

From a Dieudonné theorem concerning the Cauchy problem to an open problem in the theory of weakly Picard operators

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ABSTRACT.

Let (X, d) be a complete metric space and let $f : X \rightarrow X$ be a self operator. In this paper we study the following two problems:

Problem 1. Let f be such that its fixed points set is a singleton, i.e., $F_f = \{x^*\}$. Under which conditions the next implication does hold:

f is asymptotically regular $\Rightarrow f$ is a Picard operator?

Problem 2. Let f be such that, $F_f \neq \emptyset$. Under which conditions the following implication does hold:

f is asymptotically regular $\Rightarrow f$ is a weakly Picard operator?

The case of operators defined on a linear L^* -space is also studied.

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