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Sharp inequalities for the harmonic numbers

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ABSTRACT.

Let H_n be the nth harmonic number, and let γ be the Euler-Mascheroni constant. We prove that for all integers $n \geq 1$, the double-inequality

$$\alpha - \ln(e^{2\sqrt{2}/(4n+2+\sqrt{2})} - 1) < H_n \leq \beta - \ln(e^{2\sqrt{2}/(4n+2+\sqrt{2})} - 1)$$

holds with the best possible constants

$$\alpha = \gamma - \frac{1}{2}\ln 2 = 0.2306\ldots \quad \text{and} \quad \beta = 1 + \ln(e^{2\sqrt{2}/(6+\sqrt{2})} - 1) = 0.2331\ldots .$$

We also establish inequality for the Euler-Mascheroni constant

$$\frac{\sqrt{2}}{96n^3} - \frac{13+18\sqrt{2}}{1152n^4} < \mu_n - \gamma < \frac{\sqrt{2}}{96n^3},$$

where

$$\mu_n = H_n + \ln(e^{2\sqrt{2}/(4n+2+\sqrt{2})} - 1) + \frac{1}{2}\ln 2 .$$

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