Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem

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ABSTRACT.

In this paper we present some basic problems of the metric fixed point theory (existence, uniqueness, set-theoretic aspects (Bessaga, Janos, Rus, ...), order-theoretic aspects (Ekeland, Bronsted, Caristi, Kirk, Jachymski, ...), convergence of the succesive approximations, data dependence (general estimation, Ulam problem, dependence on the parameters, ...), well-posedness of the fixed point problem, limit shadowing property, stability, Gronwall lemmas, comparison lemmas, retractibility, ...). Following [I. A. Rus, *The theory of a metrical fixed point theorem: theoretical and applicative relevances*, Fixed Point Theory, **9** (2008), No. 2, 541–559] we define the relevance of a metrical fixed point theorem by the impact of the theorem on these basic problems. Some case studies are presented.

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