto$	$\mathcal{O}_{(1111)}$	4
sgn_2	$\mapsto$	$\mathcal{O}_{(211)}$	2
sgn_1	$\mapsto$	$\mathcal{O}^*_{(22)}$	2
σ	$\mapsto$	$\mathcal{O}_{(22)}$	1
1_W	$\mapsto$	$\mathcal{O}_{(4)}$	0

Here, $b_E^* = (\dim G - \dim T - \dim \mathcal{O}_E)/2$; furthermore, $\mathcal{L}_E \cong \overline{\mathbb{Q}}_\ell$ unless explicitly stated otherwise. Hence, these data give rise to the first two cases in Example 5.2.

We shall now explain, following Lusztig [51, 1.2], how the cardinalities of the sets $(\mathcal{O} \cap B\dot{w}B)^F$ (see Section 3) can be effectively computed. For this purpose, it will be convenient to introduce the following notation.

Definition 5.9. For any $E \in \mathrm{Irr}(W)$ and $w \in W$, we set

$$\beta_E^w := |G^F/B^F| \sum_{1 \leqslant i \leqslant d} |O_{u_i} \cap B^F \dot{w} B^F| \, Y_{\iota_E}(u_i),$$

where $u_1, \ldots, u_d$ are representatives of the G^F-conjugacy classes contained in $\mathcal{O}_E^F$. Note that, by Remark 5.4(b), we have

$$\beta_E^w = \pm |G^F/B^F| |(\mathcal{O}_E \cap B\dot{w}B)^F| \qquad \text{if } \mathcal{L}_E \cong \overline{\mathbb{Q}}_\ell.$$

We will now rewrite the expression for β_E^w using various results from the representation theory of G^F. First, by Remark 3.6(b), we obtain

$$\begin{aligned} \beta_E^w &= \sum_{1 \leqslant i \leqslant d} \sum_{V \in \mathrm{Irr}(\mathcal{H}_q)} |O_{u_i}| \, \mathrm{trace}(u_i, \rho_V) \, \mathrm{trace}(T_w, V) \, Y_{\iota_E}(u_i) \\ &= \sum_{V \in \mathrm{Irr}(\mathcal{H}_q)} \mathrm{trace}(T_w, V) \sum_{u \in G^F_{\mathrm{uni}}} \chi_V(u) \, Y_{\iota_E}(u), \end{aligned}$$

where χ_V denotes the character of ρ_V. Now, by definition, χ_V is a constituent of the character of the permutation module $\mathbb{C}[G^F/B^F]$, and the latter is known to be equal to R_T^1; see [8, 7.2.4]. But then the multiplicity of χ_V in any $R_{T_w}^\theta$ is 0 unless $\theta = 1$; see [8, 7.3.8]. Consequently, we can write

$$\chi_V = \Big(\sum_{E' \in \mathrm{Irr}(W)} \langle R_{E'}, \chi_V \rangle \, R_{E'} \Big) + \psi_V,$$

where $\langle R_{E'}, \chi_V \rangle$ denotes the multiplicity of χ_V in the decomposition of $R_{E'}$ as a linear combination of irreducible characters; furthermore, ψ_V is a class function which is orthogonal to all $R_{T_w}^\theta$. We now use Theorem 5.5 to evaluate χ_V on unipotent elements. Let $u \in G^F$ be unipotent. Then

$$\chi_V(u) = \psi_V(u) + \sum_{E', E'' \in \mathrm{Irr}(W)} \langle R_{E'}, \chi_V \rangle \, p^*_{E''.E'}(q) \, Y_{\iota_{E''}}(u).$$

Consequently, we obtain

$$\begin{aligned} \sum_{u \in G^F_{\mathrm{uni}}} \chi_V(u) Y_{\iota_E}(u) &= \sum_{u \in G^F_{\mathrm{uni}}} \psi_V(u) Y_{\iota_E}(u) \\ &\quad + \sum_{E', E'' \in \mathrm{Irr}(W)} \langle R_{E'}, \chi_V \rangle \, p^*_{E'',E'}(q) \, \lambda^*_{E'',E}(q). \end{aligned}$$

Finally, by Remark 5.6, we can write Y_{ι_E} as a linear combination of Green functions. Since ψ_V is orthogonal to all $R_{T_w}^\theta$, it follows that

$$\sum_{u \in G^F_{\mathrm{uni}}} \psi_V(u) Y_{\iota_E}(u) = 0.$$

Thus, we have shown the following formula which is a slight variation of the one obtained by Lusztig [51, 1.2(c)]; this is the key to the explicit computation of β_E^w.

Lemma 5.10. *For any $E \in \mathrm{Irr}(W)$ and $w \in W$, we have:*

$$\beta_E^w = \sum_{V \in \mathrm{Irr}(\mathcal{H}_q)} \sum_{E', E'' \in \mathrm{Irr}(W)} \mathrm{trace}(T_w, V) \, \langle R_{E'}, \chi_V \rangle \, p^*_{E'',E'}(q) \, \lambda^*_{E'',E}(q).$$

In the above formula, the terms "trace(T_w, V)" can also be seen to be specialisations of some well-defined polynomials. For this purpose, we introduce the generic Iwahori–Hecke algebra $\mathcal{H}$ associated with W. This algebra is defined over the ring of Laurent polynomials $A = \mathbb{Z}[u^{1/2}, u^{-1/2}]$; it has an A-basis $\{T_w \mid w \in W\}$ and the multiplication is given as follows, where $s \in S$ and $w \in W$:

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ uT_{sw} + (u-1)T_w & \text{if } l(sw) < l(w); \end{cases}$$

see [27, §4.4]. Thus, we have $\mathcal{H}_q \cong \mathbb{C} \otimes_A \mathcal{H}$ where $\mathbb{C}$ is considered as an A-module via the specialisation $A \to \mathbb{C}$, $u^{1/2} \mapsto q^{1/2}$; here, $q^{1/2}$ is a fixed square root of q in $\mathbb{C}$. Let K be the field of fractions of A and $\mathcal{H}_K$ be the K-algebra obtained by extending scalars from A to K. Then it is known that $\mathcal{H}_K$ is split semisimple and that there is a bijection $\mathrm{Irr}(W) \leftrightarrow \mathrm{Irr}(\mathcal{H}_K)$, $E \leftrightarrow E_u$, such that

$$\mathrm{trace}(w, E) = \mathrm{trace}(T_w, E_u)|_{u^{1/2} \to 1} \qquad \text{for all } w \in W;$$

see [27, 8.1.7, 9.3.5] or [47, Chap. 20]. Now, following [27, 8.2.9], we define the *character table* of $\mathcal{H}$ by

$$X(\mathcal{H}) := \big(\text{trace}(T_{w_C}, E_u)\big)_{E\in\text{Irr}(W),\, C\in\text{Cl}(W)},$$

where $w_C \in C_{\min}$ for each $C \in \text{Cl}(W)$. (By [27, 8.2.6], this does not depend on the choice of the elements w_C; by [27, 9.3.5], the entries of $X(\mathcal{H})$ are in $\mathbb{Z}[u^{1/2}]$.)

Finally, since the algebra $\mathcal{H}_q$ is semisimple, we also have a bijection $\text{Irr}(\mathcal{H}_q) \leftrightarrow \text{Irr}(\mathcal{H}_K)$, $V \leftrightarrow V_u$, such that

$$\text{trace}(T_w, V) = \text{trace}(T_w, V_u)|_{u^{1/2}\to q^{1/2}} \qquad \text{for all } w \in W.$$

Composing this bijection with the previous bijection $E \leftrightarrow E_u$, we obtain a bijection $\text{Irr}(W) \leftrightarrow \text{Irr}(\mathcal{H}_q)$, $E \leftrightarrow E_q$. We now define the matrix

$$\Upsilon_W := \big(\langle R_E, \chi_{E'_q}\rangle\big)_{E,E'\in\text{Irr}(W)}.$$

The entries of this matrix are explicitly described by Lusztig's multiplicity formula [41, Main Theorem 4.23], together with the information in [38, 1.5] (for types E_7, E_8) and [41, 12.6] (in all remaining cases). It turns out that Υ_W is given by certain *non-abelian Fourier transformations* associated to the various families of $\text{Irr}(W)$; in particular, Υ_W only depends on W, but not on p or q.

Now the three matrices Λ^*, P^*, Υ_W have rows and columns labelled by $\text{Irr}(W)$; furthermore, $X(\mathcal{H})$ has rows labelled by $\text{Irr}(W)$ and columns labelled by $\text{Cl}(W)$. Consequently, it makes sense to consider the following product

$$\Xi^* := \Lambda^* \cdot P^* \cdot \Upsilon_W \cdot X(\mathcal{H}),$$

which is a matrix with entries in $\mathbb{Q}[u^{1/2}, u^{-1/2}]$, which has rows labelled by $\text{Irr}(W)$ and columns labelled by $\text{Cl}(W)$. Then Lemma 5.10 can be re-stated as follows.

Corollary 5.11. *Let $E \in \text{Irr}(W)$ and $w \in C_{\min}$ for some $C \in \text{Cl}(W)$. Then we have:*

$$\boxed{\beta_E^w = (E, C)\text{-entry of the matrix } \Xi^*|_{u^{1/2}\to q^{1/2}}}$$

In particular, the numbers β_E^w are given by "polynomials in q".

Remark 5.12. The advantage of working with β_E^w is that then we obtain a true expression in terms of polynomials in q, as above. (In the original setting of [51, 1.2], one has to distinguish congruence classes of q modulo 3 in type E_8.)

Following Lusztig [51, 1.2], we are now in a position to write a computer program for computing β_E^w and, hence, the cardinalities $|(\mathcal{O} \cap B\dot{w}B)^F|$. Note that:

- The explicit knowledge of the Springer correspondence (see Remark 4.10) can be turned into a GAP/CHEVIE program which, given any G, determines the partition $\text{Irr}(W) = \mathcal{I}_1^* \sqcup \ldots \sqcup \mathcal{I}_r^*$ and the numbers $b_1^* \geqslant \ldots \geqslant b_r^*$ required for running the algorithm in Lemma 5.1. (Lübeck [34] provides an electronic library of tables of Green functions.)
- The character tables of $\mathcal{H}$ are known for all types of W; see Chapters 10 and 11 of [27]. For any given W, they are explicitly available in GAP through an already existing CHEVIE function.
- The Fourier matrices Υ_W are explicitly known by [38], [41]. They are available in GAP through Michel's [55] development version of CHEVIE.

It then remains to combine all these various pieces (data and algorithms) into a GAP program for determining β_E^w. In this way, the verification of Theorem 3.1 for a given G is reduced to a purely mechanical computation.

Remark 5.13. Using the methods described above, Lusztig [51, 1.2] has verified Theorem 3.1 for G of exceptional type; as remarked in [52, 4.8], this works both in good and in bad characteristic. The computations also yield the following property of the entries of the matrix Ξ^*. Let $C \in \text{Cl}(W)$ be cuspidal; let $\mathcal{O}$ be a unipotent class in G and $E \in \text{Irr}(W)$ be such that $\iota_E = (\mathcal{O}, \overline{\mathbb{Q}}_\ell)$. Then we have:

(a) The (E, C)-entry $\Xi^*_{E,C}$ is divisible by D_W; recall that we defined $D_W = u^{l(w_0)}(u-1)^{|S|}\sum_{w\in W} u^{l(w)}$ (hence, we have $D_w(q) = G_{\text{ad}}^F$).

(b) If $\mathcal{O} = \mathcal{O}_C$, then the constant term of the polynomial $\Xi^*_{E,C}/D_W$ is 1; otherwise, the constant term is 0.

In fact, the further results in [51], [52] provide a general proof of (a), (b), assuming that $\mathcal{O} = \mathcal{O}_C$. See [52, 4.4(a)] for an explicit formula for $\Xi^*_{E,C}$ in this case.

We illustrate all this with our usual example $G = \text{Sp}_4(\overline{\mathbb{F}}_p)$. Recall that we write $\text{Irr}(W) = \{\text{sgn}, \text{sgn}_2, \text{sgn}_1, \sigma, 1_W\}$. Then Υ_W is given by

$$\Upsilon = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(where the rows and columns are labelled by $\text{Irr}(W)$ as specified above). All the remaining pieces of information are already contained in the examples considered earlier; see Example 3.9 for the character table $X(\mathcal{H})$. The results

TABLE 4. The numbers $|G^F/B^F|^{-1}\beta_E^w$ for $G = \mathrm{Sp}_4(\overline{\mathbb{F}}_p)$

$p \neq 2$	$\mathcal{O}_{(1111)}$	$\mathcal{O}_{(211)}$	$\mathcal{O}_{(22)}, \mathcal{L} \not\cong \overline{\mathbb{Q}}_\ell$	$\mathcal{O}_{(22)}$	$\mathcal{O}_{(4)}$
1	1	q^2-1	q^3-q	$2q^3-2q^2$	$q^4-2q^3+q^2$
t	0	q^3-q^2	0	$q^4-2q^3+q^2$	$q^5-2q^4+q^3$
$stst$	0	0	0	$q^6-2q^5+q^4$	$q^8-2q^7+q^6$
s	0	0	q^4-q^3	q^4-q^3	$q^5-2q^4+q^3$
st	0	0	0	0	$q^6-2q^5+q^4$

$p = 2$	$\mathcal{O}_{(1111)}$	$\mathcal{O}_{(211)}$	$\mathcal{O}^*_{(22)}$	$\mathcal{O}_{(22)}$	$\mathcal{O}_{(4)}$
1	1	q^2-1	q^2-1	$2q^3-3q^2+1$	$q^4-2q^3+q^2$
t	0	q^3-q^2	0	$q^4-2q^3+q^2$	$q^5-2q^4+q^3$
$stst$	0	0	0	$q^6-2q^5+q^4$	$q^8-2q^7+q^6$
s	0	0	q^3-q^2	$q^4-2q^3+q^2$	$q^5-2q^4+q^3$
st	0	0	0	0	$q^6-2q^5+q^4$

are contained in Table 4. First of all note that this is, of course, consistent with the computations in Example 3.9. Furthermore, the entries in the rows corresponding to the two cuspidal classes (with representatives st and $stst$) are divisible by $|B^F|$, which implies that the properties (a) and (b) in Remark 5.13 hold.

Remark 5.14. Finally, we wish to state a conjecture concerning a general finite Coxeter group W. We place ourselves in the setting of Conjecture 5.3 where P, Λ are computed with respect to the partition of $\mathrm{Irr}(W)$ into Lusztig's families, using the equal parameter weight function L_0. We form again the matrix

$$\Xi := \Lambda \cdot P \cdot \Upsilon_W \cdot X(\mathcal{H}) \qquad \text{(with entries in } \mathbb{Q}(u^{1/2})\text{)}.$$

Analogues of the Fourier matrix Υ_W for W of type $I_2(m)$, H_3 and H_4 have been constructed by Lusztig [45, §3], Broué–Malle [6, 7.3] and Malle [54], respectively.

Now let $C \in \mathrm{Cl}(W)$. Then we conjecture that there is a unique family of $\mathrm{Irr}(W)$, denoted by $\mathcal{F}_C$, with the following properties:

(a) *For some $w \in C_{\min}$ and some $E \in \mathcal{F}$, the (E, C)-entry of Ξ is non-zero.*

(b) *For any $w' \in C_{\min}$ and any $E' \in \mathrm{Irr}(W)$, we have that the (E', C)-entry is zero unless $\mathcal{F}_C \preceq_{L_0} \mathcal{F}'$, where $\mathcal{F}'$ is the family containing E'.*

(Here, $\preceq_{L_0}$ is the partial order as in Definition 4.5.) Furthermore, we expect that the assignment $C \mapsto \mathcal{F}_C$ defines a surjective map from $\mathrm{Cl}(W)$ to the set of families of $\mathrm{Irr}(W)$. In particular, we would obtain a natural partition of $\mathrm{Cl}(W)$ into pieces which are indexed by the families of $\mathrm{Irr}(W)$; a similar idea has been formulated by Lusztig [50, 1.4] (for W of crystallographic type).

For example, if $W \cong \mathfrak{S}_n$ (type A_{n-1}), then the above conjecture is equivalent to Theorem 3.1. In this case, all families are singleton sets; furthermore, both $\mathrm{Cl}(W)$ and $\mathrm{Irr}(W)$ are naturally parametrised by the partitions of n. The map $C \mapsto \mathcal{F}_C$ is given by sending the conjugacy class of W consisting of elements of cycle type $\lambda \vdash n$ to the family consisting of the irreducible representation labelled by λ.

Using the computational methods described above, one can check that the conjecture holds for all W of exceptional type. The resulting maps $C \mapsto \mathcal{F}_C$ for types H_3, H_4 are described in Table 5, where we use the following conventions. In the first column, a family $\mathcal{F}$ is specified by the unique "special" representation in $\mathcal{F}$; see [27, App. C]. A non-cuspidal class $C \in \mathrm{Cl}(W)$ is specified as (w) where $w \in C_{\min}$. Representatives for cuspidal classes have already been described in Table 1; so, here, $\#n$ refers to the class with number n in that table.

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TABLE 5. The map from $\mathrm{Cl}(W)$ to families in types H_3 and H_4

H_3	C such that $\mathcal{F}_C = \mathcal{F}$
$1_r'$	e
3_s	$(1), (1212), w_0$
$5_r'$	(13)
$4_r'$	$(23), \#9$
5_r	(12)
$3_s'$	$\#8$
1_r	$\#6$

H_4	C such that $\mathcal{F}_C = \mathcal{F}$
$1_r'$	e
$4_t'$	$(1), (1212), (121213212132123), w_0$
$9_s'$	$(13), (12124), \#33$
$16_r'$	$(23), (121232123), \#30, \#31, \#32$
$25_r'$	(134)
$36_{rr}'$	(12)
$\overline{24}_s$	$(124), (243), (12123), \#19, \#21, \#22,$ $\#23, \#24, \#25, \#26, \#27, \#28, \#29$
36_{rr}	(123)
25_r	$\#18$
16_{rr}	$\#17$
9_s	$\#15$
4_t	$\#14$
1_r	$\#11$

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