

Existence and approximation of fixed points in convex metric spaces

HAFIZ FUKHAR-UD-DIN

ABSTRACT.

A fixed point theorem for a generalized nonexpansive mapping is established in a convex metric space introduced by Takahashi [*A convexity in metric spaces and nonexpansive mappings*, Kodai Math. Sem. Rep., 22 (1970), 142–149]. Our theorem generalizes simultaneously the fixed point theorem of Bose and Laskar [*Fixed point theorems for certain class of mappings*, Jour. Math. Phy. Sci., 19 (1985), 503–509] and the well-known fixed point theorem of Goebel and Kirk [*A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., 35 (1972), 171–174] on a nonlinear domain. The fixed point obtained is approximated by averaging Krasnosel'skii iterations of the mapping. Our results substantially improve and extend several known results in uniformly convex Banach spaces and $CAT(0)$ spaces.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEPARTMENT OF MATHEMATICS
THE ISLAMIA UNIVERSITY OF BAHAWALPUR
BAHAWALPUR 63100, PAKISTAN
E-mail address: hfdin@kfupm.edu.sa; hfdin@yahoo.com