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The Picard-Lindelöf's theorem at a regular singular point

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ABSTRACT.

We consider initial value problems of the form

$$\begin{cases} (D(x)\mathbf{y})' = \mathbf{f}(x,\mathbf{y}), & x \in [-a,b], \qquad a,b \geq 0, \\ D(0)\mathbf{y}(0) = D(0)\tilde{\mathbf{y}}_{\mathbf{0}}, & \mathbf{y} \in \mathcal{C}[-a,b], \qquad \tilde{\mathbf{y}}_{\mathbf{0}} \in \mathbb{C}^n, \end{cases}$$

where $\mathbf{f} : [-a,b] \times U \to \mathbb{C}^n$ is a continuous function in its variables and $U \subset \mathbb{C}^n$ is an open set. $D(x)$ is an $n \times n$ diagonal matrix whose first $n-m$ diagonal entries are 1 and the last m diagonal entries are x, with $m = 0,1,2,\ldots$ or n. This is an initial value problem where the initial condition is given at a regular singular point of the system of differential equations. The main result of this paper is an existence and uniqueness theorem for the solution of this initial value problem. It is shown that this problem has a unique solution and the Picard-Lindelöf's expansion converges to that solution if the function $\mathbf{F}(\mathbf{y},x) := xD^{-1}(x)\mathbf{f}(x,\mathbf{y})$ is Lipschitz continuous in the variables $\mathbf{y}$ with Lipschitz constant L of the form $L = N + Mx^p$ for a certain $p > 0$, $M > 0$ and $0 \leq N < 1$. When we add the condition $\mathbf{y}^{(s)} \in \mathcal{C}[-a,b]$, $s \in \mathbb{N}$, to the formulation of the problem and the Taylor polynomial of $\mathbf{y}$ at $x = 0$ and degree $s-1$ is available from the differential equation, then the same conclusion is true with a less restrictive condition upon N: $0 \leq N < s+1$. The standard Picard-Lindelöf's theorem is the particular case of the problem studied here obtained for $m = 0$ ($D(x)$ is the identity matrix), $N = 0$, $p = 1$ and M is the Lipschitz constant of $\mathbf{f}(x,\mathbf{y})$.

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