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Operators commuting with multi-parameter shift semigroups

OLEH LOPUSHANSKY and SERGII SHARYN

ABSTRACT. Using operators of cross-correlation with ultradistributions supported by a positive cone, we describe a commutative algebra of shift-invariant continuous linear operators, commuting with contraction multi-parameter semigroups over a Banach space. Thereby, we generalize classic Schwartz's and Hörmander's theorems on shift-invariant operators.

1. INTRODUCTION

The well-known Schwartz structure theorem for shift-invariant operators [15] claims that every continuous linear operator $L\colon \mathcal{D}(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ commuting with the shift group $\tau_s\colon \varphi(\cdot) \mapsto \varphi(\,\cdot - s)$, $s \in \mathbb{R}^n$, is necessarily the convolution operator with some distribution $f \in \mathcal{D}'(\mathbb{R}^n)$, i.e., $L\varphi = f * \varphi$. Hörmander in [8] establishes a similar structure theorem for shift-invariant operators in the Lebesgue spaces $L^p(\mathbb{R}^n)$. Such operators over test and generalized functions were discussed in an overview paper [18]. An extension of Hörmander's result on Lorentz and Hardy spaces was considered in [2] and [17], respectively. Structure of operators from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ into the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ that commute with a discrete subgroup of translations was studied in [4]. Shift-invariant operators over the space of polynomial ultradistributions were considered in [12]. For other results and references on the topic we refer the reader to [3, 5, 9, 19].

Our goal is a generalization of the structure theorems for two cases: first one for a (C_0)-semigroup $T\colon \mathbb{R}^n_+ \ni s \longmapsto T_s$ of shifts and second one for contraction (C_0)-semigroups $\{U_t : t \in \mathbb{R}^n_+\}$ of operators on a Banach space. In the main Theorems 4.1 and 5.2 we describe shift-invariant operators for scalar and operator semigroups, respectively.

It is essential that hereinafter functions and distributions are defined on the cone $\mathbb{R}^n_+$ (instead of $\mathbb{R}^n$). As a consequence, the shift-invariant property is considered under the cross-correlation (instead of the convolution). Further, we will consider only the case of Roumieu ultradistributions.

Let $\mathcal{G}'_+$ be the convolution algebra of Roumieu ultradistributions with supports in the positive cone $\mathbb{R}^n_+$ and $\mathcal{G}_+$ be its predual space of Gevrey ultradifferentiable functions with compact supports in $\mathbb{R}^n_+$. We consider the cross-correlation operator

$$K_f\colon \mathcal{G}_+ \ni \varphi \longmapsto K_f\varphi, \qquad [K_f\varphi](s) := \langle f, T_s\varphi\rangle, \qquad f \in \mathcal{G}'_+.$$

In Theorem 4.1 we prove that the algebra $\mathcal{G}'_+$ has the isomorphic representation

$$\mathcal{G}'_+ \ni f \longmapsto K_f \in \mathscr{L}(\mathcal{G}_+)$$

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Corresponding author: Sharyn Sergii; sharyn.sergii@gmail.com

onto the commutant $[T]^c$ of the shift semigroup T (see formula (3.2)) in space of linear continuous operators on $\mathcal{G}_+$.

Let us consider the set of n-parameter contraction (C_0)-semigroups $\{U_t\colon t \in \mathbb{R}^n_+\}$ on a Banach space E and a set $\mathcal{A}$ of their generators. In section 5 we investigate the isomorphic representation of the algebra $\mathcal{G}'_+$

$$\mathcal{G}'_+ \ni f \longmapsto \mathcal{F} \circ (I \otimes K_f) \circ \mathcal{F}^{-1}, \qquad \mathcal{F}\colon E \otimes_{\mathfrak{p}} \mathcal{G}_+ \ni x \longmapsto \widehat{x} \in \widehat{\mathcal{G}},$$

where $E \otimes_{\mathfrak{p}} \mathcal{G}_+$ is the complete projective tensor product. Here $\widehat{\mathcal{G}}$ denotes the subspace of E-valued functions

$$\widehat{x}\colon \mathcal{A} \ni A \longmapsto \widehat{x}(A) \in E, \qquad \widehat{x}(A) = \int_{\mathbb{R}^n_+} U_t(A)x(t)\,dt,$$

determined by the Hille-Phillips calculus, where $U_t(A)$ means that the semigroup is generated by operator A. In Theorem 5.2 we prove that there exists an algebraic isomorphism from $\mathcal{G}'_+$ into the commutant of the semigroup $\mathcal{F} \circ (I \otimes T) \circ \mathcal{F}^{-1}$.

2. Ultradifferentiable functions and ultradistributions

We use the short notations: $t^k = t_1^{k_1} \cdot \ldots \cdot t_n^{k_n}$, $k^{k\beta} = k_1^{k_1\beta} \cdot \ldots \cdot k_n^{k_n\beta}$, $|k| = k_1 + \cdots + k_n$ for any $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and a real $\beta > 1$. Let $\partial^k = \partial_1^{k_1} \ldots \partial_n^{k_n}$, where $\partial_j^{k_j} = \partial^{k_j}/\partial t_j^{k_j}$ $(j = 1, \ldots, n)$. For $\mu, \nu \in \mathbb{R}^n$, the relation $\mu \prec \nu$ (resp. $\mu \succ \nu$) means that $\mu_1 < \nu_1, \ldots, \mu_n < \nu_n$ (resp. $\mu_1 > \nu_1, \ldots, \mu_n > \nu_n$). Let $[\mu, \nu] = \times_{j=1}^n [\mu_j, \nu_j]$ and $(\mu, \nu) = \times_{j=1}^n (\mu_j, \nu_j)$ for any $\mu \prec \nu$. In what follows $t \to \infty$ (resp. $t \to 0$) means that $t_j \to \infty$ (resp. $t_j \to 0$) for all $j = 1, \ldots, n$. By $\mathbb{R}^n_+ = \times_{j=1}^n [0, \infty)$ and $\operatorname{int} \mathbb{R}^n_+ = \times_{j=1}^n (0, \infty)$ we denote the positive cone and its interior, respectively.

Let $\mathcal{E}(\mathbb{R}^n_+)$ be the class of infinitely smooth complex valued functions φ defined on $\operatorname{int} \mathbb{R}^n_+$ such that $\partial^k \varphi$ for all $k \in \mathbb{Z}^n_+$ have continuous limits at the boundary of $\mathbb{R}^n_+$. Fix a $\beta > 1$. A function $\varphi \in \mathcal{E}(\mathbb{R}^n_+)$ is called (see [11]) Gevrey ultradifferentiable if for every interval $[\mu, \nu] \subset \operatorname{int} \mathbb{R}^n_+$ there exist constants $h > 0$ and $C > 0$ such that the inequality $\sup_{t \in [\mu, \nu]} |\partial^k \varphi(t)| \le C h^{|k|} k^{k\beta}$ holds for all $k \in \mathbb{Z}^n_+$.

We denote by $\mathcal{E}^\beta_+ = \mathcal{E}^\beta(\mathbb{R}^n_+)$ the vector space of all Gevrey ultradifferentiable functions on $\operatorname{int} \mathbb{R}^n_+$. By virtue of Seeley's theorem [16] every function $\varphi \in \mathcal{E}^\beta_+$ has an infinitely smooth extension $\tilde{\varphi}$ on $\mathbb{R}^n$ such that $\partial^k \tilde{\varphi}(t) = \partial^k \varphi(t)$ for all $k \in \mathbb{Z}^n_+$ and $t \in \mathbb{R}^n_+$. Denote by $\mathcal{G}^\beta_+ \subset \mathcal{E}^\beta_+$ the subspace of Gevrey ultradifferentiable functions with compact supports. In the sequel, to simplify notations we will write $\mathcal{G}_+$ instead of $\mathcal{G}^\beta_+$ omitting the fixed β.

For a fixed $h > 0$, we consider the subspace $\mathcal{G}^h_\nu \subset \mathcal{G}_+$ of all functions φ vanishing outside $[0, \nu]$ for which there exists a constant $C = C(\varphi) > 0$ such that $\sup_{t \in [0, \nu]} |\partial^k \varphi(t)| \le C h^{|k|} k^{k\beta}$ holds for all $k \in \mathbb{Z}^n_+$. As is known [11, p. 38], the space $\mathcal{G}^h_\nu$ endowed with the norm

$$\|\varphi\|_{\mathcal{G}^h_\nu} = \sup_{k \in \mathbb{Z}^n_+,\ t \in [0, \nu]} \frac{|\partial^k \varphi(t)|}{h^{|k|} k^{k\beta}}$$

is complete and the inclusions $\mathcal{G}^h_\nu \looparrowright \mathcal{G}^l_\nu$ with $h < l$ are compact. Moreover, if $\mu \prec \nu$ then $\mathcal{G}^h_\mu$ is a closed subspace in $\mathcal{G}^h_\nu$ [11, p.40]. We equip the space $\mathcal{G}_+ = \bigcup_{\nu \succ 0,\, h > 0} \mathcal{G}^h_\nu$ with the inductive limit topology under the compact inclusions $\mathcal{G}^h_\mu \looparrowright \mathcal{G}^l_\nu$. Hence,

$$\mathcal{G}_+ \simeq \liminf_{\nu, h \to \infty} \mathcal{G}^h_\nu \tag{2.1}$$

is a nuclear (DFS)-space [11, p. 28]. Note that the space $\mathcal{G}_+$ is a topological algebra with respect to the pointwise multiplication and $\|\varphi\psi\|_{\mathcal{G}_\nu^{h+l}} \le \|\varphi\|_{\mathcal{G}_\nu^h}\|\psi\|_{\mathcal{G}_\nu^l}$ (see [11, p. 41]).

The strong dual $\mathcal{G}'_+$ of $\mathcal{G}_+$ is a nuclear (FS)-space. The space $\mathcal{G}'_+$ consists of Roumieu ultradistributions on $\mathbb{R}^n$ supported by $\mathbb{R}^n_+$ and it is a convolution algebra with Dirac δ as the unit element. Remind, that the convolution is defined by $\langle f * g, \varphi\rangle = \langle f, g \star \varphi\rangle$, where $(g \star \varphi)(x) := \langle g, \varphi(\,\cdot + x)\rangle$ with $f, g \in \mathcal{G}'_+$ and $\varphi \in \mathcal{G}_+$ (see e.g. [11, 2.5]).

Let $\langle f, \varphi\rangle$ denote the value of $f \in \mathcal{G}'_+$ on $\varphi \in \mathcal{G}_+$. Sometimes we write $\langle f(t), \varphi(t)\rangle$, where the "argument" of an ultradistribution f means the variable of a function φ on which the functional f acts.

3. Cross-correlation and shift semigroups

For a linear topological space X, we denote by $\mathscr{L}(X)$ the space of all continuous linear operators on X. We write I for the identity operator in $\mathscr{L}(X)$. We endow $\mathscr{L}(X)$ with the locally convex topology of uniform convergence on bounded subsets of X. We define the commutant of a subset $S \subset \mathscr{L}(X)$ to be $[S]^c := \{B \in \mathscr{L}(X)\colon BA = AB, \forall A \in S\}$.

An n-parameter family $\{U_t : t \in \mathbb{R}^n_+\}$ of bounded linear operators on a complex Banach space $(E, \|\cdot\|)$ is called an n-parameter semigroup of operators (see [1, 7]) if it is a mapping $U : \mathbb{R}^n_+ \ni t \longmapsto U_t \in \mathscr{L}(E)$ such that $U_0 = I$ and $U_{t+s} = U_t \circ U_s$ for all $t, s \in \mathbb{R}^n_+$.

We call a semigroup $\{U_t : t \in \mathbb{R}^n_+\}$ strongly continuous (or C_0-semigroup) if the equality $\lim_{\mathbb{R}^n_+ \ni t \to 0} \|U_t x - x\| = 0$ holds for all $x \in E$.

For any n-parameter semigroup $\{U_t : t = (t_1, \ldots, t_n) \in \mathbb{R}^n_+\}$ we define the marginal one-parameter semigroups $\{U^{(j)}_{t_j} : t_j \in \mathbb{R}_+\}$, which commute with each other, where

$$U^{(j)}_{t_j}\colon [0, \infty) \ni t_j \longmapsto U_{(0,\ldots,0,t_j,0,\ldots,0)} \in \mathscr{L}(E), \quad j = 1, \ldots, n.$$

Any n-parameter semigroup U_t may be represented as a composition of the associated marginal one-parameter semigroups, i.e. $U_t = U^{(1)}_{t_1} \circ \ldots \circ U^{(n)}_{t_n}$.

For all $j = 1, \ldots, n$, let the generator A_j of the jth marginal semigroup $\{U^{(j)}_{t_j} : t_j \in \mathbb{R}_+\}$ be defined as

$$A_j x := \lim_{t_j \to +0} t_j^{-1}\big(U^{(j)}_{t_j} x - x\big) = \partial^1_j U^{(j)}_{t_j} x \mid_{t_j = +0}, \qquad x \in \mathfrak{D}(A_j),$$

where $\mathfrak{D}(A_j)$ consists of all $x \in E$ for which the above limit exists.

We denote the generator of $\{U_t \colon t \in \mathbb{R}^n_+\}$ by $A := (A_1, \ldots, A_n)$. Let us denote $\mathfrak{D}(A) := \bigcap_{j=1}^n \mathfrak{D}(A_j)$. If U_t is a (C_0)-semigroup then the following properties hold (see [1, Propositions 1.1 8-9]):

(i) if $x \in \mathfrak{D}(A_j)$ then $U_t x \in \mathfrak{D}(A_j)$ and $A_j U_t x = U_t A_j x$;
(ii) $U_t x \in \mathfrak{D}(A)$ for any $x \in E$, $t \in \operatorname{int} \mathbb{R}^n_+$, and $\mathfrak{D}(A)$ is dense in E;
(iii) $A_i A_j x = A_j A_i x$, $(i, j = 1, \ldots, n)$ for all $x \in \mathfrak{D}(A)$.

We also consider the n-parameter (C_0)-semigroup of shifts over the space $\mathcal{G}_+$,

$$T\colon \mathbb{R}^n_+ \ni s \longmapsto T_s \in \mathscr{L}(\mathcal{G}_+), \qquad T_s\varphi(t) = \varphi(t + s), \qquad t \in \mathbb{R}^n_+, \quad \varphi \in \mathcal{G}_+. \tag{3.2}$$

It is obvious that $\operatorname{supp} \varphi(\,\cdot + s) = \operatorname{supp} \varphi - s$ for any function φ, defined on $\mathbb{R}^n$. Therefore, $\operatorname{supp} T_s\varphi = (\operatorname{supp} \varphi - s) \cap \mathbb{R}^n_+$ with $s \in \mathbb{R}^n_+$. Hence, the inequalities $\|T_s\varphi\|_{\mathcal{G}^h_\nu} \le \|\varphi\|_{\mathcal{G}^h_\nu}$ for all $s \in \mathbb{R}^n_+$ and all indexes h, ν hold. Clearly, each derivative ∂_j $(j = 1, \ldots, n)$ generates the corresponding marginal one-parameter semigroup

$$T^{(j)}_{s_j} : [0, \infty) \ni s_j \longmapsto T_{(0,\ldots,0,s_j,0,\ldots,0)} \in \mathscr{L}(\mathcal{G}_+).$$

The regular property (see [14]) of the inductive limit (2.1) implies that the semigroup T is equibounded. So, T is equicontinuous by the Banach-Steinhaus theorem.

Definition 3.1. For any ultradistribution $f \in \mathcal{G}'_+$, the cross-correlation operator over the space $\mathcal{G}_+$ is defined to be

$$K_f \colon \mathcal{G}_+ \ni \varphi \longmapsto K_f\varphi, \qquad K_f\varphi(s) := \langle f, T_s\varphi\rangle, \qquad s \in \mathbb{R}^n_+.$$

Let $E \otimes_{\mathfrak{p}} \mathcal{G}_+$ be a completion of the algebraic tensor product $E \otimes \mathcal{G}_+$ with respect to the projective tensor topology. We can treat elements of $E \otimes_{\mathfrak{p}} \mathcal{G}_+$ as a E-valued ultradifferentiable functions $x : t \longmapsto x(t)$ with compact supports in $\mathbb{R}^n_+$. From the known (see [6]) Grothendieck's isomorphism $E \otimes_{\mathfrak{p}} \liminf_{h,\nu\to\infty} \mathcal{G}^h_\nu \simeq \liminf_{h,\nu\to\infty} E \otimes_{\mathfrak{p}} \mathcal{G}^h_\nu$ it follows the isomorphism

$$E \otimes_{\mathfrak{p}} \mathcal{G}_+ \simeq \liminf_{h,\nu\to\infty} E \otimes_{\mathfrak{p}} \mathcal{G}^h_\nu.$$

So, for every $x \in E \otimes_{\mathfrak{p}} \mathcal{G}_+$ there exist $\nu \in \mathbb{R}^n_+$ and $h > 0$ such that $x \in E \otimes_{\mathfrak{p}} \mathcal{G}^h_\nu$, where each space $E \otimes_{\mathfrak{p}} \mathcal{G}^h_\nu$ is equipped with the norm

$$\|x\|_{E\otimes_{\mathfrak{p}}\mathcal{G}^h_\nu} = \sup_{k\in\mathbb{Z}^n_+,\ t\in[0,\nu]} \frac{\|\partial^k x(t)\|}{h^{|k|}k^{k\beta}}.$$

Hence, from theorem about representation of projective tensor product (see [14, Th. III. 6.4]) it follows that every $x \in E \otimes_{\mathfrak{p}} \mathcal{G}_+$ can be expanded (in general, not uniquely) in an absolutely convergent series

$$x = \sum_{j\in\mathbb{N}} \lambda_j x_j \otimes \varphi_j, \qquad \lambda_j \in \mathbb{C}, \quad x_j \in E, \quad \varphi_j \in \mathcal{G}^h_\nu, \tag{3.3}$$

for some $\nu \in \mathbb{R}^n_+$ and $h > 0$, where $\sum_j |\lambda_j| < \infty$ and the sequences $\{x_j\}$ and $\{\varphi_j\}$ are convergent to zero in the corresponding spaces.

Let $K \in \mathscr{L}(\mathcal{G}_+)$. Using (3.3), we can define the tensor product $I \otimes K \in \mathscr{L}(E \otimes_{\mathfrak{p}} \mathcal{G}_+)$ as follows

$$(I \otimes K)x = \sum_{j\in\mathbb{N}} \lambda_j x_j \otimes K\varphi_j. \tag{3.4}$$

In case $K = T_s$ we often use the short notation $x(t+s)$ instead of $(I \otimes T_s)x(t)$. We can now define analogously the action

$$\langle f, x\rangle := \sum_{j\in\mathbb{N}} \lambda_j x_j \langle f, \varphi_j\rangle \tag{3.5}$$

for any ultradistribution $f \in \mathcal{G}'_+$ and $x \in E \otimes_{\mathfrak{p}} \mathcal{G}_+$. It is well-known [14, III.6.4] that these definitions are independent of representations of elements $x \in E \otimes_{\mathfrak{p}} \mathcal{G}_+$ in the form (3.3).

We say that an operator $I \otimes K$ with $K \in \mathscr{L}(\mathcal{G}_+)$ is invariant with respect to shift operators $I \otimes T = \{I \otimes T_s \colon s \in \mathbb{R}^n_+\}$ if

$$I \otimes (K \circ T_s) = I \otimes (T_s \circ K) \quad \text{for all} \quad s \in \mathbb{R}^n_+.$$

Definition 3.2. For any ultradistribution $f \in \mathcal{G}'_+$, the cross-correlation operator over the space $E \otimes_{\mathfrak{p}} \mathcal{G}_+$ is defined to be

$$I \otimes K_f \colon E \otimes_{\mathfrak{p}} \mathcal{G}_+ \ni x \longmapsto (I \otimes K_f)x.$$

Here $(I \otimes K_f)x(s) = \sum_{j\in\mathbb{N}} \lambda_j x_j \otimes K_f\varphi_j = \langle f, (I \otimes T_s)x\rangle$ for all $s \in \mathbb{R}^n_+$ by continuity of a functional $f \in \mathcal{G}'_+$.

4. OPERATORS COMMUTING WITH SHIFT SEMIGROUPS

The next theorem, based on the cross-correlation notion, is a generalization of known structure theorems about shift-invariant operators.

Theorem 4.1. (i) *The mapping $\mathcal{K}\colon \mathcal{G}'_+ \ni f \longmapsto K_f \in \mathscr{L}(\mathcal{G}_+)$ produces a topological isomorphism from the convolution algebra $\mathcal{G}'_+$ onto the commutant $[T]^c$ of the shift semigroup T, i.e.*

$$K_{f*g} = K_f \circ K_g, \qquad f, g \in \mathcal{G}'_+, \tag{4.6}$$

where $*$ denotes the convolution in $\mathcal{G}'_+$. In particular, K_δ is the identity in $\mathscr{L}(\mathcal{G}_+)$.

(ii) *For any $f \in \mathcal{G}'_+$ the operator $I \otimes K_f$ is invariant with respect to shift operators $I \otimes T$. Conversely, for any $K \in \mathscr{L}(\mathcal{G}_+)$ such that $I \otimes K$ is invariant with respect to $I \otimes T$ there exists a unique $f \in \mathcal{G}'_+$ such that for all $x \in E \otimes_{\mathfrak{p}} \mathcal{G}_+$,*

$$K = K_f \qquad \text{and} \qquad (I \otimes K)x = (I \otimes K_f)x.$$

Proof. (i) Check that K_f is a linear continuous operator. It is clear that

$$\operatorname{supp} K_f\varphi \neq \varnothing \iff \operatorname{supp} f \cap \operatorname{supp} \varphi(\,\cdot + s) \neq \varnothing \iff \exists\, t_0 \in \operatorname{supp} f \cap \operatorname{supp} \varphi(\,\cdot + s).$$

Since $t_0 \in \operatorname{supp} \varphi(\,\cdot + s) \iff t_0 + s \in \operatorname{supp} \varphi \iff s \in \operatorname{supp} \varphi - t_0$, $s \in \operatorname{supp} \varphi - \operatorname{supp} f$. So, $\operatorname{supp} K_f\varphi \subset (\operatorname{supp} \varphi - \operatorname{supp} f) \cap \mathbb{R}^n_+ \subset [0, \nu]$ for some $\nu \succ 0$.

Prove that $K_f \in \mathscr{L}(\mathcal{G}_+)$. Let $\{\varphi_m\} \subset \mathcal{G}_+$ be a sequence for which there exists $[0, \nu] \subset \mathbb{R}^n_+$ such that $\operatorname{supp} \varphi_m \subset [0, \nu]$ for all $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} \sup_{t\in[0,\nu]} \frac{|\partial^k \varphi_m(t)|}{h^{|k|} k^{k\beta}} = 0$$

for all $k \in \mathbb{Z}^n_+$ with some $h > 0$. From the continuity of $f \in \mathcal{G}'_+$ and $T_s \in \mathscr{L}(\mathcal{G}_+)$ it follows that $\partial^k K_f \varphi = K_f \partial^k \varphi$ for all $k \in \mathbb{Z}^n_+$. So, we obtain

$$\lim_{m\to\infty} \sup_{t\in[0,\nu]} \frac{|\partial^k K_f \varphi_m(t)|}{h^{|k|} k^{k\beta}} = \left| \left\langle f, \lim_{m\to\infty} \sup_{t\in[0,\nu]} \frac{\partial^k \varphi_m(\,\cdot + t)}{h^{|k|} k^{k\beta}} \right\rangle \right| = 0$$

for all $k \in \mathbb{Z}^n_+$ and some $h > 0$. Using the isomorphism (2.1), we obtain $K_f \in \mathscr{L}(\mathcal{G}_+)$.

The following equalities

$$(K_f T_s \varphi)(t) = \langle f(r), \varphi(r + t + s)\rangle = T_s \langle f(r), \varphi(r + t)\rangle = (T_s K_f \varphi)(t) \tag{4.7}$$

hold for all $t, s \in \mathbb{R}^n_+$ and $\varphi \in \mathcal{G}_+$. Hence, for any $f \in \mathcal{G}'_+$ we have $K_f \in [T]^c$.

Let now $K \in \mathscr{L}(\mathcal{G}_+)$ be an arbitrary operator with the property

$$(KT_s)\varphi(t) = (T_s K)\varphi(t), \qquad \varphi \in \mathcal{G}_+, \quad t, s \in \mathbb{R}^n_+. \tag{4.8}$$

It is easy to see that the functional $\langle f_0, \varphi \rangle := (K\varphi)(0)$ belongs to $\mathcal{G}'_+$. By cross-correlation definition $(K_{f_0}\varphi)(0) = \langle f_0, \varphi\rangle$, i.e. $(K\varphi)(0) = (K_{f_0}\varphi)(0)$ for all $\varphi \in \mathcal{G}_+$. Substituting $T_s\varphi$ instead of φ and using the property (4.8), we get that $K = K_{f_0}$ and hence that the image of $\mathcal{K}$ coincides with the commutant $[T]^c$.

If $K_f\varphi(s) = \langle f, T_s\varphi\rangle = 0$ for all $\varphi \in \mathcal{G}_+$ then $f = 0$. Hence, the mapping $\mathcal{K}$ is injective.

Since $\mathcal{G}_+$ is a Montel space [14, IV.5], the topologies on $\mathscr{L}(\mathcal{G}_+)$ of uniform convergence on compacts and on bounded sets coincide. By barrelledness of the spaces $\mathcal{G}'_+$ and $\mathcal{G}_+$ [14, II.7] the map $\mathcal{G}'_+ \times \mathcal{G}_+ \ni (f, \varphi) \longmapsto K_f\varphi \in \mathcal{G}_+$ is equicontinuous, because it is separately continuous. Hence, $\mathcal{K}$ is continuous. Moreover, $\mathcal{K}$ has the closed image $[T]^c$. Since $\mathcal{G}_+$ is a nuclear (DFS)-space, we have $\mathscr{L}(\mathcal{G}_+) \simeq \mathcal{G}_+ \otimes_{\mathfrak{p}} \mathcal{G}'_+$ (see [14, IV.9.4]) where $\mathcal{G}'_+$ is a Fréchet space as a strong dual of (DFS)-space. So,

$$\mathcal{G}_+ \otimes_{\mathfrak{p}} \mathcal{G}'_+ \simeq \liminf_{h,\nu\to\infty} \mathcal{G}^h_\nu \otimes_{\mathfrak{p}} \mathcal{G}'_+$$

by virtue of the isomorphism (2.1). On the other hand, in $\mathcal{G}'_+$ there exists a countable base of closed absolutely convex bounded sets $\{B_n\}$ such that $\mathcal{G}'_+ = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$, where $\mathcal{B}_n := \mathbb{C}\cdot B_n$ is a subspace with the norm $\|x\|_n = \inf\{|\lambda|\colon x\in\lambda B_n\}$. From the completeness of $\mathcal{G}'_+$ and the closedness of B_n it follows that $\mathcal{B}_n$ is a Banach space for all $n\in\mathbb{N}$. From boundedness of B_n it follows that the embeddings $\mathcal{B}_n \looparrowright \mathcal{G}'_+$ are continuous [14, II.8.4]. So, the identical mapping $\liminf_{n\to\infty}\mathcal{B}_n \longrightarrow \mathcal{G}'_+$ is continuous. As a consequence, we obtain the isomorphism

$$\mathcal{G}_+ \otimes_{\mathfrak{p}} \mathcal{G}'_+ \simeq \operatorname*{lim\,ind}_{h,\nu,n\to\infty} \mathcal{G}^h_\nu \otimes_{\mathfrak{p}} \mathcal{B}_n,$$

i.e. $\mathscr{L}(\mathcal{G}_+)$ is ultrabornological space [10]. Now, Open Mapping Theorem (see [13]) implies that $\mathcal{K}$ is a topological isomorphism from $\mathcal{G}'_+$ onto $[T]^c$.

Check the equality (4.6). The convolution definition implies

$$(K_{f*g}\varphi)(t) = \langle f*g, T_t\varphi\rangle = \big\langle f(r), \langle g(s), \varphi(t+r+s)\rangle\big\rangle = \langle f, T_t(K_g\varphi)\rangle = (K_fK_g)\varphi(t).$$

with $t,r,s\in\mathbb{R}^n_+$. In particular, $K_f\circ K_\delta = K_{f*\delta} = K_f = K_{\delta*f} = K_\delta\circ K_f$ for all $f\in\mathcal{G}'_+$, so K_δ is the identity.

(ii) The equality $\sum_j \lambda_j x_j\otimes(K_fT_s)\varphi_j = \sum_j \lambda_j x_j\otimes(T_sK_f)\varphi_j$ holds for all $f\in\mathcal{G}'_+$ and $x\in E\otimes_{\mathfrak{p}}\mathcal{G}_+$ via (3.3) and (4.7). Therefore, $I\otimes(K_f\circ T_s) = I\otimes(T_s\circ K_f)$, i.e. $I\otimes K_f$ is invariant with respect to the shift operators $I\otimes T$.

Conversely, let $K\in\mathscr{L}(\mathcal{G}_+)$ be an operator such that $I\otimes(K\circ T_s) = I\otimes(T_s\circ K)$ for all $s\in\mathbb{R}^n_+$. For any $\varphi\in\mathcal{G}_+$, let $f_0\colon\varphi\longmapsto(K\varphi)(0)$. Definitions (3.4) and (3.5) imply

$$[(I\otimes K)x](0) = \langle f_0, x\rangle = [(I\otimes K_{f_0})x](0).$$

Substituting $(I\otimes T_s)x$ instead of x and using that $I\otimes K$ is invariant with respect to the shift operators $I\otimes T$, we obtain $(I\otimes K)x = (I\otimes K_{f_0})x$ for all $x\in E\otimes_{\mathfrak{p}}\mathcal{G}_+$. Consequently, $K = K_{f_0}$. □

5. Shift-invariant operators commuting with operator semigroups

Consider the set of n-parameter contraction C_0-semigroups $\{U_t : t\in\mathbb{R}^n_+\}$ on a complex Banach space $(E,\|\cdot\|)$, i.e. semigroups satisfying the condition

$$\sup_{t\in\mathbb{R}^n_+}\|U_t\|_{\mathscr{L}(E)} \le 1, \tag{5.9}$$

and let $\mathcal{A}$ be the set of their generators. To emphasize the fact that a semigroup $\{U_t : t\in\mathbb{R}^n_+\}$ is generated by an operator $A\in\mathcal{A}$ we will use the notation $\{U_t(A) : t\in\mathbb{R}^n_+\}$ for the semigroup. Consider the space

$$\widehat{\mathcal{G}} = \{\widehat{x}\colon\mathcal{A}\longrightarrow E : x\in E\otimes_{\mathfrak{p}}\mathcal{G}_+\}$$

of E-valued functions

$$\widehat{x}\colon\mathcal{A}\ni A\longmapsto\widehat{x}(A)\in E \quad\text{with}\quad \widehat{x}(A) = \int_{\mathbb{R}^n_+} U_t(A)x(t)\,dt, \tag{5.10}$$

determined by the Hille-Phillips calculus [7, Chapter 15], where the integral is understood in the Bochner sense. The integral in (5.10) is well-defined, because the integrand is a continuous E-valued function $t\longmapsto U_t(A)x(t)$ with a compact support.

Let us determine the linear mapping

$$\mathcal{F}\colon E\otimes_{\mathfrak{p}}\mathcal{G}_+\ni x\longmapsto\widehat{x}\in\widehat{\mathcal{G}}.$$

If the assumption (5.9) holds, then the mapping $\mathcal{F}$ is an isomorphism by virtue of [7, Theorem 15.2.1]. Indeed, the semigroups $\mathbb{R}^n_+\ni t\longmapsto e^{-(\lambda,t)}I$ with $\operatorname{Re}\lambda\in\operatorname{int}\mathbb{R}^n_+$ satisfy the condition (5.9). Therefore, their generators $-\lambda I$ belong to $\mathcal{A}$. Note that $\widehat{x}(-\lambda I) = \int_{\mathbb{R}^n_+} e^{-(\lambda,t)}x(t)\,dt$ is the Laplace transform of an E-valued function $x \in E \otimes_{\mathfrak{p}} \mathcal{G}_+$. Particularly, it follows that if $\widehat{x} \equiv 0$ then $x \equiv 0$, i.e., $\operatorname{Ker}\mathcal{F} = \{0\}$.

We endow the space $\widehat{\mathcal{G}}$ with the strongest locally convex topology induced by $\mathcal{F}$. Namely, let $\mathcal{G}^h_\nu(E) = \left\{\widehat{x}\colon x \in E \otimes_{\mathfrak{p}} \mathcal{G}^h_\nu\right\}$ be a Banach space endowed with the inductive topology under the mapping $E \otimes_{\mathfrak{p}} \mathcal{G}^h_\nu \ni x \longmapsto \widehat{x}$ with fixed h and ν. Then from (2.1) it follows that $\widehat{\mathcal{G}}$ has the structure of the inductive limit $\widehat{\mathcal{G}} = \lim\operatorname{ind}_{h,\nu\to\infty} \mathcal{G}^h_\nu(E)$ under the continuous embeddings $\mathcal{G}^h_\nu(E) \looparrowright \mathcal{G}^l_\mu(E)$ with $h < l$ and $\nu \prec \mu$. So the mapping $\mathcal{F}$ is a topological isomorphism.

Consider the n-parameter semigroup on the space $\widehat{\mathcal{G}}$

$$\widehat{T}\colon \mathbb{R}^n_+ \ni s \longmapsto \widehat{T}_s \in \mathscr{L}(\widehat{\mathcal{G}}), \qquad \widehat{T}_s = \mathcal{F} \circ (I \otimes T_s) \circ \mathcal{F}^{-1}, \tag{5.11}$$

where $\mathcal{F}^{-1}$ means the inverse map of $\mathcal{F}$. Since $\mathcal{F}$ is a topological isomorphism and the semigroup $\mathbb{R}^n_+ \ni s \longmapsto (I \otimes T_s)x \in E \otimes_{\mathfrak{p}} \mathcal{G}_+$ is continuous for all $x \in E \otimes_{\mathfrak{p}} \mathcal{G}_+$, we have that the semigroup $\widehat{T}$ on $\widehat{\mathcal{G}}$ has the (C_0)-property and its generator is densely defined.

Theorem 5.2. *The mapping*

$$\mathcal{G}'_+ \ni f \longmapsto \widehat{K}_f \in \mathscr{L}(\widehat{\mathcal{G}}), \qquad \widehat{K}_f = \mathcal{F} \circ (I \otimes K_f) \circ \mathcal{F}^{-1},$$

is an algebraic isomorphism of the convolution algebra $\mathcal{G}'_+$ and the subalgebra of all operators $\widehat{K} = \mathcal{F} \circ (I \otimes K) \circ \mathcal{F}^{-1}$ for some $K \in \mathscr{L}(\mathcal{G}_+)$ in the commutant $[\widehat{T}]^c$ on $\widehat{\mathcal{G}}$. In particular, the equality $\widehat{K}_{f*g} = \widehat{K}_f \circ \widehat{K}_g$ for all $f, g \in \mathcal{G}'_+$ holds and $\widehat{K}_\delta$ is the identity in $\mathscr{L}(\widehat{\mathcal{G}})$.

Proof. For any $f \in \mathcal{G}'_+$ the diagram

$$\begin{tikzcd}
\widehat{\mathcal{G}} \ni \widehat{x} \arrow[r] & \widehat{K}_f\widehat{x} \in \widehat{\mathcal{G}} \\
E \otimes_{\mathfrak{p}} \mathcal{G}_+ \ni x \arrow[u, "\mathcal{F}"] \arrow[r] & (I \otimes K_f)x \in E \otimes_{\mathfrak{p}} \mathcal{G}_+ \arrow[u, "\mathcal{F}"]
\end{tikzcd}$$

is commutative. Continuity of the mappings $I \otimes K_f$ and $\mathcal{F}$ and openness of the mapping $\mathcal{F}^{-1}$ imply that $\widehat{K}_f \in \mathscr{L}(\widehat{\mathcal{G}})$. It follows that the equalities

$$[\mathcal{F}(I \otimes K_f)x](A) = \int_{\mathbb{R}^n_+} U_t(A)(I \otimes K_f)x(t)\,dt = \widehat{K}_f\widehat{x}(A)$$

are valid for all $A \in \mathcal{A}$. Consequently, the equalities

$$\widehat{K}_f\widehat{T}_r\widehat{x}(A) = \int_{\mathbb{R}^n_+} U_t(A)(I \otimes K_f)x(t+r)\,dt = \widehat{T}_r\widehat{K}_f\widehat{x}(A)$$

hold for all $r \in \mathbb{R}^n_+$ and $\widehat{x} \in \widehat{\mathcal{G}}$. Hence, for any $f \in \mathcal{G}'_+$ we have that $\widehat{K}_f$ belongs to the commutant of the semigroup $\widehat{T}$ in $\mathscr{L}(\widehat{\mathcal{G}})$.

Conversely, let $\widehat{K} = \mathcal{F} \circ (I \otimes K) \circ \mathcal{F}^{-1}$ belongs to the commutant $[\widehat{T}]^c$ of $\widehat{T}$. Then

$$\begin{aligned}
\mathcal{F} \circ (I \otimes (K \circ T_s)) \circ \mathcal{F}^{-1} &= \mathcal{F} \circ (I \otimes K) \circ (I \otimes T_s) \circ \mathcal{F}^{-1} \\
&= \mathcal{F} \circ (I \otimes K) \circ \mathcal{F}^{-1} \circ \mathcal{F} \circ (I \otimes T_s) \circ \mathcal{F}^{-1} = \widehat{K} \circ \widehat{T}_s \\
&= \widehat{T}_s \circ \widehat{K} = \mathcal{F} \circ (I \otimes T_s) \circ \mathcal{F}^{-1} \circ \mathcal{F} \circ (I \otimes K) \circ \mathcal{F}^{-1} \\
&= \mathcal{F} \circ (I \otimes T_s) \circ (I \otimes K) \circ \mathcal{F}^{-1} = \mathcal{F} \circ (I \otimes (T_s \circ K)) \circ \mathcal{F}^{-1},
\end{aligned}$$

therefore $K \in [T]^c$. By Theorem 4.1 there exists a unique $f \in \mathcal{G}'_+$ such that $K = K_f$, i.e., $\widehat{K}\widehat{x} = \widehat{K}\mathcal{F}x = \mathcal{F}(I \otimes K_f)x = \widehat{K}_f\widehat{x}$, $\widehat{x} \in \widehat{\mathcal{G}}$. Hence, $\widehat{K} = \widehat{K}_f$.

Since $K_\delta = I$, we obtain $\widehat{K}_\delta = \mathcal{F} \circ (I \otimes K_\delta) \circ \mathcal{F}^{-1} = \mathcal{F} \circ \mathcal{F}^{-1}$, so the operator $\widehat{K}_\delta$ is the identity in $\mathscr{L}(\widehat{\mathcal{G}})$.

From the properties of cross-correlation (see Theorem 4.1) it follows

$$\widehat{K}_f\widehat{K}_g\widehat{x}(A) = \int_{\mathbb{R}^n_+} U_t(A)(I \otimes K_f K_g)x(t)\,dt = \int_{\mathbb{R}^n_+} U_t(A)(I \otimes K_{f*g})x(t)\,dt = \widehat{U}_{f*g}\widehat{x}(A),$$

so the mapping $f \longmapsto \widehat{K}_f$ is an algebraic isomorphism. $\square$

In the same way as in the proof of Theorem 4.1 we can prove that the algebraic isomorphism in Theorem 5.2 is topological.

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UNIVERSITY OF RZESZÓW
FACULTY OF MATHEMATICS AND NATURAL SCIENCES
16A REJTANA, 35-959 RZESZÓW, POLAND
E-mail address: ovlopusz@ur.edu.pl

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES
PRECARPATHIAN NATIONAL UNIVERSITY
57 SHEVCHENKA, 76-018 IVANO-FRANKIVSK, UKRAINE
E-mail address: sharyn.sergii@gmail.com