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Distribution of some quadratic linear recurrence sequences modulo 1

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ABSTRACT.

We show that if a is an even integer then for every $\xi \in \mathbb{R}$ the smallest limit point of the sequence $||\xi a^n||_{n=1}^{\infty}$ does not exceed $|a|/(2|a|+2)$ and this bound is best possible in the sense that for some ξ this constant cannot be improved. Similar (best possible) bound is also obtained for the smallest limit point of the sequence $||\xi x_n||_{n=1}^{\infty}$, where $(x_n)_{n=1}^{\infty}$ satisfies the second order linear recurrence $x_n = ax_{n-1} + bx_{n-2}$ with $a, b \in \mathbb{N}$ satisfying $a \geqslant b$. For the Fibonacci sequence $(F_n)_{n=1}^{\infty}$ our result implies that $\sup_{\xi \in \mathbb{R}} \liminf_{n\to\infty} ||\xi F_n|| = 1/5$, and e.g., in case when $a \geqslant 3$ is an odd integer, $b = 1$ and $\theta := a/2 + \sqrt{a^2/4+1}$ it shows that $\sup_{\xi\in\mathbb{R}} \liminf_{n\to\infty} ||\xi\theta^n|| = (a-1)/2a$.

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