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Book Review

INTEGRABILITY OF CUBIC SYSTEMS WITH INVARIANT STRAIGHT LINES AND INVARIANT CONICS, BY D. COZMA, EDITURA ŞTIINŢA, CHIŞINĂU, 2013, 240 PAGES, ISBN 978-9975-67-906-0

One of the most important problems in the qualitative theory of differential equations is the local 16th Hilbert problem which deals with the estimation of the number of limit cycles, i.e., isolated periodic solutions, that can bifurcate from a singular point of a center or a focus type, when the coefficients of the system of differential equations are perturbed by an arbitrary small amount, called the *cyclicity problem*. Some aspects of this problem were initiated by Poincaré [1] and Lyapunov [2] and it remains to be one of the most difficult problems to be solved in the list given by Hilbert [3] at the beginning of the past century. This problem was included by Smale in his millennium list of the most important problems for the XXI century [4]. An important step in solving the cyclicity problem is the problem of distinguishing between a center and a focus, called the center-focus problem. Although it dates from the end of the 19th century, it is completely solved only for quadratic systems, cubic symmetric systems and a few particular cases of families of polynomial systems of higher degree. The center-focus problem asks for the criteria which determine whether a singular point whose linear part give a center, really is a center.

This monograph is intended to approach the reader to the center-focus and cyclicity problems. It contains the results obtained by the author in the last twenty years in solving of the center-focus problem. The methods presented are effectively exploited for real polynomial systems of differential equations with invariant algebraic curves. Although the idea of integration of real polynomial systems by using their invariant algebraic curves goes back to Darboux [5], it is still relatively scantily used.

The Cozma's monograph consists of preface, 5 chapters and a list of references containing 140 bibliography items. The material is exposed coherent and the work contains the solutions of some integrability cases of an important mathematical problem, the center-focus problem. The author gives an efficient computational algorithm for computing the Lyapunov quantities, which was applied for the cases studied in the work.

Chapter 1 contains an analyze of the bibliography and a brief summary of the concepts related to the theory of polynomial differential systems with invariant algebraic curves and is devoted to the center-focus and integrability problems. The main problem stated in this chapter is the ILC-problem (I - invariant algebraic curves, L - Lyapunov quantities, C - center), i.e. the relation between algebraic solutions and Lyapunov quantities which implies a singular point to be a center. The author presents two main mechanisms for proving the existence of a center in a polynomial system with invariant algebraic curves, Darboux integrability and rational reversibility. Advances in computational algebra and computer technology made possible the application of these two mechanisms in solving

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of the center-focus problem for cubic differential systems with invariant straight lines and invariant conics.

In Chapter 2 all cubic systems with a singular point $O(0,0)$ of a center or a focus type and having two distinct invariant straight lines are classified in three classes. For every class the sufficient center conditions are obtained by using the method of Darboux integrability and the rational reversibility.

Chapter 3 is devoted to the problem of the center for cubic systems with a singular point $O(0,0)$ of a center or a focus type having four invariant straight lines. It is proved that if a cubic system has four invariant straight lines and the first two Lyapunov quantities vanish, then the origin $O(0,0)$ is a center. It is also demonstrated that the cubic system with four invariant straight lines and a center is always Darboux integrable.

Chapter 4 is dedicated to the problem of the center for cubic systems with a singular point $O(0,0)$ of a center or a focus type having at least three invariant straight lines. It is shown that if a cubic system has three invariant straight lines and the first seven Lyapunov quantities vanish, then the origin is a center. It is established that the cubic system with three invariant straight lines and a center is either Darboux integrable or rationally reversible.

In Chapter 5 conditions for the existence of two invariant straight lines and one invariant conic, in cubic systems with a singular point $O(0,0)$ of a center or a focus type, were found. It is proved that if a cubic system has two invariant straight lines and one invariant conic and the first four Lyapunov quantities vanish, then the origin is a center. It is also demonstrated that the cubic system with a center having two invariant straight lines and one invariant conic is always Darboux integrable.

Hence, in the Cozma's monograph, the local 16th Hilbert problem is solved for polynomial cubic differential systems with some invariant straight lines and invariant conics. This work presents a unified treatment of interesting cases of one important mathematical problem as the center-focus problem. All the results are included with detailed proofs, many of them simplified or rewritten on purpose for the book. The text is self-contained.

The work will be of interest to graduate students in mathematics and physics and researches concerned with the qualitative theory of dynamical systems. At the same time, researchers dealing with the theory of integrability and the theory of limit cycles of polynomial systems will find new results in the cyclicity problem.

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