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A fixed point theorem for a Ćirić-Berinde type mapping in orbitally complete metric spaces

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ABSTRACT. In this paper, we introduce the notion of Ćirić-Berinde type almost set-valued contraction mappings and give a fixed point theorem for these mappings in orbitally complete metric spaces.

1. INTRODUCTION

Banach's contraction principle [5] is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also, it is a powerful tool in the study on finding fixed points of mappings defined on metric spaces. In [27], it enounced in the setting of metric spaces. It is generalized and extended in many directions by the authors [1-4, 19, 20, 22-29].

Ćirić [21] introduced the concept of quasi-contraction mappings, and proved that a quasi-contraction mapping defined on complete metric spaces has a unique fixed point. In the recent years, Berinde [6-15] obtained valuable achievements on fixed point theory. In [7], he introduced the notion of Ćirić almost contraction mappings and obtained a fixed point theorem for these mappings. He proved the following theorem.

Theorem 1.1. [7] *Let (X, d) be a complete metric space. Suppose that a mapping $T : X \to X$ is Ćirić almost contraction, that is, T satisfies the following condition:*

$$d(Tx, Ty) \leq \alpha \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} + Ld(y, Tx),$$

for all $x, y \in X$, where $\alpha \in [0, 1)$ and $L \geq 0$.

Then

(1) *$Fix(T) \neq \emptyset$, where $Fix(T)$ is the set of all fixed points of T;*

(2) *For any $x_0 \in X$, the Picard iteration $\{Tx_n\}$ is convergent to some $x^* \in Fix(T)$;*

(3) *The following estimate holds*

$$d(x_n, x^*) \leq \frac{\alpha^n}{(1-\alpha)^2} d(x, Tx), n = 1, 2, \cdots .$$

The study of fixed point theory for set-valued maps continues to attract the interest of mathematicians. Interest in such theory stems, perhaps, from its usefulness in real world problems, such as in Game Theory; and its applications in other areas of mathematics such as in differential equations with discontinuous right hand sides. For more details on this, the reader may consult any of the following references: Chidume et. al. [16], [17] and [18].

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Let (X, d) be a metric space. We denote by $CB(X)$ the family of nonempty closed and bounded subsets of (X, d). Let $H(\cdot, \cdot)$ be the Hausdorff distance on $CB(X)$, i.e.,

$$H(A, B) = max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}, \text{ for } A, B \in CB(X),$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from the point a to the subset B.

Recently, the author [4] obtained the following result:

Theorem 1.2. [4] *Let (X, d) be a complete metric space. Suppose that a set-valued mapping $T : X \to CB(X)$ satisfies the following condition:*

$$H(Tx, Ty)) \leq kM(x, y) \tag{1.1}$$

for all $x, y \in X$, where $k \in \left[0, \frac{1}{2}\right)$ and $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

Then T has a fixed point in X, that is, there exists an $x^* \in X$ such that $x^* \in Tx^*$.

In this paper, we introduce the concept of Ćirić-Berinde type almost set-valued contraction mappings and establish a new fixed point theorem for these mappings in orbitally complete metric spaces.

Lemma 1.1. *Let (X, d) be a metric space. Suppose that $A, B \in CB(X)$ and $c > 0$. If $H(A, B) < c$ and $a \in A$, then there exists $b \in B$ such that $d(a, b) < c$.*

2. Fixed point theorems

Let (X, d) be a metric space and $T : X \to CB(X)$ be a set-valued mapping. Then,
(1) X is called *T-orbitally complete* if any Cauchy subsequence $\{x_{n(k)}\}$ of

$$\{x_0, x_1 \in Tx_0, x_2 \in Tx_1, \cdots\}, x_0 \in X$$

converges in X.
(2) T is called *Ćirić-Berinde type almost set-valued contraction* if

$$H(Tx, Ty) \leq \frac{\alpha d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta M(x, y) + Ld(y, Tx) \tag{2.1}$$

for all $x, y \in X$, where $\alpha, \beta \geq 0$, $\alpha + 2\beta < 1$, $L \geq 0$.

Theorem 2.3. *Let (X, d) be a metric space, and let $T : X \to CB(X)$ be a given set-valued mapping. Suppose that X is T-orbitally complete. If T is a Ćirić-Berinde type almost set-valued contraction mapping, then T has a fixed point in X.*

Proof. Let $x_0 \in X$, and let $x_1 \in Tx_0$. Let $c > 0$ be such that $d(x_0, x_1) < c$.

From (2.1) we have

$$\begin{aligned}
& H(Tx_0, Tx_1) \\
& \leq \frac{\alpha d(x_1, Tx_1)[1 + d(x_0, Tx_0)]}{1 + d(x_0, x_1)} \\
& + \beta \max\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), d(x_0, Tx_1), d(x_1, Tx_0)\} \\
& + Ld(x_1, Tx_0) \\
& \leq \frac{\alpha d(x_1, Tx_1)[1 + d(x_0, x_1)]}{1 + d(x_0, x_1)} \\
& + \beta \max\{d(x_0, x_1), d(x_0, x_1), d(x_1, Tx_1), d(x_0, x_1) + d(x_1, Tx_1), d(x_1, x_1)\} \\
& + Ld(x_1, x_1) \\
& \leq \alpha d(x_1, Tx_1) \\
& + \beta \max\{d(x_0, x_1), d(x_1, Tx_1), d(x_0, x_1) + d(x_1, Tx_1), 0\} \\
& \leq \alpha H(Tx_0, Tx_1) \\
& + \beta \max\{d(x_0, x_1), H(Tx_0, Tx_1), d(x_0, x_1) + H(Tx_0, Tx_1), 0\} \\
& = \alpha H(Tx_0, Tx_1) + \beta\{d(x_0, x_1) + H(Tx_0, Tx_1)\}.
\end{aligned}$$

Thus we have $H(Tx_0, Tx_1) \leq rd(x_0, x_1) < rc$, where $r = \dfrac{\beta}{1 - \alpha - \beta}$.
By Lemma 1.1, we can choose $x_2 \in Tx_1$ such that

$$d(x_1, x_2) < rc.$$

Again, from (2.1) we have

$$\begin{aligned}
& H(Tx_1, Tx_2) \\
& \leq \frac{\alpha d(x_2, Tx_2)[1 + d(x_1, Tx_1)]}{1 + d(x_1, x_2)} \\
& + \beta \max\{d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), d(x_1, Tx_2), d(x_2, Tx_1)\} \\
& + Ld(x_2, Tx_1) \\
& \leq \frac{\alpha d(x_2, Tx_2)[1 + d(x_1, x_2)]}{1 + d(x_1, x_2)} \\
& + \beta \max\{d(x_1, x_2), d(x_1, x_2), d(x_2, Tx_2), d(x_1, Tx_2), d(x_2, x_2)\} \\
& + Ld(x_2, x_2) \\
& \leq \alpha d(x_2, Tx_2) + \beta \max\{d(x_1, x_2), d(x_2, Tx_2), d(x_1, x_2) + d(x_2, Tx_2), 0\} \\
& \leq \alpha H(Tx_1, Tx_2) + \beta \max\{d(x_1, x_2), H(Tx_1, Tx_2), d(x_1, x_2) + H(Tx_1, Tx_2), 0\} \\
& = \alpha H(Tx_1, Tx_2) + \beta\{d(x_1, x_2) + H(Tx_1, Tx_2)\}.
\end{aligned}$$

Thus we have $H(Tx_1, Tx_2) \leq rd(x_1, x_2) < r^2 c$.
By Lemma 1.1, we can choose $x_3 \in Tx_2$ such that

$$d(x_2, x_3) < r^2 c.$$

Continuing this process, we obtain a sequence $\{x_n\} \subset X$ such that

$$x_{n+1} \in Tx_n$$

and

$$d(x_n, x_{n+1}) \le r^n c$$

for all $n = 0, 1, 2, \cdots$.

For $m > n$, we obtain

$$\begin{aligned} & d(x_n, x_m) \\ & \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ & \le (r^n + r^{n+1} + \cdots + r^{m-1})c \\ & \le \frac{r^n}{1-r} c. \end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence in X.

By the T-orbitally completeness of X, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

From (2.1) we have

$$\begin{aligned} & d(x_{n+1}, Tz) \\ & \le H(Tx_n, Tz) \\ & \le \frac{\alpha d(z, Tz)[1 + d(x_n, Tx_n)]}{1 + d(x_n, z)} \\ & + \beta \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(x_n, Tz), d(z, Tx_n)\} \\ & + Ld(z, Tx_n) \\ & \le \frac{\alpha d(z, Tz)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, z)} \\ & + \beta \max\{d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), d(x_n, Tz), d(z, x_{n+1})\} \\ & + Ld(z, x_{n+1}). \end{aligned}$$

Letting $n \to \infty$ in above, we have $d(z, Tz) \le (\alpha+\beta) d(z, Tz)$. Since $\alpha+\beta < 1$, $d(z, Tz) = 0$. Thus, $z \in Tz$. □

By Theorem 2.3, we have the following corollaries.

Corollary 2.1. *Let (X, d) be a metric space, and let $T : X \to CB(X)$ be a given set-valued mapping. Suppose that X is T-orbitally complete.*

Assume that T satisfies the following condition:

$$H(Tx, Ty) \le \frac{\alpha d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta M(x, y) + LN(x, y),$$

for any $x, y \in X$, where $\alpha, \beta \ge 0$, $\alpha + 2\beta < 1$, $L \ge 0$ and
$N(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

Then T has a fixed point in X.

Corollary 2.2. *Let (X, d) be a metric space, and let $T : X \to CB(X)$ be a given set-valued mapping. Suppose that X is T-orbitally complete.*

Assume that a set-valued mapping $T : X \to CB(X)$ satisfies the following condition:

$$H(Tx, Ty) \le \frac{\alpha d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta M(x, y),$$

for any $x, y \in X$, where $\alpha, \beta \ge 0$, $\alpha + 2\beta < 1$.

Then T has a fixed point in X.

Corollary 2.3. *Let (X, d) be a metric space, and let $T : X \to CB(X)$ be a given set-valued mapping. Suppose that X is T-orbitally complete.*

Assume that a set-valued mapping $T : X \to CB(X)$ satisfies the following condition:

$$H(Tx, Ty) \leq \frac{\alpha d(y, Ty) d(x, Tx)}{1 + d(x, y)} + \beta M(x, y),$$

for any $x, y \in X$, where $\alpha, \beta \geq 0$, $\alpha + 2\beta < 1$.

Then T has a fixed point in X.

Corollary 2.4. *Let (X, d) be a metric space, and let $T : X \to CB(X)$ be a given set-valued mapping. Suppose that X is T-orbitally complete.*

Assume that a set-valued mapping $T : X \to CB(X)$ satisfies the following condition:

$$H(Tx, Ty) \leq \beta M(x, y) + L d(y, Tx),$$

for any $x, y \in X$, where $0 \leq \beta < \frac{1}{2}$ and $L \geq 0$.

Then T has a fixed point in X.

Corollary 2.5. *Let (X, d) be a metric space, and let $T : X \to CB(X)$ be a given set-valued mapping. Suppose that X is T-orbitally complete.*

Assume that a set-valued mapping $T : X \to CB(X)$ satisfies the following condition:

$$H(Tx, Ty) \leq \beta M(x, y)$$

for any $x, y \in X$, where $0 \leq \beta < \frac{1}{2}$.

Then T has a fixed point in X.

Remark 2.1. (1) Corollary 2.4 is a generalization of Theorem 2.1 in [6] and Theorem 3.2 [7] to the case of set-valued mapping and T-orbitally complete.
(2) Corollary 2.5 is a generalization of Theorem 2.2 in [4] to the case of T-orbitally complete.

Question. (1) Does the conclusion of Theorem 2.3 remain true for $\alpha + \beta < 1$ and $L \geq 0$?
(2) Does the conclusion of Corollary 2.4 remain true for $0 \leq \beta < 1$ and $L \geq 0$?

A set-valued mapping $T : X \to CB(X)$ is called *Ćirić-Berinde type strong almost set-valued contraction* if

$$H(Tx, Ty) \leq \frac{\alpha d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta M_1(x, y) + L d(y, Tx)$$

for all $x, y \in X$, where $\alpha, \beta \geq 0$, $\alpha + \beta < 1$, $L \geq 0$ and

$$M_1(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}\{d(x, Ty) + d(y, Tx)\}\}.$$

Remark 2.2. (1) Ćirić-Berinde type strong almost set-valued contraction mapping is Ćirić-Berinde type almost set-valued contraction mapping. Thus, for $\alpha, \beta \geq 0$, $\alpha + 2\beta < 1$ and $L \geq 0$, a Ćirić-Berinde type strong almost set-valued contraction mapping has a fixed point.
(2) Theorem 2.3 generalizes and improves Corollary 3.3 of [22] and Theorem 3.3 of [23].

The following example illustrates Theorem 2.3.

Example 2.1. Let $X = \left\{\frac{1}{n} : n = 1, 2, \cdots\right\} \cup \{0\}$ with the Euclidean metric d.

We define a set-valued mapping $T : X \to CB(X)$ by

$$Tx = \begin{cases} \left\{\frac{1}{n+1}\right\} & (x = \frac{1}{n}, n = 1, 2, 3, \cdots), \\ \{0\} & (x = 0). \end{cases}$$

Then, (X, d) is complete, and X is T-orbitally complete.

Let $\alpha, \beta \geq 0$, $\alpha + 2\beta < 1$ and $L = 1$.

We now show that condition (2.1) is satisfied.

We consider three cases.

Case 1. Let $x = y$. Then we have $H(Tx, Ty) = 0$. Hence condition (2.1) is satisfied.

Case 2. Let $x = 0$ and $y = \frac{1}{n}\left(\text{or } x = \frac{1}{n} \text{ and } y = 0\right)$. Then we have

$$\begin{aligned} H(Tx, Ty) &= H\left(\{0\}, \left\{\frac{1}{n+1}\right\}\right) = \frac{1}{n+1} \leq \frac{1}{n} = Ld(y, Tx) \\ &\leq \frac{\alpha d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta M(x, y) + Ld(y, Tx). \end{aligned}$$

Case 3. Let $x = \frac{1}{n}$ and $y = \frac{1}{m}(m > n)$. Then we have

$$\begin{aligned} H(Tx, Ty) &= H\left(\left\{\frac{1}{n+1}\right\}, \left\{\frac{1}{m+1}\right\}\right) \\ &= \left|\frac{1}{n+1} - \frac{1}{m+1}\right| \\ &= \frac{m - m}{(m+1)(n+1)} \\ &\leq (1 + \beta)\frac{m - m}{mn} \\ &= \beta M(x, y) + Ld(y, Tx) \\ &\leq \frac{\alpha d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta M(x, y) + Ld(y, Tx). \end{aligned}$$

Thus T satisfies all conditions in Theorem 2.3 and $0 \in T0$.

Note that condition (1.1) of Theorem 1.2 is not satisfied.

In fact, if there exists $k \in \left[0, \frac{1}{2}\right)$ such that for any $x, y \in X$

$H(Tx, Ty) \leq k\ max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$, then we have, for $x = 0$ and $y = \frac{1}{n}$ for $n = 1, 2, 3, \cdots$,

$$\begin{aligned} &\frac{1}{n+1} = H(Tx, Ty) \\ &\leq k\ max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &= k\ max\left\{\frac{1}{n}, 0, \frac{1}{n} - \frac{1}{n+1}, \frac{1}{n+1}, \frac{1}{n+1}\right\} \\ &= k\frac{1}{n}. \end{aligned}$$

Thus we obtain $k \geq \dfrac{n}{n+1}$ for $n = 1, 2, 3, \cdots$.

From this inequality, we have that $k \geq 1$. But it is not possible. Thus, condition (1.1) of Theorem 1.2 is not satisfied.

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