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Some strong differential subordinations using a differential operator

LORIANA ANDREI and MITROFAN CHOBAN

ABSTRACT. In the present paper we study the operator $RD^n_{\lambda,\alpha}f(z,\zeta)$ defined by using the extended Ruscheweyh derivative $R^nf(z,\zeta)$ and the extended generalized Sălăgean operator $D^n_\lambda f(z,\zeta)$, as $RD^n_{\lambda,\alpha} : \mathcal{A}^*_\zeta \to \mathcal{A}^*_\zeta$, $RD^n_{\lambda,\alpha}f(z,\zeta) = (1-\alpha)R^nf(z,\zeta) + \alpha D^n_\lambda f(z,\zeta)$, where $\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\}$ is the class of normalized analytic functions with $\mathcal{A}^*_{1\zeta} = \mathcal{A}^*_\zeta$. We obtain several strong differential subordinations regarding the extended operator $RD^n_{\lambda,\alpha}$. Some examples are presented.

1. INTRODUCTION

Complex-valued analytic functions have many nice properties that are not necessarily true for real-valued functions. One of the basic problems in the geometric function theory is the solution of various extremal problems and the study of certain subclasses of holomorphic complex-valued functions which are defined by differential subordination, extremal functional conditions and differential operators. In 1935 G. M. Goluzin initiated the theory of differential subordination of functions. Then distinct aspects of the subordinations of functions were considered by R. M. Robinson, T. J. Suffridge, D. J. Hallenbeck, S. T. Ruscheweyh, S. S. Miller, P. T. Mocanu, J. A. Antonino, S. Romaguera, G. St. Sălăgean, and others (see [13, 10, 11, 1]).

Denote by $U = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc of the complex plane, by $\overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ the closed unit disc of the complex plane and by $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta)\, z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\},$$

with $\mathcal{A}^*_{1\zeta} = \mathcal{A}^*_\zeta$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge 2$, and

$$\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta)\, z^n + a_{n+1}(\zeta)\, z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\},$$

for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge n$.

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Corresponding author: Mitrofan Choban; mmchoban@gmail.com

Definition 1.1. For $\lambda \geq 0$ and $m \in \mathbb{N}$ the extended generalized Sălăgean operator $D_\lambda^m : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$ is defined by:

$$\begin{aligned} D_\lambda^0 f(z,\zeta) &= f(z,\zeta) \\ D_\lambda^1 f(z,\zeta) &= D_\lambda f(z,\zeta) = (1-\lambda) f(z,\zeta) + \lambda z f_z'(z,\zeta) \\ &\dots \\ D_\lambda^{m+1} f(z,\zeta) &= (1-\lambda) D_\lambda^m f(z,\zeta) + \lambda z \left(D_\lambda^m f(z,\zeta)\right)_z' = \\ &= D_\lambda \left(D_\lambda^m f(z,\zeta)\right), \text{ for } z \in U,\ \zeta \in \overline{U}. \end{aligned}$$

Remark 1.1. If $f \in \mathcal{A}_\zeta^*$ and $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then $D_\lambda^m f(z,\zeta) = z + \sum_{j=2}^{\infty} [1+(j-1)\lambda]^m a_j(\zeta) z^j$, for $z \in U$, $\zeta \in \overline{U}$.

Definition 1.2. For $m \in \mathbb{N}$ the extended Ruscheweyh derivative $R^m : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$ is defined by:

$$\begin{aligned} R^0 f(z,\zeta) &= f(z,\zeta) \\ R^1 f(z,\zeta) &= z f_z'(z,\zeta) \\ &\dots \\ (m+1) R^{m+1} f(z,\zeta) &= z \left(R^m f(z,\zeta)\right)_z' + m R^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}. \end{aligned}$$

Remark 1.2. If $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then

$$R^m f(z,\zeta) = z + \sum_{j=2}^{\infty} \frac{(m+j-1)!}{m!\,(j-1)!} a_j(\zeta) z^j, \ z \in U,\ \zeta \in \overline{U}.$$

Definition 1.3. [3] Let $\alpha, \lambda \geq 0$, $m \in \mathbb{N}$. Denote by $RD_{\lambda,\alpha}^m$ the operator $RD_{\lambda,\alpha}^m : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$ given by

$$RD_{\lambda,\alpha}^m f(z,\zeta) = (1-\alpha) R^m f(z,\zeta) + \alpha D_\lambda^m f(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}.$$

Remark 1.3. If $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then
$RD_{\lambda,\alpha}^m f(z,\zeta) = z + \sum_{j=2}^{\infty} \left\{ \alpha [1+(j-1)\lambda]^m + (1-\alpha) \frac{(m+j-1)!}{m!(j-1)!} \right\} a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

This operator was studied also in [4], [5] and [6].

Remark 1.4. For $\alpha = 0$, $RD_{\lambda,0}^m f(z,\zeta) = R^m f(z,\zeta)$, where $z \in U$, $\zeta \in \overline{U}$ and for $\alpha = 1$, $RD_{\lambda,1}^m f(z,\zeta) = D_\lambda^m f(z,\zeta)$, where $z \in U$, $\zeta \in \overline{U}$.

For $\lambda = 1$, we obtain the operator $RD_{1,\alpha}^m f(z,\zeta) = L_\alpha^m f(z,\zeta)$ which was studied in [1] and [2].

For $m = 0$, $RD_{\lambda,\alpha}^0 f(z,\zeta) = (1-\alpha) R^0 f(z,\zeta) + \alpha D_\lambda^0 f(z,\zeta) = f(z,\zeta) = R^0 f(z,\zeta) = D_\lambda^0 f(z,\zeta)$, where $z \in U$, $\zeta \in \overline{U}$.

Generalizing the notion of differential subordinations, J. A. Antonino and S. Romaguera have introduced in [11] the notion of strong differential subordinations, which was developed by G. I. Oros and Gh. Oros in [15] and [14].

Definition 1.4. (see [15, 14]). Let $f(z,\zeta)$ and $H(z,\zeta)$ be analytic functions in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists an analytic in U function w , with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec H(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Definition 1.5. (see [10, 11, 15]). Let $f(z,\zeta)$ and $H(z,\zeta)$ be analytic functions in $U\times\overline{U}$. The function $f(z,\zeta)$ is said to be strictly subordinate to $H(z,\zeta)$ if there exists a function $w(z,\zeta)$ in $U\times\overline{U}$ such that $w(0,\zeta)=0$ and $w(z,\zeta)$ is analytic in U for each $\zeta\in\overline{U}$, $|w(z,\zeta)|<1$ and $f(z,\zeta)=H(w(z,\zeta),\zeta)$ for all $(z,\zeta)\in U\times\overline{U}$. In such a case we write $f(z,\zeta)\ll H(z,\zeta)$, $z\in U,\zeta\in\overline{U}$.

Remark 1.5. If $H(z,\zeta)\equiv H(z)$ and $f(z,\zeta)\equiv f(z)$, then the strong subordination and strict subordination becomes the usual notion of subordination $f(z)\prec H(z)$. Distinct relations of subordinations in general spaces were examined in [9].

Remark 1.6. If $H(z,\zeta)\equiv H(z)$, then the definition of strongly subordination from [11, 10, 15] is similar with the above definition of strictly subordination. In this case from $f(z,\zeta)\prec\prec H(z)$ it follows that $f(z,\zeta)\equiv f(z)$ and $f(z)\prec H(z)$.

Remark 1.7. (i) Assume that $f(z,\zeta)$ and $H(z,\zeta)$ are analytic in $U\times\overline{U}$. If $f(z,\zeta)\prec\prec H(z,\zeta)$, then $f(z,\zeta)\ll H(z,\zeta)$.

(ii) Assume that $f(z,\zeta)$ and $H(z,\zeta)$ are analytic in $U\times\overline{U}$ and $H(z,\zeta)$ is univalent in U for all $\zeta\in\overline{U}$. Then $f(z,\zeta)\ll H(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, if and only if $f(0,\zeta)=H(0,\zeta)$ and $f(U\times\{\zeta\})\subset H(U\times\{\zeta\})$ for each $\zeta\in\overline{U}$.

(iii) Assume that $f(z,\zeta)$ is analytic in $U\times\overline{U}$ for all $\zeta\in\overline{U}$ and $H(z)$ is analytic and univalent in U. Then $f(z,\zeta)\ll H(z)$, $z\in U$, $\zeta\in\overline{U}$, if and only if $f(0,\zeta)=H(0)$ for all $\zeta\in\overline{U}$ and $f(U\times\overline{U})\subset H(U)$ (see [11, 10]).

There exists functions $f(z,\zeta)$ and $H(z,\zeta)$ such that $f(z,\zeta)\ll H(z,\zeta)$ and $f(z,\zeta)$ is not strongly subordinated to $H(z,\zeta)$.

Example 1.1. Let $f(z,\zeta)=z(2+\zeta)$ and $H(z,\zeta)=z(4+\zeta^2)$. Then for $\omega(z,\zeta)=z(2+\zeta):(4+\zeta^2)=z:(2-i\zeta)$ (where i^2 = -1) we have $f(z,\zeta)=H(w(z,\zeta),\zeta)$ for all $(z,\zeta)\in U\times\overline{U}$. Hence $f\ll H$. Since $\omega(z,0)=2^{-1}z$ and $\omega(z,1)=\frac{3}{5}z$ are distinct functions, the relation $f\prec\prec H$ is not true.

Example 1.2. Let $f(z,\zeta)=z(2+\zeta)$ and $H(z,\zeta)=3z$. Then $\omega(z,\zeta)=3^{-1}z(2+\zeta)$. Since $\omega(z,0)=\frac{2z}{3}$ and $\omega(z,1)=z$ are distinct functions, $f(z,\zeta)\ll H(z)$ and the relation $f(z,\zeta)\prec\prec H(z)$ is not true.

The conditions $f(0,\zeta)=H(0,\zeta)$ for all $\zeta\in\overline{U}$ and $f(U\times\overline{U})\subset H(U\times\overline{U})$ are only necessary for relation $f\ll H$.

Example 1.3. Let $f(z,\zeta)=z(2+\zeta^2)$ and $H(z,\zeta)=z(2+\zeta)$. Since $\overline{U}=\{\zeta^2:\zeta\in\overline{U}\}$, we have $f(U\times\overline{U})=H(U\times\overline{U})$. Obviously, $f(0,\zeta)=H(0,\zeta)=0$ for each $\zeta\in\overline{U}$. The function $\omega(z,\zeta)=z(2+\zeta^2):(2+\zeta)$ is the unique function for which $f(z,\zeta)=H(\omega(z,\zeta),\zeta)$. Since $\omega(1,-1)=3$ and the function ω is continuous, then $sup\{|\omega(z,\zeta)|:(z,\zeta)\in U\times\overline{U}\}\geq 3$. Hence f is not strictly subordinated to H.

We need the following lemmas to study the strong differential subordinations.

Lemma 1.1. [12] *Let $g(z,\zeta)$ be a convex function in $U\times\overline{U}$, for all $\zeta\in\overline{U}$, and let*

$$h(z,\zeta)=g(z,\zeta)+n\alpha zg'_z(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

where $\alpha>0$ and n is a positive integer. If

$$p(z,\zeta)=g(0,\zeta)+p_n(\zeta)z^n+p_{n+1}(\zeta)z^{n+1}+\dots,\quad z\in U,\ \zeta\in\overline{U},$$

is holomorphic in $U\times\overline{U}$ and

$$p(z,\zeta)+\alpha zp'_z(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta), \;\; z \in U, \; \zeta \in \overline{U},$$

and this result is sharp.

Lemma 1.2. [13] *Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ for every $\zeta \in \overline{U}$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\gamma \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta]$ and*

$$p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta) \prec\prec h(z,\zeta), \;\; z \in U, \; \zeta \in \overline{U},$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta), \;\; z \in U, \; \zeta \in \overline{U},$$

where $g(z,\zeta) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$ is convex and it is the best dominant.

2. Main results

Extending the results obtained in [8] and [7] to the class $\mathcal{A}^*_\zeta$, we obtain the following theorems:

Theorem 2.1. *Let $g(z,\zeta)$ be a convex function, $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + \frac{z}{\delta} g'_z(z,\zeta)$, for $z \in U$, $\zeta \in \overline{U}$.*

If $\alpha > 0$, $\lambda \geq 0$, $\delta \in \mathbb{N}$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_\zeta$ and satisfies the strong differential subordination

$$\left(\frac{RD^m_{\lambda,\alpha} f(z,\zeta)}{z}\right)^{\delta-1} \left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \qquad z \in U, \; \zeta \in \overline{U}, \tag{2.1}$$

then

$$\left(\frac{RD^m_{\lambda,\alpha} f(z,\zeta)}{z}\right)^{\delta} \prec\prec g(z,\zeta), \qquad z \in U, \; \zeta \in \overline{U},$$

and this result is sharp.

Proof. By using the properties of operator $RD^m_{\lambda,\alpha}$, we have

$$RD^m_{\lambda,\alpha} f(z,\zeta) = z + \sum_{j=2}^{\infty} \left\{ \alpha \left[1 + (j-1)\lambda\right]^m + (1-\alpha) \frac{(m+j-1)!}{m!\,(j-1)!} \right\} a_j(\zeta) z^j, \; z \in U, \; \zeta \in \overline{U}.$$

Consider $p(z,\zeta) = \left(\frac{RD^m_{\lambda,\alpha} f(z,\zeta)}{z}\right)^{\delta} = \left(\frac{z + \sum_{j=2}^{\infty} \left\{\alpha[1+(j-1)\lambda]^m + (1-\alpha)\frac{(m+j-1)!}{m!(j-1)!}\right\} a_j(\zeta) z^j}{z}\right)^{\delta}$ for $z \in U$ and $\zeta \in \overline{U}$. We deduce that $p \in \mathcal{H}^*[1,1,\zeta]$.

Differentiating with respect to z we obtain

$$\left(\frac{RD^m_{\lambda,\alpha} f(z,\zeta)}{z}\right)^{\delta-1} \left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z = p(z,\zeta) + \tfrac{1}{\delta} z p'_z(z,\zeta), \; z \in U, \; \zeta \in \overline{U}.$$

Then (2.1) becomes

$$p(z,\zeta) + \frac{1}{\delta} z p'_z(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + \frac{z}{\delta} g'_z(z,\zeta), \quad \text{for } z \in U, \; \zeta \in \overline{U}.$$

By using Lemma 1.1, we have

$$p(z,\zeta) \prec\prec g(z,\zeta), \;\; z \in U, \; \zeta \in \overline{U}, \quad \text{i.e.} \quad \left(\frac{RD^m_{\lambda,\alpha} f(z,\zeta)}{z}\right)^{\delta} \prec\prec g(z,\zeta), \;\; z \in U, \; \zeta \in \overline{U}.$$

□

Theorem 2.2. *Let h be a holomorphic function which satisfies the inequality*

$$\mathrm{Re}\left(1+\frac{zh''_{z^2}(z,\zeta)}{h'_z(z,\zeta)}\right) > -\frac{1}{2},\ z\in U,\ \zeta\in\overline{U},\ \text{and}\ \ h(0,\zeta)=1,\ \zeta\in\overline{U}.$$

If $\alpha>0$, $\lambda\geq 0$, $\delta\in\mathbb{N}$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and satisfies the strong differential subordination

$$\left(\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}\right)^{\delta-1}\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z \prec\prec h(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U}, \tag{2.2}$$

then

$$\left(\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}\right)^{\delta}\prec\prec q(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{\delta}{z^\delta}\int_0^z h(t,\zeta)t^{\delta-1}dt$ is convex and it is the best dominant.

Proof. Let $p(z,\zeta)=$

$$=\left(\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}\right)^{\delta}=\left(\frac{z+\sum_{j=2}^\infty\left\{\alpha\left[1+(j-1)\lambda\right]^m+(1-\alpha)\frac{(m+j-1)!}{m!(j-1)!}\right\}a_j(\zeta)z^j}{z}\right)^{\delta}$$

$$=\left(1+\sum_{j=2}^\infty\left\{\alpha\left[1+(j-1)\lambda\right]^m+(1-\alpha)\frac{(m+j-1)!}{m!\,(j-1)!}\right\}a_j(\zeta)z^{j-1}\right)^{\delta}$$

for $z\in U$, $\zeta\in\overline{U}$, $p\in\mathcal{H}^*[1,1,\zeta]$.

Differentiating with respect to z, we obtain

$$\left(\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}\right)^{\delta-1}\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z=p(z,\zeta)+\frac{1}{\delta}zp'_z(z,\zeta),\ z\in U,\ \zeta\in\overline{U}$$

and (2.2) becomes

$$p(z,\zeta)+\frac{1}{\delta}zp'_z(z,\zeta)\prec\prec h(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1.2, we have

$$p(z,\zeta)\prec\prec q(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}\quad \text{i.e.}$$

$$\left(\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}\right)^{\delta}\prec\prec q(z,\zeta)=\frac{\delta}{z^\delta}\int_0^z h(t,\zeta)t^{\delta-1}dt,\ \ z\in U,\ \zeta\in\overline{U}$$

and q is the best dominant. □

Corollary 2.1. *Let $h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, where $0\leq\beta<1$.*

If $\alpha>0$, $\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and satisfies the strong differential subordination

$$1-\frac{RD^m_{\lambda,\alpha}f(z,\zeta)\cdot\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)''_{zz}}{\left[\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z\right]^2}\prec\prec h(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U}, \tag{2.3}$$

then

$$\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}\prec\prec q(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z,\zeta)=(2\beta-\zeta)+(1+\zeta-2\beta)\frac{\ln(1+z)}{z}$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem and considering $p(z,\zeta) = \frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}$, the strong differential subordination (2.3) becomes

$$p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}, \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1.2 for $\gamma = 1$, we have $p(z,\zeta) \prec\prec q(z,\zeta)$, i.e.

$$\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z} \prec\prec q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt =$$

$$\frac{1}{z}\int_0^z \frac{1+(2\beta-\zeta)t}{1+t}dt = \frac{1}{z}\int_0^z \left[(2\beta-\zeta) + \frac{1+\zeta-2\beta}{1+t}\right]dt$$

$$= (2\beta-\zeta) + (1+\zeta-2\beta)\frac{\ln(1+z)}{z}, \quad z \in U,\ \zeta \in \overline{U}.$$

□

Theorem 2.3. *Let g be a convex function such that $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + \frac{z}{\delta}g'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. If $\alpha > 0$, $\lambda \geq 0$, $\delta > 0$, $m \in \mathbb{N}$, $z \in U$, $\zeta \in \overline{U}$, $f \in \mathcal{A}^*_\zeta$ and the strong differential subordination*

$$z\frac{\delta+1}{\delta}\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)^2} +$$

$$\frac{z^2}{\delta}\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)^2}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)} - 2\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}\right] \prec\prec h(z,\zeta) \tag{2.4}$$

holds, where $z \in U$, $\zeta \in \overline{U}$, then

$$z\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)^2} \prec\prec g(z,\zeta),\ z \in U,\ \zeta \in \overline{U},$$

and this result is sharp.

Proof. For $f \in \mathcal{A}^*_\zeta$, $f(z,\zeta) = z + \sum_{j=2}^\infty a_j(\zeta)z^j$ we have

$RD^m_{\lambda,\alpha}f(z,\zeta) = z + \sum_{j=2}^\infty \left\{\alpha\left[1+(j-1)\lambda\right]^m + (1-\alpha)\frac{(m+j-1)!}{m!(j-1)!}\right\}a_j(\zeta)z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider $p(z,\zeta) = z\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)^2}$ and we obtain $p(z,\zeta) + \frac{z}{\delta}p'_z(z,\zeta) =$

$= z\frac{\delta+1}{\delta}\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)^2} + \frac{z^2}{\delta}\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)^2}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)} - 2\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}\right]$.

Relation (2.4) becomes

$$p(z,\zeta) + \frac{z}{\delta}p'_z(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + \frac{z}{\delta}g'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1.1, we have

$$p(z,\zeta) \prec\prec g(z,\zeta),\ z \in U,\ \zeta \in \overline{U},\ \text{i.e.}\quad z\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)^2} \prec\prec g(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

□

Theorem 2.4. *Let h be a holomorphic function which satisfies the inequality* $\operatorname{Re}\left(1+\frac{zh''_{z^2}(z,\zeta)}{h'_z(z,\zeta)}\right) > -\frac{1}{2}$, $z \in U$, $\zeta \in \overline{U}$, *and* $h(0,\zeta) = 1$. *If* $\alpha > 0$, $\lambda \geq 0$, $\delta > 0$, $m \in \mathbb{N}$, $z \in U$, $\zeta \in \overline{U}$, $f \in \mathcal{A}^*_\zeta$ *and satisfies the strong differential subordination*

$$z\frac{\delta+1}{\delta}\frac{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)}{\left(RD^{m+1}_{\lambda,\alpha}f\left(z,\zeta\right)\right)^2}+$$

$$\frac{z^2}{\delta}\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)^2}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}-2\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}\right] \prec\prec h(z,\zeta), \tag{2.5}$$

$z \in U$, $\zeta \in \overline{U}$, *then*

$$z\frac{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)}{\left(RD^{m+1}_{\lambda,\alpha}f\left(z,\zeta\right)\right)^2} \prec\prec q(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{\delta}{z^\delta}\int_0^z h(t,\zeta)t^{\delta-1}dt$ *is convex and it is the best dominant.*

Proof. Let $p(z,\zeta) = z\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)^2}$, $z \in U, \zeta \in \overline{U}, p \in \mathcal{H}^*[1,1,\zeta]$.

Differentiating with respect to z, we obtain $p\left(z,\zeta\right)+\frac{z}{\delta}p'_z\left(z,\zeta\right) =$
$= z\frac{\delta+1}{\delta}\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)^2}+\frac{z^2}{\delta}\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)^2}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}-2\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}\right]$, $z \in U$, $\zeta \in \overline{U}$, and (2.5) becomes

$$p(z,\zeta)+\frac{z}{\delta}p'_z(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 1.2, we have

$$p(z,\zeta) \prec\prec q(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.}$$

$$z\frac{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)}{\left(RD^{m+1}_{\lambda,\alpha}f\left(z,\zeta\right)\right)^2} \prec\prec q(z,\zeta) = \frac{\delta}{z^\delta}\int_0^z h(t,\zeta)t^{\delta-1}dt, \quad z \in U,\ \zeta \in \overline{U},$$

and q is the best dominant. □

Theorem 2.5. *Let g be a convex function such that $g(0,\zeta) = 0$ and let h be the function $h(z,\zeta) = g(z,\zeta)+\frac{z}{\delta}g'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

If $\alpha > 0$, $\lambda \geq 0$, $\delta > 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_\zeta$ *and the strong differential subordination*

$$z^2\frac{\delta+2}{\delta}\frac{\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z}{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)}+\frac{z^3}{\delta}\left[\frac{\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)''_{z^2}}{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)}-\left(\frac{\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z}{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)}\right)^2\right] \tag{2.6}$$

$\prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, *holds, then*

$$z^2\frac{\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z}{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)} \prec\prec g(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

This result is sharp.

Proof. Let $p(z,\zeta)=z^2\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}$. We deduce that $p\in\mathcal{H}^*[0,1,\zeta]$.

Differentiating with respect to z, we obtain

$$p(z,\zeta)+\frac{z}{\delta}p'_z(z,\zeta)=z^2\frac{\delta+2}{\delta}\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}+\frac{z^3}{\delta}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)''_{z^2}}{RD^m_{\lambda,\alpha}f(z,\zeta)}-\left(\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\right)^2\right],$$

$z\in U$, $\zeta\in\overline{U}$. Using the notation in (2.6), the strong differential subordination becomes

$$p(z,\zeta)+\frac{1}{\delta}zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+\frac{z}{\delta}g'_z(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.1, we have

$$p(z,\zeta)\prec\prec g(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\ \text{i.e.}\quad z^2\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\prec\prec g(z,\zeta),\ z\in U,\ \zeta\in\overline{U},$$

and this result is sharp. □

Theorem 2.6. *Let h be a holomorphic function which satisfies the inequality* $\mathrm{Re}\left(1+\frac{zh''_{z^2}(z,\zeta)}{h'_z(z,\zeta)}\right)>-\frac{1}{2}$, $z\in U$, $\zeta\in\overline{U}$ *and* $h(0,\zeta)=0$.

If $\alpha>0$, $\lambda\geq 0$, $\delta>0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and satisfies the strong differential subordination

$$z^2\frac{\delta+2}{\delta}\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}+\frac{z^3}{\delta}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)''_{z^2}}{RD^m_{\lambda,\alpha}f(z,\zeta)}-\left(\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\right)^2\right]\tag{2.7}$$

$\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, *then*

$$z^2\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\prec\prec q(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{\delta}{z^\delta}\int_0^z h(t,\zeta)t^{\delta-1}dt$ *is convex and it is the best dominant.*

Proof. Let $p(z,\zeta)=z^2\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}$, $z\in U$, $\zeta\in\overline{U}$, $p\in\mathcal{H}^*[0,1,\zeta]$.
Differentiating with respect to z, we obtain

$$p(z,\zeta)+\frac{z}{\delta}p'_z(z,\zeta)=z^2\frac{\delta+2}{\delta}\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}+\frac{z^3}{\delta}\left[\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)''_{z^2}}{RD^m_{\lambda,\alpha}f(z,\zeta)}-\left(\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\right)^2\right],$$

$z\in U$, $\zeta\in\overline{U}$, and (2.7) becomes

$$p(z)+\frac{1}{\delta}zp'_z(z,\zeta)\prec\prec h(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1.2, we have

$$p(z,\zeta)\prec\prec q(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\quad \text{i.e.}$$

$$z^2\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\prec\prec q(z,\zeta)=\frac{\delta}{z^\delta}\int_0^z h(t,\zeta)t^{\delta-1}dt,\ \ z\in U,\ \zeta\in\overline{U},$$

and q is the best dominant. □

Theorem 2.7. *Let g be a convex function such that $g(0,\zeta)=1$ and let h be the function*
$$h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta),\ z\in U,\zeta\in\overline{U}.$$
If $\alpha>0$, $\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and the strong differential subordination

$$1-\frac{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)''_{zz}}{\left[\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z\right]^2}\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}\tag{2.8}$$

holds, then
$$\frac{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)}{z\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z}\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$
This result is sharp.

Proof. Let $p(z,\zeta)=\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}$. We deduce that $p\in\mathcal{H}^*[1,1,\zeta]$.

Differentiating with respect to z, we obtain
$$1-\frac{RD^m_{\lambda,\alpha}f(z,\zeta)\cdot\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)''_{z^2}}{\left[\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z\right]^2}=p\left(z,\zeta\right)+zp'_z\left(z,\zeta\right),\ z\in U,\zeta\in\overline{U}.$$
Using the notation in (2.8), the strong differential subordination becomes
$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta),\ z\in U,\zeta\in\overline{U}.$$
By using Lemma 1.1, we have
$$p(z,\zeta)\prec\prec g(z,\zeta),\ z\in U,\ \zeta\in\overline{U}\quad\text{i.e.}\quad\frac{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)}{z\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z}\prec\prec g(z,\zeta),\ z\in U,\ \zeta\in\overline{U},$$
and this result is sharp. □

Theorem 2.8. *Let h be a holomorphic function which satisfies the inequality $\mathrm{Re}\left(1+\frac{zh''_{z^2}(z,\zeta)}{h'_z(z,\zeta)}\right)>-\frac{1}{2}$, $z\in U,\zeta\in\overline{U}$, and $h(0,\zeta)=1$.*
If $\alpha>0$, $\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and satisfies the strong differential subordination

$$1-\frac{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)''_{zz}}{\left[\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z\right]^2}\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\tag{2.9}$$

then
$$\frac{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)}{z\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z}\prec\prec q(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$
where $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt$ is convex and it is the best dominant.

Proof. Let $p(z,\zeta)=\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}$, $z\in U,\zeta\in\overline{U}, p\in\mathcal{H}^*[1,1,\zeta]$.

Differentiating with respect to z, we obtain $1-\frac{RD^m_{\lambda,\alpha}f(z,\zeta)\cdot\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)''_{z^2}}{\left[\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z\right]^2}=p\left(z,\zeta\right)+zp'_z\left(z,\zeta\right), z\in U,\zeta\in\overline{U}$, and (2.9) becomes
$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1.2, we have

$$p(z,\zeta) \prec\prec q(z,\zeta), \quad z \in U,\ \zeta \in \overline{U} \quad \text{i.e.}$$

$$\frac{RD^m_{\lambda,\alpha} f(z,\zeta)}{z\left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z} \prec\prec q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt, \quad z \in U,\ \zeta \in \overline{U},$$

and q is the best dominant. □

Corollary 2.2. *Let $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}$ be a convex function in $U \times \overline{U}$, where $0 \le \beta < 1$.*

If $\alpha, \lambda \ge 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_\zeta$ and satisfies the strong differential subordination

$$(2.10) \qquad 1 - \frac{RD^m_{\lambda,\alpha} f(z,\zeta) \cdot \left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)''_{zz}}{\left[\left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z\right]^2} \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

then

$$\frac{RD^m_{\lambda,\alpha} f(z,\zeta)}{z\left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z} \prec\prec q(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = (2\beta-\zeta) + (1+\zeta-2\beta)\frac{\ln(1+z)}{z}$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.8 and considering $p(z,\zeta) = \frac{RD^m_{\lambda,\alpha} f(z,\zeta)}{z\left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z}$, the strong differential subordination (2.10) becomes

$$p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}, \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1.2 for $\gamma = 1$, we have $p(z,\zeta) \prec\prec q(z,\zeta)$, i.e.

$$\frac{RD^m_{\lambda,\alpha} f(z,\zeta)}{z\left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z} \prec\prec q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt =$$

$$\frac{1}{z}\int_0^z \frac{1+(2\beta-\zeta)t}{1+t}dt = \frac{1}{z}\int_0^z \left[(2\beta-\zeta) + \frac{1+\zeta-2\beta}{1+t}\right]dt$$

$$= (2\beta-\zeta) + (1+\zeta-2\beta)\frac{\ln(1+z)}{z}, \quad z \in U,\ \zeta \in \overline{U}.$$

□

Theorem 2.9. *Let g be a convex function such that $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + zg'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

If $\alpha > 0$, $\lambda \ge 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_\zeta$ and the strong differential subordination

$$(2.11) \quad \left[\left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z\right]^2 + RD^m_{\lambda,\alpha} f(z,\zeta) \cdot \left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)''_{zz} \prec\prec h(z,\zeta), \ z \in U,\ \zeta \in \overline{U}$$

holds, then

$$\frac{RD^m_{\lambda,\alpha} f(z,\zeta) \cdot \left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z}{z} \prec\prec g(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

This result is sharp.

Proof. Let $p(z,\zeta)=\frac{RD^m_{\lambda,\alpha}f(z,\zeta)\cdot\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{z}$. We deduce that $p\in\mathcal{H}^*[1,1,\zeta]$.

Differentiating with respect to z, we obtain

$$\left[\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z\right]^2+RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)''_{z^2}=p\left(z,\zeta\right)+zp'_z\left(z,\zeta\right),z\in U,\zeta\in\overline{U}.$$

Using the notation in (2.11), the strong differential subordination becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta),\ \ z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.1, we have

$$p(z,\zeta)\prec\prec g(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U}\quad\text{i.e.}$$

$$\frac{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z}{z}\prec\prec g(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp. □

Theorem 2.10. *Let h be a holomorphic function which satisfies the inequality*

$\mathrm{Re}\left(1+\frac{zh''_{z^2}(z,\zeta)}{h'_z(z,\zeta)}\right)>-\frac{1}{2}$, $z\in U,\zeta\in\overline{U}$ *and* $h(0,\zeta)=1$.

If $\alpha>0$, $\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and satisfies the strong differential subordination

$$\left[\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z\right]^2+RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)''_{zz}\prec\prec h(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\tag{2.12}$$

then

$$\frac{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z}{z}\prec\prec q(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt$ is convex and it is the best dominant.

Proof. Let $p(z,\zeta)=\frac{RD^m_{\lambda,\alpha}f(z,\zeta)\cdot\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{z}$, $z\in U,\zeta\in\overline{U}$, $p\in\mathcal{H}^*[1,1,\zeta]$.

Differentiating with respect to z, we obtain

$$\left[\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z\right]^2+RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)''_{z^2}=p\left(z,\zeta\right)+zp'_z\left(z,\zeta\right),z\in U,\zeta\in\overline{U},$$

and (2.12) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1.2, we have

$$p(z,\zeta)\prec\prec q(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\quad\text{i.e.}$$

$$\frac{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z}{z}\prec\prec q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt,\ \ z\in U,\ \zeta\in\overline{U},$$

and q is the best dominant. □

Corollary 2.3. *Let $h(z)=\frac{1+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, where $0\leq\beta<1$.*

If $\alpha>0$, $\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and satisfies the strong differential subordination

$$\left[\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z\right]^2+RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)''_{zz}\prec\prec h(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\tag{2.13}$$

then

$$\frac{RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\cdot\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z}{z}\prec\prec q(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z,\zeta) = (2\beta - \zeta) + (1 + \zeta - 2\beta)\frac{\ln(1+z)}{z}$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.10 and considering $p(z,\zeta) = \frac{RD^m_{\lambda,\alpha} f(z,\zeta)\cdot\left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z}{z}$, the strong differential subordination (2.13) becomes

$$p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta) = \frac{1 + (2\beta - \zeta)z}{1+z}, \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1.2 for $\gamma = 1$, we have $p(z,\zeta) \prec\prec q(z,\zeta)$, i.e.

$$\frac{RD^m_{\lambda,\alpha} f(z,\zeta)\cdot\left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z}{z} \prec\prec q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt =$$

$$\frac{1}{z}\int_0^z \frac{1+(2\beta-\zeta)t}{1+t}dt = \frac{1}{z}\int_0^z \left[(2\beta-\zeta) + \frac{1+\zeta-2\beta}{1+t}\right]dt =$$

$$(2\beta - \zeta) + (1+\zeta-2\beta)\frac{\ln(1+z)}{z}, \quad z \in U,\ \zeta \in \overline{U}$$

and the proof is completed. □

Theorem 2.11. *Let g be a convex function such that $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + \frac{z}{1-\delta}g'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

If $\alpha > 0$, $\lambda \geq 0$, $\delta \in (0,1)$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_\zeta$ and the strong differential subordination

$$\left(\frac{z}{RD^m_{\lambda,\alpha} f(z,\zeta)}\right)^\delta \frac{RD^{m+1}_{\lambda,\alpha} f(z,\zeta)}{1-\delta}\left(\frac{\left(RD^{m+1}_{\lambda,\alpha} f(z,\zeta)\right)'_z}{RD^{m+1}_{\lambda,\alpha} f(z,\zeta)} - \delta\frac{\left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha} f(z,\zeta)}\right) \tag{2.14}$$

$\prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, holds, then

$$\frac{RD^{m+1}_{\lambda,\alpha} f(z,\zeta)}{z}\cdot\left(\frac{z}{RD^m_{\lambda,\alpha} f(z,\zeta)}\right)^\delta \prec\prec g(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

This result is sharp.

Proof. Let $p(z,\zeta) = \frac{RD^{m+1}_{\lambda,\alpha} f(z,\zeta)}{z}\cdot\left(\frac{z}{RD^m_{\lambda,\alpha} f(z,\zeta)}\right)^\delta$. We deduce that $p \in \mathcal{H}^*[1,1,\zeta]$. Differentiating with respect to z, we obtain

$$\left(\frac{z}{RD^m_{\lambda,\alpha} f(z,\zeta)}\right)^\delta \frac{RD^{m+1}_{\lambda,\alpha} f(z,\zeta)}{1-\delta}\left(\frac{\left(RD^{m+1}_{\lambda,\alpha} f(z,\zeta)\right)'_z}{RD^{m+1}_{\lambda,\alpha} f(z)} - \delta\frac{\left(RD^m_{\lambda,\alpha} f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha} f(z,\zeta)}\right) =$$

$p(z,\zeta) + \frac{1}{1-\delta}zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. Using the notation in (2.14), the strong differential subordination becomes

$$p(z,\zeta) + \frac{1}{1-\delta}zp'_z(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + \frac{z}{1-\delta}g'_z(z,\zeta),\ z \in U, \zeta \in \overline{U}.$$

By using Lemma 1.1, we have $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e.

$$\frac{RD^{m+1}_{\lambda,\alpha} f(z,\zeta)}{z}\cdot\left(\frac{z}{RD^m_{\lambda,\alpha} f(z,\zeta)}\right)^\delta \prec\prec g(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

and this result is sharp. □

Theorem 2.12. *Let h be a holomorphic function which satisfies the inequality*

$$\mathrm{Re}\left(1+\frac{zh''_{z^2}(z,\zeta)}{h'_z(z,\zeta)}\right)>-\frac{1}{2},\ z\in U,\ \zeta\in\overline{U} \text{ and } h(0,\zeta)=1.$$

If $\alpha>0$, $\lambda\geq 0$, $\delta\in(0,1)$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and satisfies the strong differential subordination

$$\left(\frac{z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\right)^\delta\frac{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}{1-\delta}\left(\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}-\delta\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\right)\tag{2.15}$$

$\prec\prec h(z,\zeta),\ z\in U,\ \zeta\in\overline{U}$, *then*

$$\frac{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}{z}\cdot\left(\frac{z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\right)^\delta\prec\prec q(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1-\delta}{z^{1-\delta}}\int_0^z h(t,\zeta)t^{-\delta}dt$ is convex and it is the best dominant.

Proof. Let $p(z,\zeta)=\frac{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}{z}\cdot\left(\frac{z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\right)^\delta$, $z\in U,\zeta\in\overline{U},p\in\mathcal{H}^*[1,1,\zeta]$.
Differentiating with respect to z, we obtain

$$\left(\frac{z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\right)^\delta\frac{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}{1-\delta}\left(\frac{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}-\delta\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\right)=$$

$p(z,\zeta)+\frac{1}{1-\delta}zp'_z(z,\zeta)$, $z\in U,\zeta\in\overline{U}$, and (2.15) becomes

$$p(z,\zeta)+\frac{1}{1-\delta}zp'_z(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1.2, we have

$$p(z,\zeta)\prec\prec q(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\quad \text{i.e.}$$

$$\frac{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}{z}\cdot\left(\frac{z}{RD^m_{\lambda,\alpha}f(z,\zeta)}\right)^\delta\prec\prec q(z,\zeta)=\frac{1-\delta}{z^{1-\delta}}\int_0^z h(t,\zeta)t^{-\delta}dt,\ z\in U,\ \zeta\in\overline{U},$$

and q is the best dominant. □

The cases from the following Example indicate the possible applications of the above results.

Example 2.4. Let $m=1$, $\lambda=\frac{1}{2}$, $\alpha=2$ and $\delta=1$. Then:

1. If $f(z,\zeta)$ is an analytic function, then
$RD^1_{\frac{1}{2},2}f(z,\zeta)=-R^1f(z,\zeta)+2D^1_{\frac{1}{2}}f(z,\zeta)=-zf'(z,\zeta)+2(\frac{1}{2}f(z,\zeta)+\frac{1}{2}zf'_z(z,\zeta))=f(z,\zeta)$.

2. Let $f(z,\zeta)\in\mathcal{A}^*_\zeta$, $g(z,\zeta)$ be a convex function, $g(0,\zeta)=1$ and $h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$. Theorem 2.1 affirms that the strong differential subordination $f'_z(z,\zeta)\prec\prec h(z,\zeta)$, $(e1)$, implies $f(z,\zeta):z\prec\prec g(z,\zeta)$, $(e2)$.

In particular:

2.1. For the function $f(z,\zeta)=\frac{z-\zeta\frac{z^2}{2}}{1+\frac{z}{2}}$ and the convex function $g(z,\zeta)=\frac{1-\zeta z}{1+z}$, we have $h(z,\zeta)=\frac{1-\zeta z(z+2)}{(1+z)^2}$ and the strong differential subordinations $(e1)$ and $(e2)$ are realized by the function $\omega(z)=z:2$.

2.2. For the function $f(z,\zeta)=\frac{z}{1-\zeta^2 z}$ and the convex function $g(z,\zeta)=\frac{1}{1-\zeta z}$ the strong differential subordinations $(e1)$ and $(e2)$ are not true and the strict differential subordinations $f'_z(z,\zeta)\ll h(z,\zeta)$ and $f(z,\zeta):z\ll g(z,\zeta)$ are realized by the function $\omega(z,\zeta)=\zeta z$.

3. Let $f(z,\zeta) \in \mathcal{A}^*_\zeta$, $g(z,\zeta)$ be a convex function, $g(0,\zeta) = 1$ and $h(z,\zeta) = g(z,\zeta) + zg'_z(z,\zeta)$. Theorem 2.7 affirms that the strong differential subordination $1 - \frac{f(z,\zeta)f''(z,\zeta)}{f'_z(z,\zeta)^2} \prec\prec h(z,\zeta)$, $(e3)$, implies $\frac{f(z,\zeta)}{zf'_z(z,\zeta)} : z \prec\prec g(z,\zeta)$, $(e4)$.

For the functions $f(z,\zeta) = z - \zeta z^2$ and $g(z,\zeta) = \frac{1}{1-\zeta z}$ the strong differential subordination $(e4)$ is not true. Hence, the strong differential subordination $(e3)$ is not true for the functions f and g, too. The strict differential subordinations $1 - \frac{f(z,\zeta)f''_{zz}(z,\zeta)}{f'_z(z,\zeta)^2} \ll h(z,\zeta)$ and $\frac{f(z,\zeta)}{zf'_z(z,\zeta)} \ll g(z,\zeta)$ are realized by the function $\omega(z,\zeta) = \frac{\zeta z}{1-\zeta z}$.

Remark 2.8. The above results of that section are true for the relation of the strict subordination $\ll$, too. The proofs are similar.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF ORADEA
UNIVERSITATII 1, 410087 ORADEA, ROMANIA
E-mail address: lori_andrei@yahoo.com

DEPARTMENT OF PHYSICS, MATHEMATICS AND INFORMATION TECHNOLOGIES
TIRASPOL STATE UNIVERSITY,
GH. IABLOCIKIN 5, MD2069 CHIŞINĂU, REPUBLIC OF MOLDOVA
E-mail address: mmchoban@gmail.com