Operators commuting with multi-parameter shift semigroups

OLEH LOPUSHANSKY and SERGII SHARYN

ABSTRACT. Using operators of cross-correlation with ultradistributions supported by a positive cone, we describe a commutative algebra of shift-invariant continuous linear operators, commuting with contraction multi-parameter semigroups over a Banach space. Thereby, we generalize classic Schwartz’s and Hörmander’s theorems on shift-invariant operators.

1. INTRODUCTION

The well-known Schwartz structure theorem for shift-invariant operators [15] claims that every continuous linear operator \( L : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n) \) commuting with the shift group \( \tau_s : \varphi(\cdot) \mapsto \varphi(\cdot - s), s \in \mathbb{R}^n \), is necessarily the convolution operator with some distribution \( f \in \mathcal{D}'(\mathbb{R}^n) \), i.e., \( L\varphi = f \ast \varphi \). Hörmander in [8] establishes a similar structure theorem for shift-invariant operators in the Lebesgue spaces \( L^p(\mathbb{R}^n) \). Such operators over test and generalized functions were discussed in an overview paper [18]. An extension of Hörmander’s result on Lorentz and Hardy spaces was considered in [2] and [17], respectively. Structure of operators from the Schwartz space \( S(\mathbb{R}^n) \) into the tempered distributions \( S'(\mathbb{R}^n) \) that commute with a discrete subgroup of translations was studied in [4]. Shift-invariant operators over the space of polynomial ultradistributions were considered in [12]. For other results and references on the topic we refer the reader to [3, 5, 9, 19].

Our goal is a generalization of the structure theorems for two cases: first one for a \((C_0)\)-semigroup \( T : \mathbb{R}_+^n \ni s \mapsto T_s \) of shifts and second one for contraction \((C_0)\)-semigroups \( \{U_t : t \in \mathbb{R}_+^n \} \) of operators on a Banach space. In the main Theorems 4.1 and 5.2 we describe shift-invariant operators for scalar and operator semigroups, respectively.

It is essential that hereinafter functions and distributions are defined on the cone \( \mathbb{R}_+^n \) (instead of \( \mathbb{R}^n \)). As a consequence, the shift-invariant property is considered under the cross-correlation (instead of the convolution). Further, we will consider only the case of Roumieu ultradistributions.

Let \( \mathcal{G}_+ \) be the convolution algebra of Roumieu ultradistributions with supports in the positive cone \( \mathbb{R}_+^n \) and \( \mathcal{G}'_+ \) be its predual space of Gevrey ultradifferentiable functions with compact supports in \( \mathbb{R}_+^n \). We consider the cross-correlation operator

\[
K_f : \mathcal{G}_+ \ni \varphi \mapsto K_f \varphi, \quad [K_f \varphi](s) := \langle f, T_s \varphi \rangle, \quad f \in \mathcal{G}_+.
\]

In Theorem 4.1 we prove that the algebra \( \mathcal{G}'_+ \) has the isomorphic representation

\[
\mathcal{G}'_+ \ni f \mapsto K_f \in \mathcal{L}(\mathcal{G}_+)
\]
onto the commutant $[T]^c$ of the shift semigroup $T$ (see formula (3.2)) in space of linear continuous operators on $G_+$.

Let us consider the set of $n$-parameter contraction $(C_0)$-semigroups $\{U_t : t \in \mathbb{R}^n_+\}$ on a Banach space $E$ and a set $A$ of their generators. In section 5 we investigate the isomorphic representation of the algebra $G_+'$

$$G_+' \ni f \mapsto F \circ (I \otimes K_f) \circ F^{-1}, \quad F : E \otimes_p G_+ \ni x \mapsto \hat{x} \in \hat{G},$$

where $E \otimes_p G_+$ is the complete projective tensor product. Here $\hat{G}$ denotes the subspace of $E$-valued functions

$$\hat{x} : A \ni A \mapsto \hat{x}(A) \in E, \quad \hat{x}(A) = \int_{\mathbb{R}^n_+} U_t(A)x(t) \, dt,$$

determined by the Hille-Phillips calculus, where $U_t(A)$ means that the semigroup is generated by operator $A$. In Theorem 5.2 we prove that there exists an algebraic isomorphism from $G_+'$ into the commutant of the semigroup $F \circ (I \otimes T) \circ F^{-1}$.

2. ULTRADIFFERENTIABLE FUNCTIONS AND ULTRADISTRIBUTIONS

We use the short notations: $t^k = t_1^{k_1} \cdots t_n^{k_n}, k^{\beta} = k_1^{\beta_1} \cdots k_n^{\beta_n}, |k| = k_1 + \cdots + k_n$ for any $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+, t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and a real $\beta > 1$. Let $\partial^k = \partial_1^{k_1} \cdots \partial_n^{k_n}$, where $\partial_j^{k_j} = \partial_j^{k_j}/\partial t_j^{k_j}$ ($j = 1, \ldots, n$). For $\mu, \nu \in \mathbb{R}^n$, the relation $\mu < \nu$ (resp. $\mu > \nu$) means that $\mu_1 < \nu_1, \ldots, \mu_n < \nu_n$ (resp. $\mu_1 > \nu_1, \ldots, \mu_n > \nu_n$). Let $[\mu, \nu] = \times_{j=1}^n [\mu_j, \nu_j]$ and $(\mu, \nu] = \times_{j=1}^n (\mu_j, \nu_j]$ for any $\mu < \nu$. In what follows $t \to \infty$ (resp. $t \to 0$) means that $t_j \to \infty$ (resp. $t_j \to 0$) for all $j = 1, \ldots, n$. By $\mathbb{R}^n_+ = \times_{j=1}^n [0, \infty)$ and $\text{int} \mathbb{R}^n_+ = \times_{j=1}^n (0, \infty)$ we denote the positive cone and its interior, respectively.

Let $\mathcal{E}(\mathbb{R}^n_+)$ be the class of infinitely smooth complex valued functions $\varphi$ defined on $\text{int} \mathbb{R}^n_+$ such that $\partial^k \varphi$ for all $k \in \mathbb{Z}^n_+$ have continuous limits at the boundary of $\mathbb{R}^n_+$. Fix a $\beta > 1$. A function $\varphi \in \mathcal{E}(\mathbb{R}^n_+)$ is called (see [11]) Gevrey ultradifferentiable if for every interval $[\mu, \nu] \subset \text{int} \mathbb{R}^n_+$ there exist constants $h > 0$ and $C > 0$ such that the inequality

$$\sup_{t \in [\mu, \nu]} |\partial^k \varphi(t)| \leq C h^{|k|} k^{\beta}$$

holds for all $k \in \mathbb{Z}^n_+$.

We denote by $\mathcal{E}^\beta_+ = \mathcal{E}^\beta(\mathbb{R}^n_+)$ the vector space of all Gevrey ultradifferentiable functions on $\text{int} \mathbb{R}^n_+$. By virtue of Seeley’s theorem [16] every function $\varphi \in \mathcal{E}^\beta_+$ has an infinitely smooth extension $\hat{\varphi}$ on $\mathbb{R}^n$ such that $\partial^k \hat{\varphi}(t) = \partial^k \varphi(t)$ for all $k \in \mathbb{Z}^n_+$ and $t \in \mathbb{R}^n_+$. Denote by $\mathcal{G}^\beta_+ \subset \mathcal{E}^\beta_+$ the subspace of Gevrey ultradifferentiable functions with compact supports.

In the sequel, to simplify notations we will write $\mathcal{G}^\beta_+$ instead of $\mathcal{G}^\beta_+$ omitting the fixed $\beta$.

For a fixed $h > 0$, we consider the subspace $\mathcal{G}^\beta_+ \subset \mathcal{G}^\beta_+$ of all functions $\varphi$ vanishing outside $[0, \nu]$ for which there exists a constant $C = C(\varphi) > 0$ such that $\sup_{t \in [0, \nu]} |\partial^k \varphi(t)| \leq C h^{|k|} k^{\beta}$ holds for all $k \in \mathbb{Z}^n_+$. As is known [11, p. 38], the space $\mathcal{G}^\beta_+$ endowed with the norm

$$\|\varphi\|_{\mathcal{G}^\beta_+} = \sup_{k \in \mathbb{Z}^n_+, t \in [0, \nu]} \frac{|\partial^k \varphi(t)|}{h^{|k|} k^{\beta}}$$

is complete and the inclusions $\mathcal{G}^\beta_+ \hookrightarrow \mathcal{G}^\beta_+$ with $h < l$ are compact. Moreover, if $\mu \prec \nu$ then $\mathcal{G}^\mu_+ \subset \mathcal{G}^\nu_+$ is a closed subspace in $\mathcal{G}^\nu_+$ [11, p.40]. We equip the space $\mathcal{G}^\beta_+ = \bigcup_{\nu > 0, h > 0} \mathcal{G}^\beta_+$ with the inductive limit topology under the compact inclusions $\mathcal{G}^\mu_+ \hookrightarrow \mathcal{G}^\nu_+$. Hence,

(2.1) $\mathcal{G}^\beta_+ \simeq \lim_{\nu, h \to \infty} \text{ind} \mathcal{G}^\beta_+$
is a nuclear (DFS)-space [11, p. 28]. Note that the space $G^+_-$ is a topological algebra with respect to the pointwise multiplication and $\|\varphi \psi\|_{G^+_{p+t}} \leq \|\varphi\|_{G^+_p} \|\psi\|_{G^+_t}$ (see [11, p. 41]).

The strong dual $G'_+$ of $G_+$ is a nuclear (FS)-space. The space $G'_+$ consists of Roumieu ultradistributions on $\mathbb{R}^n$ supported by $\mathbb{R}^+_n$ and it is a convolution algebra with Dirac $\delta$ as the unit element. Remind, that the convolution is defined by $(f * g, \varphi) = (f, g * \varphi)$, where $(g * \varphi)(x) := \langle g, \varphi(\cdot + x) \rangle$ with $g, \varphi \in G'_+$ and $\varphi \in G_+$ (see e.g. [11, 2.5]).

Let $(f, \varphi)$ denote the value of $f \in G'_+$ on $\varphi \in G_+$. Sometimes we write $(f(t), \varphi(t))$, where the “argument” of an ultradistribution $f$ means the variable of a function $\varphi$ on which the functional $f$ acts.

### 3. CROSS-CORRELATION AND SHIFT SEMIGROUPS

For a linear topological space $X$, we denote by $\mathcal{L}(X)$ the space of all continuous linear operators on $X$. We write $I$ for the identity operator in $\mathcal{L}(X)$. We endow $\mathcal{L}(X)$ with the locally convex topology of uniform convergence on bounded subsets of $X$. We define the commutant of a subset $S \subset \mathcal{L}(X)$ to be $S^c := \{ B \in \mathcal{L}(X) : BA = AB, \forall A \in S \}$.

An $n$-parameter family $\{U_t : t \in \mathbb{R}^+_n\}$ of bounded linear operators on a complex Banach space $(E, \|\cdot\|)$ is called an $n$-parameter semigroup of operators (see [1, 7]) if it is a mapping $U : \mathbb{R}^+_n \ni t \mapsto U_t \in \mathcal{L}(E)$ such that $U_0 = I$ and $U_{t+s} = U_t \circ U_s$ for all $t, s \in \mathbb{R}^+_n$.

We call a semigroup $\{U_t : t \in \mathbb{R}^+_n\}$ strongly continuous (or $C_0$-semigroup) if the equality $\lim_{t \to 0} \|U_t x - x\| = 0$ holds for all $x \in E$.

For any $n$-parameter semigroup $\{U_t : t = (t_1, \ldots, t_n) \in \mathbb{R}^+_n\}$ we define the marginal one-parameter semigroups $\{U^{(j)}_{t_j} : t_j \in \mathbb{R}^+_1\}$, which commute with each other, where

$$U^{(j)}_{t_j} : [0, \infty) \ni t_j \mapsto U_{(0, \ldots, 0, t_j, 0, \ldots, 0)} \in \mathcal{L}(E), \quad j = 1, \ldots, n.$$  

Any $n$-parameter semigroup $U_t$ may be represented as a composition of the associated marginal one-parameter semigroups, i.e. $U_t = U^{(1)}_{t_1} \circ \cdots \circ U^{(n)}_{t_n}$.

For all $j = 1, \ldots, n$, let the generator $A_j$ of the $j$th marginal semigroup $\{U^{(j)}_{t_j} : t_j \in \mathbb{R}^+_1\}$ be defined as

$$A_j x := \lim_{t_j \to +0} t_j^{-1} (U^{(j)}_{t_j} x - x) = \partial^1_{t_j} U^{(j)}_{t_j} x \big|_{t_j = +0}, \quad x \in \mathcal{D}(A_j),$$

where $\mathcal{D}(A_j)$ consists of all $x \in E$ for which the above limit exists.

We denote the generator of $\{U_t : t \in \mathbb{R}^+_n\}$ by $A := (A_1, \ldots, A_n)$. Let us denote $\mathcal{D}(A) := \bigcap_{j=1}^n \mathcal{D}(A_j)$. If $U_t$ is a $(C_0)$-semigroup then the following properties hold (see [1, Propositions 1.1.8-9]):

(i) if $x \in \mathcal{D}(A_j)$ then $U_t x \in \mathcal{D}(A_j)$ and $A_j U_t x = U_t A_j x$;
(ii) $U_t x \in \mathcal{D}(A)$ for any $x \in E, t \in \text{int} \mathbb{R}^+_n$, and $\mathcal{D}(A)$ is dense in $E$;
(iii) $A_i A_j x = A_j A_i x, \quad (i, j = 1, \ldots, n)$ for all $x \in \mathcal{D}(A)$.

We also consider the $n$-parameter $(C_0)$-semigroup of shifts over the space $G_+$,

$$T : \mathbb{R}^+_n \ni s \mapsto T_s \in \mathcal{L}(G_+), \quad T_s \varphi(t) = \varphi(t + s), \quad t \in \mathbb{R}^+_n, \quad \varphi \in G_+.$$  

It is obvious that $\text{supp} \varphi(\cdot + s) = \text{supp} \varphi - s$ for any function $\varphi$, defined on $\mathbb{R}^n$. Therefore, $\text{supp} T_s \varphi = (\text{supp} \varphi - s) \cap \mathbb{R}^+_n$ with $s \in \mathbb{R}^+_n$. Hence, the inequalities $\|T_s \varphi\|_{G^+_h} \leq \|\varphi\|_{G^+_0}$ for all $s \in \mathbb{R}^+_n$ and all indexes $h, \nu$ hold. Clearly, each derivative $\partial^j (j = 1, \ldots, n)$ generates the corresponding marginal one-parameter semigroup

$$T^{(j)}_{s_j} : [0, \infty) \ni s_j \mapsto T_{(0, \ldots, 0, s_j, 0, \ldots, 0)} \in \mathcal{L}(G_+).$$
The regular property (see [14]) of the inductive limit (2.1) implies that the semigroup $T$ is equibounded. So, $T$ is equicontinuous by the Banach-Steinhaus theorem.

**Definition 3.1.** For any ultradistribution $f \in \mathcal{G}_+^\prime$, the cross-correlation operator over the space $\mathcal{G}_+$ is defined to be

$$K_f : \mathcal{G}_+ \ni \varphi \mapsto K_f \varphi, \quad K_f \varphi(s) := \langle f, T_s \varphi \rangle, \quad s \in \mathbb{R}_+^n.$$ 

Let $E \otimes_p \mathcal{G}_+$ be a completion of the algebraic tensor product $E \otimes \mathcal{G}_+$ with respect to the projective tensor topology. We can treat elements of $E \otimes_p \mathcal{G}_+$ as an $E$-valued ultradifferentiable functions $x : t \mapsto x(t)$ with compact supports in $\mathbb{R}_+^n$. From the known (see [6]) Grothendieck’s isomorphism $E \otimes_p \lim \text{ind} \, \mathcal{G}_+^h \simeq \lim \text{ind} \, E \otimes_p \mathcal{G}_+^h$ it follows the isomorphism

$$E \otimes_p \mathcal{G}_+ \simeq \lim \text{ind} \, E \otimes_p \mathcal{G}_+^h.$$ 

So, for every $x \in E \otimes_p \mathcal{G}_+$ there exist $\nu \in \mathbb{R}_+^n$ and $h > 0$ such that $x \in E \otimes_p \mathcal{G}_+^h$, where each space $E \otimes_p \mathcal{G}_+^h$ is equipped with the norm

$$\|x\|_{E \otimes_p \mathcal{G}_+^h} = \sup_{k \in \mathbb{Z}_+^n, \, t \in [0, \nu]} \|\partial^k x(t)\|_{h^{|k|}k!}.$$ 

Hence, from theorem about representation of projective tensor product (see [14, Th. III. 6.4]) it follows that every $x \in E \otimes_p \mathcal{G}_+$ can be expanded (in general, not uniquely) in an absolutely convergent series

$$x = \sum_{j \in \mathbb{N}} \lambda_j x_j \otimes \varphi_j, \quad \lambda_j \in \mathbb{C}, \quad x_j \in E, \quad \varphi_j \in \mathcal{G}_+^h,$$

for some $\nu \in \mathbb{R}_+^n$ and $h > 0$, where $\sum_j |\lambda_j| < \infty$ and the sequences $\{x_j\}$ and $\{\varphi_j\}$ are convergent to zero in the corresponding spaces.

Let $K \in \mathcal{L}(\mathcal{G}_+)$. Using (3.3), we can define the tensor product $I \otimes K \in \mathcal{L}(E \otimes_p \mathcal{G}_+)$ as follows

$$(I \otimes K)x = \sum_{j \in \mathbb{N}} \lambda_j x_j \otimes K \varphi_j.$$ 

In case $K = T_s$ we often use the short notation $x(t + s)$ instead of $(I \otimes T_s)x(t)$. We can now define analogously the action

$$(f, x) := \sum_{j \in \mathbb{N}} \lambda_j x_j \langle f, \varphi_j \rangle$$

for any ultradistribution $f \in \mathcal{G}_+^\prime$ and $x \in E \otimes_p \mathcal{G}_+$. It is well-known [14, III.6.4] that these definitions are independent of representations of elements $x \in E \otimes_p \mathcal{G}_+$ in the form (3.3).

We say that an operator $I \otimes K$ with $K \in \mathcal{L}(\mathcal{G}_+)$ is invariant with respect to shift operators $I \otimes T = \{I \otimes T_s : s \in \mathbb{R}_+^n\}$ if

$$I \otimes (K \circ T_s) = I \otimes (T_s \circ K) \quad \text{for all} \quad s \in \mathbb{R}_+^n.$$ 

**Definition 3.2.** For any ultradistribution $f \in \mathcal{G}_+^\prime$, the cross-correlation operator over the space $E \otimes_p \mathcal{G}_+$ is defined to be

$$I \otimes K_f : E \otimes_p \mathcal{G}_+ \ni x \mapsto (I \otimes K_f)x.$$ 

Here $(I \otimes K_f)x(s) = \sum_{j \in \mathbb{N}} \lambda_j x_j \otimes K_f \varphi_j = \langle f, (I \otimes T_s)x \rangle$ for all $s \in \mathbb{R}_+^n$ by continuity of a functional $f \in \mathcal{G}_+^\prime$. 

4. OPERATORS COMMUTING WITH SHIFT SEMIGROUPS

The next theorem, based on the cross-correlation notion, is a generalization of known structure theorems about shift-invariant operators.

**Theorem 4.1.** (i) The mapping \( K : \mathcal{G}_+^0 \ni f \mapsto K_f \in \mathcal{L}(\mathcal{G}_+) \) produces a topological isomorphism from the convolution algebra \( \mathcal{G}_+^0 \) onto the commutant \([T]^c\) of the shift semigroup \( T \), i.e.

\[
K_{f*g} = K_f \circ K_g, \quad f, g \in \mathcal{G}_+^0,
\]

where \(*\) denotes the convolution in \( \mathcal{G}_+^0 \). In particular, \( K_\delta \) is the identity in \( \mathcal{L}(\mathcal{G}_+) \).

(ii) For any \( f \in \mathcal{G}_+^0 \) the operator \( I \otimes K_f \) is invariant with respect to shift operators \( I \otimes T \). Conversely, for any \( K \in \mathcal{L}(\mathcal{G}_+) \) such that \( I \otimes K \) is invariant with respect to \( I \otimes T \) there exists a unique \( f \in \mathcal{G}_+^0 \) such that for all \( x \in E \otimes_p \mathcal{G}_+^0 \),

\[
K = K_f \quad \text{and} \quad (I \otimes K)x = (I \otimes K_f)x.
\]

**Proof.** (i) Check that \( K_f \) is a linear continuous operator. It is clear that

\[
\text{supp } K_f \varphi \neq \emptyset \iff \text{supp } f \cap \text{supp } \varphi(\cdot + s) \neq \emptyset \iff \exists t_0 \in \text{supp } f \cap \text{supp } \varphi(\cdot + s).
\]

Since \( t_0 \in \text{supp } \varphi(\cdot + s) \iff t_0 + s \in \text{supp } \varphi \iff s \in \text{supp } \varphi - t_0, s \in \text{supp } \varphi - \text{supp } f \). So, \( \text{supp } K_f \varphi \subset (\text{supp } \varphi - \text{supp } f) \cap \mathbb{R}_+^n \subset [0, \nu] \) for some \( \nu > 0 \).

Prove that \( K_f \in \mathcal{L}(\mathcal{G}_+) \). Let \( \{\varphi_m\} \subset \mathcal{G}_+ \) be a sequence for which there exists \([0, \nu] \subset \mathbb{R}_+^n\) such that \( \text{supp } \varphi_m \subset [0, \nu] \) for all \( m \in \mathbb{N} \) and

\[
\lim_{m \to \infty} \sup_{t \in [0, \nu]} \frac{|\partial^k \varphi_m(t)|}{h^{|k|} k^{|k\beta|}} = 0
\]

for all \( k \in \mathbb{Z}_+^n \) with some \( h > 0 \). From the continuity of \( f \in \mathcal{G}_+^0 \) and \( T_s \in \mathcal{L}(\mathcal{G}_+) \) it follows that \( \partial^k K_f \varphi = K_f \partial^k \varphi \) for all \( k \in \mathbb{Z}_+^n \). So, we obtain

\[
\lim_{m \to \infty} \sup_{t \in [0, \nu]} \frac{|\partial^k K_f \varphi_m(t)|}{h^{|k|} k^{|k\beta|}} = \left| \left\langle f, \lim_{m \to \infty} \sup_{t \in [0, \nu]} \frac{\partial^k \varphi_m(\cdot + t)}{h^{|k|} k^{|k\beta|}} \right\rangle \right| = 0
\]

for all \( k \in \mathbb{Z}_+^n \) and some \( h > 0 \). Using the isomorphism (2.1), we obtain \( K_f \in \mathcal{L}(\mathcal{G}_+) \).

The following equalities

\[
(K_f T_s \varphi)(t) = \langle f(r), \varphi(r + t + s) \rangle = T_s \langle f(r), \varphi(r + t) \rangle = (T_s K_f \varphi)(t)
\]

hold for all \( t, s \in \mathbb{R}_+^n \) and \( \varphi \in \mathcal{G}_+^0 \). Hence, for any \( f \in \mathcal{G}_+^0 \) we have \( K_f \in [T]^c \).

Let now \( K \in \mathcal{L}(\mathcal{G}_+) \) be an arbitrary operator with the property

\[
(KT_s) \varphi(t) = (T_s K) \varphi(t), \quad \varphi \in \mathcal{G}_+, \quad t, s \in \mathbb{R}_+^n.
\]

It is easy to see that the functional \( \langle f_0, \varphi \rangle := (K \varphi)(0) \) belongs to \( \mathcal{G}_+^0 \). By cross-correlation definition \( (Kf_0 \varphi)(0) = \langle f_0, \varphi \rangle \), i.e. \( (K \varphi)(0) = (Kf_0 \varphi)(0) \) for all \( \varphi \in \mathcal{G}_+^0 \). Substituting \( T_s \varphi \) instead of \( \varphi \) and using the property (4.8), we get that \( K = Kf_0 \) and hence that the image of \( K \) coincides with the commutant \([T]^c\).

If \( K_f \varphi(s) = \langle f, T_s \varphi \rangle = 0 \) for all \( \varphi \in \mathcal{G}_+^0 \) then \( f = 0 \). Hence, the mapping \( K \) is injective.

Since \( \mathcal{G}_+ \) is a Montel space [14, IV.5], the topologies on \( \mathcal{L}(\mathcal{G}_+) \) of uniform convergence on compacta and on bounded sets coincide. By barreledness of the spaces \( \mathcal{G}_+^0 \) and \( \mathcal{G}_+ \) [14, II.7] the map \( \mathcal{G}_+^0 \times \mathcal{G}_+ \ni (f, \varphi) \mapsto K_f \varphi \in \mathcal{G}_+^0 \) is equicontinuous, because it is separately continuous. Hence, \( K \) is continuous. Moreover, \( K \) has the closed image \([T]^c\). Since \( \mathcal{G}_+ \) is a nuclear \((DFS)\)-space, we have \( \mathcal{L}(\mathcal{G}_+) \simeq \mathcal{G}_+ \otimes_p \mathcal{G}_+^0 \) (see [14, IV.9.4]) where \( \mathcal{G}_+^0 \) is a Fréchet space as a strong dual of \((DFS)\)-space. So,

\[
\mathcal{G}_+ \otimes_p \mathcal{G}_+^0 \simeq \lim_{h, \nu \to \infty} \mathcal{G}_+^h \otimes_p \mathcal{G}_+^0
\]
by virtue of the isomorphism (2.1). On the other hand, in $G'_+$ there exists a countable base of closed absolutely convex bounded sets $\{B_n\}$ such that $G'_+ = \bigcup_{n \in \mathbb{N}} B_n$, where $B_n := \mathbb{C} \cdot B_n$ is a subspace with the norm $\|x\|_n = \inf \{\|x\| : x \in \lambda B_n\}$. From the completeness of $G'_+$ and the closedness of $B_n$ it follows that $B_n$ is a Banach space for all $n \in \mathbb{N}$. From boundedness of $B_n$ it follows that the embeddings $B_n \hookrightarrow G'_+$ are continuous [14, II.8.4]. So, the identical mapping $\lim \text{ind}_{n \to \infty} B_n \rightarrow G'_+$ is continuous. As a consequence, we obtain the isomorphism

$$G_+ \otimes_p G'_+ \simeq \lim \text{ind}_{h,\nu,n \to \infty} G^h_+ \otimes_p B_n,$$

i.e. $\mathcal{L}(G_+)$ is ultrabornological space [10]. Now, Open Mapping Theorem (see [13]) implies that $K$ is a topological isomorphism from $G'_+$ onto $[T]^c$.

Check the equality (4.6). The convolution definition implies

$$(Kf*g)(t) = \langle f \ast g, T_t \varphi \rangle = \langle f(r), \langle g(s), \varphi(t + r + s) \rangle \rangle = \langle f, T_t(K_g \varphi) \rangle = (K_fK_g) \varphi(t),$$

with $t, r, s \in \mathbb{R}^n$. In particular, $K_f \circ K_\delta = K_f \ast \delta = K_f = \delta \ast K_f$ for all $f \in G'_+$, so $K_\delta$ is the identity.

(ii) The equality $\sum_j \lambda_j x_j \otimes (K_fT_s) \varphi_j = \sum_j \lambda_j x_j \otimes (T_sK_f) \varphi_j$ holds for all $f \in G'_+$ and $x \in E \otimes_p G_+$ via (3.3) and (4.7). Therefore, $I \otimes (K_f \circ T_s) = I \otimes (T_s \circ K_f)$, i.e. $I \otimes K_f$ is invariant with respect to the shift operators $I \otimes T$.

Conversely, let $K \in \mathcal{L}(G_+)$ be an operator such that $I \otimes (K \circ T_s) = I \otimes (T_s \circ K)$ for all $s \in \mathbb{R}^n$. For any $\varphi \in G_+$, let $f_0 : \varphi \mapsto (K \varphi)(0)$. Definitions (3.4) and (3.5) imply

$$[(I \otimes K)x](0) = (f_0, x) = [(I \otimes Kf_0)x](0).$$

Substituting $(I \otimes T_s)x$ instead of $x$ and using that $I \otimes K$ is invariant with respect to the shift operators $I \otimes T$, we obtain $(I \otimes K)x = (I \otimes Kf_0)x$ for all $x \in E \otimes_p G_+$. Consequently, $K = Kf_0$. \hfill \square

5. Shift-invariant operators commuting with operator semigroups

Consider the set of $n$-parameter contraction $C_0$-semigroups $\{U_t : t \in \mathbb{R}^n_+\}$ on a complex Banach space $(E, \| \cdot \|)$, i.e. semigroups satisfying the condition

$$\sup_{t \in \mathbb{R}^n_+} \|U_t\|_{\mathcal{L}(E)} \leq 1,$$

and let $A$ be the set of their generators. To emphasize the fact that a semigroup $\{U_t : t \in \mathbb{R}^n_+\}$ is generated by an operator $A 
\in A$ we will use the notation $\{U_t(A) : t \in \mathbb{R}^n_+\}$ for the semigroup. Consider the space

$$\hat{G} = \{\hat{x} : A \longrightarrow E : x \in E \otimes_p G_+\}$$

of $E$-valued functions

$$\hat{x} : A \ni A \mapsto \hat{x}(A) \in E \quad \text{with} \quad \hat{x}(A) = \int_{\mathbb{R}^n_+} U_t(A)x(t) \, dt,$$

determined by the Hille-Phillips calculus [7, Chapter 15], where the integral is understood in the Bochner sense. The integral in (5.10) is well-defined, because the integrand is a continuous $E$-valued function $t \mapsto U_t(A)x(t)$ with a compact support.

Let us determine the linear mapping

$$\mathcal{F} : E \otimes_p G_+ \ni x \mapsto \hat{x} \in \hat{G}.$$

If the assumption (5.9) holds, then the mapping $\mathcal{F}$ is an isomorphism by virtue of [7, Theorem 15.2.1]. Indeed, the semigroups $\mathbb{R}^n_+ \ni t \mapsto e^{-(\lambda,t)}I$ with $\Re \lambda \in \text{int} \mathbb{R}^n_+$ satisfy
the condition (5.9). Therefore, their generators $-\lambda I$ belong to $A$. Note that $\hat{x}(-\lambda I) = \int_{\mathbb{R}^+} e^{-\lambda t} x(t) \, dt$ is the Laplace transform of an $E$-valued function $x \in E \otimes_p \mathcal{G}_p$. Particularly, it follows that if $\hat{x} \equiv 0$ then $x \equiv 0$, i.e., $\text{Ker} \, \mathcal{F} = \{0\}$.

We endow the space $\hat{G}$ with the strongest locally convex topology induced by $\mathcal{F}$. Namely, let $\mathcal{G}^h_p(E) = \{ \hat{x} : x \in E \otimes_p \mathcal{G}^h_p \}$ be a Banach space endowed with the inductive topology under the mapping $E \otimes_p \mathcal{G}^h_p \ni x \mapsto \hat{x}$ with fixed $h$ and $\nu$. Then from (2.1) it follows that $\hat{G}$ has the structure of the inductive limit $\hat{G} = \lim \text{ind}_{h, \nu \to \infty} \mathcal{G}^h_p(E)$ under the continuous embeddings $\mathcal{G}^h_p(E) \ni \mathcal{G}^\mu_p(E)$ with $h < l$ and $\nu < \mu$. So the mapping $F$ is a topological isomorphism.

Consider the $n$-parameter semigroup on the space $\hat{G}$

\begin{equation}
\hat{T} : \mathbb{R}_+^n \ni s \mapsto \hat{T}_s \in \mathcal{L}(\hat{G}), \quad \hat{T}_s = \mathcal{F} \circ (I \otimes T_s) \circ \mathcal{F}^{-1},
\end{equation}

where $\mathcal{F}^{-1}$ means the inverse map of $\mathcal{F}$. Since $\mathcal{F}$ is a topological isomorphism and the semigroup $\mathbb{R}_+^n \ni s \mapsto (I \otimes T_s)x \in E \otimes_p \mathcal{G}_p$ is continuous for all $x \in E \otimes_p \mathcal{G}_p$, we have that the semigroup $\hat{T}$ on $\hat{G}$ has the $(C_0)$-property and its generator is densely defined.

Theorem 5.2. The mapping

$$\mathcal{G}^\prime_p \ni f \mapsto \hat{K}_f \in \mathcal{L}(\hat{G}), \quad \hat{K}_f = \mathcal{F} \circ (I \otimes K_f) \circ \mathcal{F}^{-1},$$

is an algebraic isomorphism of the convolution algebra $\mathcal{G}^\prime_p$ and the subalgebra of all operators $\hat{K} = \mathcal{F} \circ (I \otimes K) \circ \mathcal{F}^{-1}$ for some $K \in \mathcal{L}(\mathcal{G}_p)$ in the commutant $[\hat{T}]^c$ on $\hat{G}$. In particular, the equality $\hat{K}_{fg} = \hat{K}_f \circ \hat{K}_g$ for all $f, g \in \mathcal{G}^\prime_p$ holds and $\hat{K}_\delta$ is the identity in $\mathcal{L}(\hat{G})$.

Proof. For any $f \in \mathcal{G}^\prime_p$ the diagram

$$
\begin{array}{ccc}
\hat{G} & \ni & \hat{x} \\
\mathcal{F} \uparrow & & \mathcal{F} \uparrow \\
E \otimes_p \mathcal{G}_p & \ni & x \\
\mathcal{F} \circ (I \otimes K_f) \circ \mathcal{F}^{-1} & \mathcal{F} \circ (I \otimes K_f) \circ \mathcal{F}^{-1} & \mathcal{F} \circ (I \otimes K_f) \circ \mathcal{F}^{-1}
\end{array}
$$

is commutative. Continuity of the mappings $I \otimes K_f$ and $\mathcal{F}$ and openness of the mapping $\mathcal{F}^{-1}$ imply that $\hat{K}_f \in \mathcal{L}(\hat{G})$. It follows that the equalities

$$[\mathcal{F}(I \otimes K_f)](A) = \int_{\mathbb{R}_+^n} U_t(A)(I \otimes K_f)x(t) \, dt = \hat{K}_f \hat{x}(A)$$

are valid for all $A \in A$. Consequently, the equalities

$$\hat{K}_f \hat{T}_r \hat{x}(A) = \int_{\mathbb{R}_+^n} U_t(A)(I \otimes K_f)x(t + r) \, dt = \hat{T}_r \hat{K}_f \hat{x}(A)$$

hold for all $r \in \mathbb{R}_+^n$ and $\hat{x} \in \hat{G}$. Hence, for any $f \in \mathcal{G}^\prime_p$ we have that $\hat{K}_f$ belongs to the commutant of the semigroup $\hat{T}$ in $\mathcal{L}(\hat{G})$.

Conversely, let $\hat{K} = \mathcal{F} \circ (I \otimes K) \circ \mathcal{F}^{-1}$ belongs to the commutant $[\hat{T}]^c$ of $\hat{T}$. Then

$$\mathcal{F} \circ (I \otimes (K \circ T_s)) \circ \mathcal{F}^{-1} = \mathcal{F} \circ (I \otimes K) \circ (I \otimes T_s) \circ \mathcal{F}^{-1}$$

$$= \mathcal{F} \circ (I \otimes K) \circ \mathcal{F}^{-1} \circ \mathcal{F} \circ (I \otimes T_s) \circ \mathcal{F}^{-1} = \hat{K} \circ \hat{T}_s$$

$$= \hat{T}_s \circ \hat{K} = \mathcal{F} \circ (I \otimes T_s) \circ \mathcal{F}^{-1} \circ \mathcal{F} \circ (I \otimes K) \circ \mathcal{F}^{-1}$$

$$= \mathcal{F} \circ (I \otimes T_s) \circ (I \otimes K) \circ \mathcal{F}^{-1} = \mathcal{F} \circ (I \otimes (T_s \circ K)) \circ \mathcal{F}^{-1},$$
therefore $K \in [T]^c$. By Theorem 4.1 there exists a unique $f \in G'_+$ such that $K = K_f$, i.e.,
\[ \hat{K} = \hat{K}_f = \mathcal{F}(I \otimes K_f) = \hat{f} \hat{K} = \hat{K}_f \hat{f} \in \mathcal{G}. \]
Hence, $\hat{K} = \hat{K}_f$.

Since $K_\delta = I$, we obtain $\hat{K}_\delta = \mathcal{F} \circ (I \otimes K_\delta) \circ \mathcal{F}^{-1} = \mathcal{F} \circ \mathcal{F}^{-1}$, so the operator $\hat{K}_\delta$ is the identity in $L'(\mathcal{G})$.

From the properties of cross-correlation (see Theorem 4.1) it follows
\[ \hat{K}_f \hat{K}_g \hat{x}(A) = \int_{\mathbb{R}^n_+} U_t(A) (I \otimes K_f K_g) x(t) \, dt = \int_{\mathbb{R}^n_+} U_t(A) (I \otimes K_{f*g}) x(t) \, dt = \hat{U}_{f*g} \hat{x}(A), \]
so the mapping $f \mapsto \hat{K}_f$ is an algebraic isomorphism.

In the same way as in the proof of Theorem 4.1 we can prove that the algebraic isomorphism in Theorem 5.2 is topological.

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**References**

[2] Colzani, L. and Sjörgren, P., Translation-invariant operators on Lorentz spaces $L(1, q)$ with $0 < q < 1$, Studia Math., 132 (1999), No. 2, 101–124
[12] Lopushansky, O. and Sharyn, S., Polynomial ultradistributions on cone $\mathbb{R}^d_+$, Topology, 48 (2009), No. 2-4, 80–90

University of Rzeszów
Faculty of Mathematics and Natural Sciences
16A Reitana, 35-959 Rzeszów, Poland
E-mail address: ovlopusz@ur.edu.pl

Department of Mathematics and Computer Sciences
Precarpathian National University
57 Shevchenka, 76-018 Ivano-Frankivsk, Ukraine
E-mail address: sharyn.sergii@gmail.com