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A remark on a theorem of Browder

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ABSTRACT.

This work deals with a Browder type theorem, and some of its consequences.We consider $\langle X, Y\rangle$ a dual pair of real normed spaces, C a weakly closed convex subset of X containing 0_X, and L a function from C into Y which is monotone, weakly continuous on the line segments in C, and coercive.
In the article „*Nonlinear monotone operators and convex sets in Banach spaces*", Bull. Amer. Math. Soc., **71** (1965), F. E. Browder proved the existence of solutions for variational inequalities with such an operator L provided that $X = E$ is a reflexive Banach space, and $Y = E'$ is its dual space.
It is the object of this note to remark that a similar result is valid when $Y = E$ is a Banach space (not necessary reflexive) and $X = E'$ (for example in the case of the Lebesgue spaces $E = L^1(T)$, and $E' = L^\infty(T)$). Moreover we shall show that the Browder's theorem is a consequence of this result, and we shall also prove a Stampacchia type theorem.

1. PRELIMINARIES

In this part of the work we present the general setting of our problem, and we sketch the principal steps of the Minty-Browder technique (in conformity with [1], and [3]).

Remark 1.1. (i). We denote by $\langle X, Y\rangle$ a real dual system, and by $\langle\ ,\ \rangle_X : X \times Y \to \mathbb{R}$ the corresponding pairing.

(ii). Unless otherwise explicitly stated we shall consider on X (respectively Y) the weak topology denoted by $\sigma(X, Y)$ (respectively $\sigma(Y, X)$).

(iii). We assume that Γ is a closed, and convex subset of Y containing 0_Y.

(iv). Let Σ be a class of functions from Γ into X which is closed under translations of X (i.e. for all $x \in X$, and $T \in \Sigma$ the mapping $(\gamma \mapsto T\gamma + x) : \Gamma \to X$ is also contained in Σ).

Definition 1.1. For T a map from Γ into X, and x_0 a point of X we consider the following systems of inequations:

$$\{\langle T\phi - x_0, \phi - \gamma\rangle_X \le 0\}_{\gamma\in\Gamma} \tag{1.1}$$

$$\{\langle T\phi, \phi - \gamma\rangle_X \le 0\}_{\gamma\in\Gamma} \tag{1.2}$$

$$\{\langle T\gamma - x_0, \gamma - \phi\rangle_X \ge 0\}_{\gamma\in\Gamma} \tag{1.3}$$

$$\{\langle T\gamma, \gamma - \phi\rangle_X \ge 0\}_{\gamma\in\Gamma} \tag{1.4}$$

where it is obvious that ϕ is the unknown term.

(i). By $Sol(T; x_0)$ we denote the set of the solutions of the system (1.1), i.e.

$$Sol(T; x_0) := \{\phi \in \Gamma : \langle T\phi - x_0, \phi - \gamma\rangle \le 0, \text{ for all } \gamma \in \Gamma\}.$$

In particularly $Sol(T; 0_X) =: Sol(T)$.

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(ii). Similarly $\widetilde{Sol}(T;x_0)$ denotes the set of the solutions in Γ of the system (1.3), i.e.

$$\widetilde{Sol}(T;x_0) := \{\phi \in \Gamma : \langle T\gamma - x_0, \gamma - \phi\rangle_X \geq 0, \text{ for all } \gamma \in \Gamma\},$$

and $\widetilde{Sol}(T) := \widetilde{Sol}(T;0_X)$.

Lemma 1.1. *The following assertions are equivalent.*

(i). For each $x_0 \in X$, and $T \in \Sigma$ the set $Sol(T;x_0)$ (respectively $\widetilde{Sol}(T;x_0)$) is non-empty.

(ii). For each $T \in \Sigma$ the set $Sol(T)$ (respectively $\widetilde{Sol}(T)$) is non-empty.

Definition 1.2. (i). If M is a non-empty subset of $X \times Y$ such that

$$\langle x_1 - x_2, y_1 - y_2\rangle_X \geq 0$$

for all (x_1, y_1), and (x_2, y_2) of M, then M is called *monotone*. Moreover if M is maximal with respect to the inclusion relation, then we say that M is *maximal monotone*.

(ii). Let T be a function from Γ into X such that the set $G_T := \{(T\gamma, \gamma) : \gamma \in \Gamma\}$ is monotone. Then T is called a *monotone operator*.

(iii). If $T : \Gamma \to X$ is such that

$$(t \mapsto \langle T((1-t)\gamma_1 + t\gamma_2), y\rangle_X) : [0,1] \to \mathbb{R}$$

is continuous for all $\gamma_1, \gamma_2 \in \Gamma$ and $y \in Y$, then we shall say that T is *(weakly) continuous on the line segments* in Γ.

Remark 1.2. (i). The following classes of functions from Γ into X are closed under translations of X:

$$\{T : T \text{ is monotone }\}, \text{and } \{T : T \text{ is weakly continuous on the line segments}\}.$$

(ii). We also remark that if T is monotone (respectively weakly continuous on the line segments) and $\gamma_0 \in \Gamma$, then

$$T_{\gamma_0} : \Gamma - \gamma_0 \to X, T_{\gamma_0}(\gamma) = T(\gamma + \gamma_0), \ \forall \gamma \in \Gamma - \gamma_0$$

is monotone (respectively weakly continuous on the line segments).

Proposition 1.1. *(i). For a monotone operator T from Γ into X, and a point x_0 in X, it holds $Sol(T;x_0) \subset \widetilde{Sol}(T;x_0)$.*

(ii). If T is monotone and weakly continuous on the line segments in Γ then

$$Sol(T;x_0) = \widetilde{Sol}(T;x_0)$$

for every $x_0 \in X$.

Proposition 1.2. ([1]). *Let T be monotone and weakly continuous on the line segments from Γ into X, and*

$$S_T := \{(x, \gamma) \in X \times \Gamma : \gamma \in Sol(T;x)\}.$$

If 0_Y is an interior point of Γ, then S_T is a maximal monotone subset of $X \times Y$.

Remark 1.3. We consider T as in the previous proposition, and we assume that γ_0 is an interior point of Γ. In view of Remark 1.2.(ii) it follows that S_T is a maximal monotone subset of $X \times Y$.

Theorem 1.1. (Minty, [3]). *For all real Hilbert space X let us consider the dual system $\langle X, X\rangle$ which is defined by the scalar product, and G a maximal monotone subset of $X \times X$. Then for every $n \in \mathbb{N}^\star$ there exists $(h_n, \chi_n) \in G$ such that $h_n + n\chi_n = 0_X$.*

Definition 1.3. Suppose that $(X, \|\cdot\|)$ is a real Banach space, $Y = X'$ is the topological dual of X, $\langle\cdot,\cdot\rangle_X : X \times X' \to \mathbb{R}$ is the canonical pairing, and $\gamma_0 \in \Gamma$. The function T from Γ into X is called *coercive* with respect to γ_0 iff

$$\liminf_{\gamma\in\Gamma,\ \|\gamma\|\to\infty} \frac{\langle T\gamma, \gamma-\gamma_0\rangle_X}{\|\gamma\|} = \infty.$$

If $\gamma_0 = 0_Y$, then T is called *coercive*.

Proposition 1.3. *For T a monotone operator from Γ into X, and $\gamma_0 \in \Gamma$ the following assertions are equivalent.*

(i). The function T is coercive with respect to γ_0.

(ii). There exists a lower bounded function $\varphi : (0,\infty) \to (-\infty,\infty)$ such that

$$\lim_{r\to\infty} \varphi(r) = \infty, \text{ and } \varphi(\|\gamma-\gamma_0\|)\cdot\|\gamma-\gamma_0\| \le \langle T\gamma, \gamma-\gamma_0\rangle_X\,,\ \forall\gamma\in\Gamma.$$

Corollary 1.1. *If T is coercive with respect to $\gamma_0 \in \Gamma$, and monotone, then, for all $x_0 \in X$, $Sol(T;x_0)$ is a bounded subset of X' (hence by the Banach-Alaoglu theorem* ([4]) *it is a weakly relatively compact subset of X').*

Remark 1.4. It is obvious that the class of the monotone, weakly continuous on the line segments, and coercive functions from Γ into X is closed under the translations of X.

2. A Browder type theorem and some consequences

From now on $(X, \|\ \|)$ is a real Banach space, X' is topological dual of X, and $\langle X, X'\rangle$ is the corresponding dual system with respect to the canonical pairing $\langle x, x'\rangle = x'(x)$ for all $x \in X$ and $x' \in X'$.

Suppose that Γ is a convex weakly closed subset of X' containing $0_{X'}$, and T is a function from Γ into X which is monotone, coercive, and weakly continuous on the line segments in Γ and $T0_{X'} = 0_X$.

Proposition 2.4. ([1]). *Suppose that X is a finite dimensional space and X' is spanned by Γ. Then for all $x_0 \in X$ the set $Sol(T;x_0)$ is non-empty.*

Remark 2.5. We shall denote by $Sub_f X'$ the class of all finite dimensional subspaces of X', and for every $H \in Sub_f X'$ let γ_H be a solution in $\Gamma\cap H$ of the system (1.2) for $(H, H', \Gamma\cap H, T|_{\Gamma\cap H})$. Hence

$$\langle T\gamma_H, \gamma_H - \gamma\rangle_X \le 0,\ \forall\gamma\in\Gamma\cap H.$$

Obviously since $\langle T0_{X'}, 0_{X'}\rangle_X = 0$, if $H\cap\Gamma = \{0_{X'}\}$, then $\gamma_H = 0_{X'}$.

Theorem 2.2. (a Browder type theorem). *For all $x_0 \in X$ the set $Sol(T;x_0)$ is non-empty.*

Proof. (similar to ([1]). Let $\mathcal{F} := \{\gamma_H : H \in Sub_f X'\}$ (where γ_H is as in the previous remark). Since $\mathcal{F}$ is contained in the set $\{\gamma\in\Gamma : \langle T\gamma,\gamma\rangle_X \le 0\}$, by Corollary 1.1, $\mathcal{F}$ is weakly relatively compact subset of X', and the weak closure of $\mathcal{F}$ (denoted $\overline{\mathcal{F}}^{\sigma}$) is contained in Γ.

If for all $H \in Sub_f X'$,

$$\mathcal{A}_H := \{\gamma_E : E \in Sub_f X' \text{ and } E \subset H\}\,,$$

then $\mathcal{A}_H$ is a non-empty weakly relatively compact subset of $\mathcal{F}$, and for all $H_1, H_2 \in Sub_f X'$ such that $H_1 \subset H_2$ we have $\mathcal{A}_{H_1} \subset \mathcal{A}_{H_2}$. Moreover since

$$\mathcal{A}_{H_1\cap H_2} \subset \mathcal{A}_{H_1} \cap \mathcal{A}_{H_2},\ \ \forall\, H_1, H_2 \in Sub_f X',$$

the family $\left\{\overline{\mathcal{A}}_H^\sigma : H \in Sub_f X'\right\}$ has the finite intersection property. Therefore by the weak compactness the sets $\left\{\overline{\mathcal{A}}_H^\sigma\right\}_{H\in Sub_f X'}$ have at least a common point γ_0 (which is in Γ).

So that for all $H \in Sub_f X'$ in view of Proposition 1.1.(ii) it follows that

$$\forall\gamma \in \Gamma \cap H, \langle T\gamma_H, \gamma_H - \gamma\rangle_X \leq 0 \Leftrightarrow \forall\gamma \in \Gamma \cap H,\ \langle T\gamma, \gamma - \gamma_H\rangle_X \geq 0.$$

Moreover for all $H \in Sub_f X'$ and $\gamma \in \Gamma \cap H$

$$\langle T\gamma, \gamma - \gamma_0\rangle_X \geq 0 \Rightarrow \forall\gamma \in \Gamma, \langle T\gamma, \gamma - \gamma_0\rangle_X \geq 0.$$

Therefore $Sol(T) = \widetilde{Sol}(T) \neq \emptyset$. (Proposition 1.1.(ii)), and $Sol(T; x_0) \neq \emptyset$ (Lemma 1.1.(ii)). □

Corollary 2.2. (Browder's theorem). *Assume that X is a real reflexive Banach space, C is a convex closed subset of X containing 0_X, and T is a function from C into X' such that it is monotone, coercive and weakly continuous on the line segments of C. Then*

$$\forall x_0' \in X',\ \exists c_0 \in C \text{ such that } \langle c_0 - c, Tc_0 - x_0'\rangle_X \leq 0,\ \forall c \in C.$$

Proof. If J denotes the canonical linear isometry from X onto X'',

$$\Gamma := J(C), \text{ and } T_J := T \circ J^{-1}$$

then we apply Theorem 2.2 to the system (X', X'', Γ, T_J). So that

$$\forall x_0' \in X', \exists\gamma_0 \in \Gamma \text{ such that } \langle T_J\gamma_0 - x_0', \gamma_0 - \gamma\rangle_{X'} \leq 0,\ \forall\gamma \in \Gamma.$$

Since there exists $c_0 \in C$, and, for all $\gamma \in \Gamma$, there exists $c \in C$ such that $Jc_0 = \gamma_0$, and $Jc = \gamma$ it follows that

$$\langle c_0 - c, Tc_0 - x_0'\rangle_X = \langle T_J\gamma_0 - x_0', \gamma_0 - \gamma\rangle_{X'} \leq 0,\ \forall c \in C. \qquad \square$$

Remark 2.6. Let $(X, \|\cdot\|)$ be a real Banach space, Γ a non-empty convex weakly closed subset of X', and $a : \Gamma \times X' \to \mathbb{R}$ which satisfies the following properties (as in [2]).

(S_1). (a). For all $\gamma \in \Gamma$ the function $(x' \mapsto a(\gamma, x')) : X' \to \mathbb{R}$ is a linear weakly continuous functional.

(b). For all $x' \in X'$ the mapping $(\gamma \mapsto a(\gamma, x')) : \Gamma \to \mathbb{R}$ is continuous on the line segments in Γ.

(S_2). There exists $\alpha \in (0, \infty)$ and $p \in (1, \infty)$ such that

$$a(\gamma, \gamma) \geq \alpha\|\gamma\|^p,\ \forall\gamma \in \Gamma.$$

(S_3). For all $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$ we have that

$$a(\gamma_1, \gamma_1 - \gamma_2) - a(\gamma_2, \gamma_1 - \gamma_2) \geq 0.$$

Corollary 2.3. (a Stampacchia type theorem). *Under the hypothesis of the previous remark for all $x_0 \in X$ there exists an element γ_0 from Γ such that*

$$a(\gamma_0, \gamma_0 - \gamma) \leq \langle x_0, \gamma_0 - \gamma\rangle_X,\ \forall\gamma \in \Gamma.$$

Proof. According to the condition (S_1).(a) for all $\gamma \in \Gamma$ there exists an unique element (say $T\gamma$) from X such that

$$a(\gamma, x') = \langle T\gamma, x'\rangle_X,\ \forall x' \in X'.$$

Moreover the mapping $T : \Gamma \to X$ is weakly continuous on the line segments in Γ (in view of (S_1).(b)), is coercive (according to (S_2)), and monotone (by condition (S_3)), hence we can apply Theorem 2.2. □

Remark 2.7. (i). The element γ_0 from the previous corollary is unique provided that the inequality (S_3) is strict.

(ii). Suppose that X is a real Hilbert space, and a is a bilinear form in X.

(a). Under the conditions of Remark 2.6, if we assume that a is separately continuous, then Corollary 2.3 becomes the Stampacchia's theorem ([1], and [5]).

(b). If moreover $\Gamma = X$, and a is continuous, then the Lax-Milgram theorem follows from Corollary 2.3.

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