CARPATHIAN J. MATH. **31** (2015), No. 2, 165 - 172
Online version available at http://carpathian.ubm.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

On the geometry behind a recurrent relation

CRISTIAN COBELI[1] and ALEXANDRU ZAHARESCU[2]

ABSTRACT. We consider a certain linear recursive relation with integer parameters and study some of its algebraic and geometric properties, with the purpose of estimating the number of chains of valences in the Farey series.

1. INTRODUCTION

In the present paper we study some of the significant properties and consequences of a recurrent construction involving a sequence of polynomials that appears naturally in different contexts where the algebra and geometry are linked with the arithmetical features of integers. As in [18], let $p_{-1}(\cdot) = 0$, $p_0(\cdot) = 1$, and then recursively, for any integer $r \geq 1$ and variables $X_1, X_2, \dots X_r$, let

$$p_r(X_1, \dots, X_r) = X_r p_{r-1}(X_1, \dots, X_{r-1}) - p_{r-2}(X_1, \dots, X_{r-2}). \tag{R}$$

We write $\mathbf{X} = (X_1, \dots, X_r)$ when the *order* (or *length*) r is understood from the context and $p_r(\mathbf{X})$ instead of $p_r(X_1, \dots, X_r)$. A few polynomials of small orders satisfying (R) are:

$$\begin{aligned}
p_1(\mathbf{X}) =& X_1; \qquad p_2(\mathbf{X}) = X_1X_2 - 1; \qquad p_3(\mathbf{X}) = X_1X_2X_3 - X_1 - X_3;\\
p_4(\mathbf{X}) =& X_1X_2X_3X_4 - X_1X_2 - X_1X_4 - X_3X_4 + 1;\\
p_5(\mathbf{X}) =& X_1X_2X_3X_4X_5 - X_1X_2X_3 - X_1X_2X_5 - X_1X_4X_5 - X_3X_4X_5 + X_1 + X_3 + X_5\\
p_6(\mathbf{X}) =& X_1X_2X_3X_4X_5X_6 - X_1X_2X_3X_6 - X_1X_4X_5X_6\\
& - X_1X_2X_5X_6 - X_1X_2X_5X_6 - X_1X_2X_3X_4 - X_3X_4X_5X_6\\
& + X_1X_2 + X_5X_6 + X_1X_4 + X_3X_6 + X_1X_6 + X_3X_4 - 1\,.
\end{aligned}$$

Relation (R) has many nice properties. For example, it produces the symmetry

$$p_r(X_1, \dots, X_r) = p_r(X_r, \dots, X_1)\,. \tag{1.1}$$

Notice also the alternation in the signs of the monomials of $p_r(\mathbf{X})$ for values of r of the same parity. The polynomials $p_r(\mathbf{X})$ will be used with suitable values $X_j = k_j$. We call an r-tuple $\mathbf{k} = (k_1, \dots, k_r)$ of positive integers *admissible* if there exists an integer $Q \geq 1$ and integers $1 \leq q_0, q_1, \dots, q_{r+1} \leq Q$ with the following properties:

$$\gcd(q_j, q_{j+1}) = 1, \quad \text{for } 0 \leq j \leq r; \tag{1.2a}$$

$$q_j + q_{j+1} > Q, \quad \text{for } 0 \leq j \leq r; \tag{1.2b}$$

$$k_j q_j = q_{j-1} + q_{j+1}, \quad \text{for } 1 \leq j \leq r. \tag{1.2c}$$

We call the components of $\mathbf{k}$ *valences* and say that they are generated by the *denominators* $q_0, q_1, \dots, q_{r+1}$. An r-tuple of consecutive valences will also be called a *chain of valences*.

Received: 30.09.2013; In revised form: 09.09.2014; Accepted: 22.09.2014
2010 *Mathematics Subject Classification.* 11B57, 11B37.
Key words and phrases. *Farey sequences, tessellations, geometry of recurrent relations.*
Corresponding author: Cristian Cobeli; cristian.cobeli@imar.ro

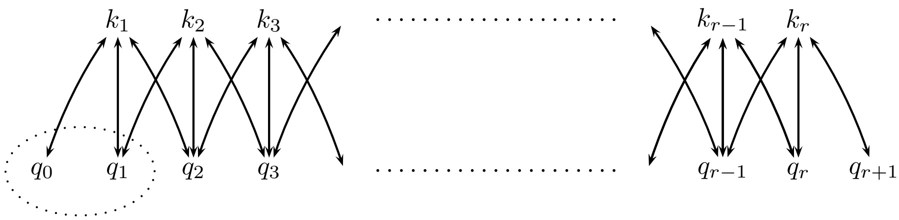

Figure 1. A chain of valences and their generators.

Notice that it would be enough to only require in (1.2a) that two neighbor denominators are relatively prime, since by (1.2c) the same property radiates recursively to all the other pairs of neighbor denominators.

We remark that by relations (1.2a)-(1.2c) it follows that any admissible r-tuple $\mathbf{k}$ can be extended to an admissible sequence $(\!(\mathbf{k})\!)$ that is infinite on both ends. (We call an infinite sequence admissible if all its r-subchains of consecutive valences are admissible.) Notice that the extension is not unique. There is a close connection between the sequence of polynomials defined by relation (R) and Farey sequences. For more details the reader is referred to [18, Section 6].

A few experiments reveal a peculiar property of $(\!(\mathbf{k})\!)$. One may find in $(\!(\mathbf{k})\!)$ components indefinitely large, but in any neighborhood of such a component all the others are comparatively small. And the larger a valence is, the larger is its neighborhood with only small components. Here are a few examples of admissible chains $\mathbf{k}$, that shed some light on this phenomenon:

$$[11, 1, 2, 2, 3, 1, 5, 1, 3, 2, 2, 1, 12];$$
$$[10, 1, 2, 3, 1, 5, 1, 4, 1, 3, 2, 2, 1, 15];$$
$$[16, 1, 2, 2, 2, 3, 1, 5, 1, 3, 1, 7, 1, 3, 1, 5, 1, 3, 2, 2, 2, 1, 16];$$
$$[1, 6, 1, 3, 1, 5, 1, 4, 1, 3, 2, 2, 2, 2, 2, 2, 1, 28, 1, 2, 2, 2, 2, 2, 3].$$

For any chain of valences $\mathbf{k}$, we define the norm of $\mathbf{k}$, to be its largest component. We denote the norm of $\mathbf{k}$ by $\|\mathbf{k}\|$. Let $\mathcal{A}_r$ be the set of admissible chains of valences of length r. Our aim is to estimate the size of $\mathcal{A}_r$. The main result below unveils the following peculiar fact: for each positive integer r, the number of admissible chains of length r and norm at most x grows almost linearly as a function of x.

Theorem 1.1. *For any integer $r \geq 1$, we have*

$$\sum_{\substack{\mathbf{k}\in\mathcal{A}_r \\ ||\mathbf{k}||\leq x}} 1 = rx + O_r(1). \tag{1.3}$$

Remark 1.1. We found that for n positive integer and sufficiently large the difference

$$\delta_r(n) := \#\{\mathbf{k} \in \mathcal{A}_r\colon\ \|\mathbf{k}\| \leq n\} - rn$$

becomes constant. We denote this constant, which depends only on r, by $C(r)$. The first twenty five values of $C(r)$ are: $C(1) = 0$; $C(2) = 3$; $C(3) = 15$; $C(4) = 41$; $C(5) = 84$; $C(6) = 153$; $C(7) = 247$; $C(8) = 367$; $C(9) = 523$; $C(10) = 721$; $C(11) = 961$; $C(12) = 1251$; $C(13) = 1588$; $C(14) = 1983$; $C(15) = 2437$; $C(16) = 2963$; $C(17) = 3548$; $C(18) = 4219$; $C(19) = 4954$; $C(20) = 5761$.

Open problem. We leave open the question of whether there exists a closed formula for $C(r)$ for all r, or at least for r large enough.

2. The Farey sequence

About two hundred years ago Haros and Farey observed (see also [11], [17] and the references therein) that by arranging the subunitary fractions with denominators at most a given $Q \geq 1$ in ascending order, the finite sequence obtained has remarkable properties. Thus, if $a'/q' < a''/q''$ are consecutive fractions then $a''q' - a'q'' = 1$ and $q' + q'' > Q$. Given $Q \geq 1$, let

$$\mathfrak{F}_{Q} := \left\{ \frac{a}{q} \in [0,1] \colon \ \gcd(a,q) = 1, \ q \leq Q \right\}.$$

Arranged in ascending order, this is the Farey sequence of order Q. For example the sequence of Farey fractions of order 8 is

$$\mathfrak{F}_{8} := \left\{ \frac{0}{1}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{1}{1} \right\}.$$

Relations (1.2a)-(1.2c) are crystallized from some basic features of a Farey sequence. If $a'/q', a''/q'', a'''/q'''$ are consecutive Farey fractions, then $(a'+a''')/a'' = (q'+q''')/q'' \in \mathbb{N}^*$, any neighbor denominators are relatively prime, and their sum is greater than the order. Thus, if $a_0/q_0, a_1/q_1, \dots, a_{r+1}/q_{r+1}$ are consecutive fractions in $\mathfrak{F}_{Q}$ and $\mathbf{k} = (k_1, \dots, k_r)$ is the chain of the associated valences to the fractions, in [18] it is shown that

$$\frac{a_{r+1}}{q_{r+1}} - \frac{a_0}{q_0} = \frac{p_r(\mathbf{k})}{q_0 q_{r+1}}.$$

In recent years the authors of [5], [6], [2], [8], [12], [7], [13], [19], [20], [9], [10], [18], [4], [14], [15], [3], [16], [1] investigated various questions on the distribution of Farey fractions, the tessellations and their polygonal tiles.

3. Germs, Tiles and Tessellations

Any chain of valences has many corresponding chains of denominators (actually infinitely many as $Q \to \infty$), but conversely, exactly one chain of valences corresponds to a given chain of admissible denominators.

In the following, if $(k_1, \dots, k_r)$ is a chain of valences with denominators $(q_0, q_1, \dots, q_r)$, we shall call the pair (q_0, q_1) a pair of *integer germs* of $\mathbf{k}$.

For a given $\mathbf{k} \in \mathcal{A}_r$ and Q sufficiently large, let $\mathcal{T}^Q[\mathbf{k}] = \mathcal{T}_r^Q[\mathbf{k}]$ be the set of integer germs of $\mathbf{k}$. Since this set depends on Q and we are interested in all admissible chains, independent of the size of their germs, it is natural to let Q approach infinity and move the problem to a bounded frame. Starting with two variables x, y, we put $x_{-1} = x$, $x_0 = y$ and then define

$$x_j = x_j(k_1, \dots, k_j; x, y) := k_j x_{j-1} - x_{j-2}, \quad \text{for } j \geq 1, \tag{RR}$$

where $k_j = \left[\frac{1+x_{j-2}}{x_{j-1}} \right]$. The connection with relation (R) is given in the next lemma. (Observe also that when $x = 0$ and $y = 1$ (RR) produces the same sequence as (R).)

Lemma 3.1. *We have:*

$$x_j = p_j(k_1, \dots, k_j) y - p_{j-1}(k_2, \dots, k_j) x, \qquad \text{for } j \geq 1. \tag{3.1}$$

Proof. The proof is by induction. For $j = 1$, relation (3.1) coincides with (RR). For $j \geq 2$, using (R) and the induction hypothesis, we have:

$$
\begin{aligned}
&p_{j+1}(k_1, \ldots, k_{j+1})y - p_j(k_2, \ldots, k_{j+1})x \\
=&\big(k_{j+1}p_j(k_1, \ldots, k_j) - p_{j-1}(k_1, \ldots, k_{j-1})\big)y - \big(k_{j+1}p_{j-1}(k_2, \ldots, k_j) - p_{j-2}(k_2, \ldots, k_{j-1})\big)x \\
=&k_{j+1}(k_j x_{j-1} - x_{j-2}) - (k_{j-1}x_{j-2} - x_{j-3}) \\
=&k_{j+1}x_j - x_{j-1} = x_j\,.
\end{aligned}
$$

This completes the proof of the lemma. □

The next result expresses the integers defined by (RR) in the language of inequalities.

Lemma 3.2. *For any $j \geq 1$, the equality $k_j = \left[\frac{1+x_{j-2}}{x_{j-1}}\right]$ is equivalent to*

$$
\frac{p_{j-1}(k_2, \ldots, k_j + 1)x + 1}{p_j(k_1, \ldots, k_j)} < y \leq \frac{p_{j-1}(k_2, \ldots, k_j)x + 1}{p_j(k_1, \ldots, k_j)}\,. \tag{3.2}
$$

Proof. The lemma follows by translating $k_j = \left[\frac{1+x_{j-2}}{x_{j-1}}\right]$ into

$$
\frac{1 + x_{j-2}}{x_{j-1}} - 1 < k_j \leq \frac{1 + x_{j-2}}{x_{j-1}}\,,
$$

and inserting here the information provided by (3.1). □

The *Farey triangle* is defined by

$$
\mathcal{T} = \big\{(x, y)\colon\ x + y > 1,\ \text{and } 0 < x, y \leq 1\big\}\,,
$$

and

$$
\mathcal{T}_r[k_1, \ldots, k_r] = \Big\{(x, y) \in \mathcal{T}\colon\ k_j = \left[\tfrac{1+x_{j-2}}{x_{j-1}}\right],\ \text{for } 1 \leq j \leq r\Big\}\,. \tag{3.3}
$$

We call $\mathcal{T}_r[\mathbf{k}]$ the *tile* of $\mathbf{k}$. Similarly we say that any pair $(x, y) \in \mathcal{T}_r[\mathbf{k}]$ is a *germ* of $\mathbf{k}$.

Notice that one can write

$$
\mathcal{T}_j[k_1, \ldots, k_j] = \left\{(x, y) \in \mathcal{T}_{j-1}[k_1, \ldots, k_{j-1}]\colon\ k_j = \left[\frac{x+1}{y}\right]\right\}, \quad \text{for } j \geq 1, \tag{3.4}
$$

with $\mathcal{T}_0[\cdot] := \mathcal{T}$. This shows that any tile of any admissible chain is a convex polygon. It is easy to see that any two tiles are disjoint, and the set of all tiles $\mathcal{T}_r[\mathbf{k}]$ with $\mathbf{k} \in \mathcal{A}_r$ form a partition of $\mathcal{T}$, which we call the *tessellation* of order r. In this language, our main problem is to estimate the number of tiles in such a tessellation.

The expression (3.4) gives also an algorithm to find germs of $\mathbf{k}$: Calculate $\mathcal{T}_r[\mathbf{k}]$; if it is empty, then $\mathbf{k}$ is not admissible. Otherwise choose Q sufficiently large and pick a pair of relatively prime integers $(q_0, q_1) \in Q\mathcal{T}_r[\mathbf{k}]$. This is an integer germ of $\mathbf{k}$ and any other germ is obtainable by this method. In conclusion, given a chain of valences $\mathbf{k}$ and Q sufficiently large, the polygon $Q\mathcal{T}[\mathbf{k}]$ contains plenty generators of $\mathbf{k}$.

4. Small orders

Here we look at the size of valences for several small orders.

`Case` $r = 1$. Any positive integer is a valence, and the exact shape of the polygons $\mathcal{T}_1[\mathbf{k}]$ can be calculated easily by the definition (see the first case of relation (5.4)). Thus we have

$$
\sum_{\substack{\mathbf{k} \in \mathcal{A}_1 \\ 1 \leq k_1 \leq K}} 1 = K, \quad \text{for } K \geq 1\,. \tag{4.1}
$$

`Case` $r = 2$. Let k and l be two consecutive valences and suppose they are generated by q_1, q_2, q_3, q_4, that is,

$$kq_2 = q_1 + q_3\,, \qquad lq_3 = q_2 + q_4\,. \tag{4.2}$$

Lemma 4.3. *The smallest of any two consecutive neighbor valences cannot be larger than* 3.

Proof. By (4.2) and the fact that the sum of consecutive denominators of Farey fractions in $\mathfrak{F}_Q$ is larger than Q, it follows that

$$\min(k,l)Q < \min(k,l)(q_2+q_3) \le kq_2 + lq_3 = (q_1+q_3) + (q_2+q_4) \le 4Q\,,$$

which gives $\min(k,l) \le 3$, as required. □

Lemma 4.4. *There are no two neighbor valences both equal to* 1.

Proof. Let q_1, q_2, q_3, q_4 be consecutive denominators in $\mathfrak{F}_Q$, for some Q, and assume that (4.2) holds with $k = l = 1$. Then, adding the two relations, we obtain $q_1 + q_4 = 0$, a contradiction which completes the proof of the lemma. □

Lemma 4.5. *Let* (k,l) *be two neighbor valences in* $((\mathbf{k}))$. *Then, if one of* k *or* l *is* ≥ 5, *than the other is equal to* 1.

Proof. Since the pairs (k,l) and (l,k) are either both admissible or not, we may assume that $k \ge 5$. Let d_1 be the bottom edge of the quadrilateral $\mathcal{T}_1[k]$, and let d_2 be the top line of the strip that should intersect $\mathcal{T}_1[k]$ in order to have a nonempty $\mathcal{T}_2[k,l]$.

Our aim is to show that, for any $l \ge 2$, in the triangle $\mathcal{T}$, the line d_1 with equation $y = \frac{lx+1}{kl-1}$ is under the line d_2, whose equation is $y = \frac{x+1}{k+1}$. On $x = 1$ this is true since $(l+1)/(kl-1) \le 2/(k+1)$, which follows by our assumption that $4 \le (k-1)(l-1)$.

The slope of d_2 is greater than that of d_1, and this completes the proof of the lemma. □

Inspecting all the pairs (k_1,k_2) in the remaining cases, one finds the set of pairs of neighbor valences presented in Table 1 . In particular, this gives

$$\sum_{\substack{\mathbf{k}\in\mathcal{A}_2\\ 1\le k_1,k_2\le K}} 1 = 2K+3, \quad \text{for } K \ge 4\,. \tag{4.3}$$

`Case` $r = 3$. The larger the order, the larger the noise, in other words, many triples of consecutive valences occur. We check first only the end points of a triple.

Lemma 4.6. *Let* (k,l,m) *be three consecutive valences. Then,* $\min(k,m) < 8$.

Proof. Suppose k, l and m are generated by q_1, q_2, q_3, q_4, q_5, that is,

$$kq_2 = q_1 + q_3, \tag{4.4}$$

$$lq_3 = q_2 + q_4\,, \tag{4.5}$$

$$mq_4 = q_3 + q_5\,. \tag{4.6}$$

We split the argument in three parts.

`Case 1.` Suppose $mq_4 - kq_3 \ge 0$. By (4.4) and (4.6) we obtain $kq_2 + mq_4 \le 4Q$. Then

$$kQ < k(q_2+q_3) \le kq_2 + kq_3 + mq_4 - kq_3 < (q_1+q_3) + (q_3+q_5) \le 4Q\,,$$

which gives that $k < 4$.

`Case 2.` Suppose $kq_2 - mq_3 \ge 0$. By symmetry, or proceeding similarly as in `Case 1`, it follows that $l < 4$.

`Case 3.` Now assume that $mq_4 - kq_3 < 0$ and $kq_2 - mq_3 < 0$. Then

$$mQ < m(q_3 + q_4) < (k+m)q_3,$$
$$kQ < k(q_2 + q_3) < (k+m)q_3,$$

which give $Q < 2q_3$. Then, by (4.5), $Q/2 < q_3 \leq lq_3 = q_2 + q_4$. This implies

$$\min(k,m)Q/2 < \min(k,m)(q_2+q_4) \leq kq_2 + mq_4 \leq 4Q\,,$$

that is, $\min(k,m) < 8$, as claimed. □

On combining Lemma 4.6 with (4.3) and the analysis for the remaining triples summarized in Table 1, we obtain (1.3) for $r = 3$ with the error term $C(3) = 15$:

$$\sum_{\substack{\mathbf{k}\in\mathcal{A}_3 \\ 1\leq k_1,k_2,k_3\leq K}} 1 = 3K + 15, \quad \text{for } K \geq 4\,. \tag{4.7}$$

TABLE 1. Chains of valences. In the second column only one of $\mathbf{k}$ and its reverse $\reflectbox{$\mathbf{k}$}$ is included.

r	Chains of admissible valences of length r
1	(k) for $k \geq 1$
2	$(1,k)$ for $k \geq 2$; $(2,2)$; $(2,3)$; $(2,4)$
3	$(1,k,1)$ for $k \geq 3$; $(2,1,k)$ for $k \geq 6$; $(2,2,2)$; $(2,3,2)$; $(4,1,4)$;
	$(1,2,2)$; $(1,2,3)$; $(1,2,4)$; $(1,3,2)$; $(1,4,2)$; $(2,2,3)$;
	$(3,1,4)$; $(3,1,5)$; $(3,1,6)$; $(3,1,7)$; $(3,1,8)$; $(4,1,5)$
4	$(1,k,1,2)$ for $k \geq 6$; $(2,2,1,k)$ for $k \geq 10$; $(2,2,2,2)$;
	$(1,2,2,2)$; $(1,2,2,3)$; $(1,2,3,1)$; $(1,2,3,2)$; $(1,2,4,1)$; $(1,3,2,2)$;
	$(1,3,1,5)$; $(1,3,1,6)$; $(1,3,1,7)$; $(1,3,1,8)$; $(1,4,1,4)$; $(1,4,1,5)$; $(1,5,1,4)$;
	$(1,4,1,3)$; $(1,5,1,3)$; $(1,6,1,3)$; $(1,7,1,3)$; $(1,8,1,3)$;
	$(2,2,2,3)$; $(2,2,3,2)$; $(2,3,1,4)$; $(2,3,1,5)$; $(2,3,1,6)$; $(2,4,1,3)$; $(2,4,1,4)$;
	$(3,2,1,7)$; $(3,2,1,8)$; $(3,2,1,9)$; $(3,2,1,10)$; $(3,2,1,11)$; $(3,2,1,12)$;
	$(4,2,1,6)$; $(4,2,1,7)$; $(4,2,1,8)$

5. COMPLETION OF THE PROOF OF THEOREM 1.1

By induction, we show that at most one component of an admissible r-tuple can be excessively large. Thus, for a given $\mathbf{k} = (k_1, \dots, k_r) \in \mathcal{A}_r$ we have to show that the minimum of k_1 and k_r can not exceed a certain margin, while bounds for the other components $k_2, \dots, k_{r-1}$ follow by the induction hypothesis. For this it is helpful to see that $\mathcal{T}_r[\mathbf{k}]$ lies at the intersection between $\mathcal{T}_{r-1}[k_1, \dots, k_{r-1}]$ and the angular region defined by the last condition in the definition of $\mathcal{T}_r[\mathbf{k}]$:

$$\mathcal{V}(k_1, \dots, k_r) := \left\{(x,y) \in \mathcal{T} \colon\ k_r = \left[\frac{1+x_{r-2}}{x_{r-1}}\right]\right\},$$

in which the $x_j = x_j(k_1, \dots, k_j)$, for $j \geq 1$, are defined by (3.1). We claim that if both k_1 and k_r were large enough, then the intersection $\mathcal{T}_{r-1}[k_1, \dots, k_{r-1}] \cap \mathcal{V}(k_1, \dots, k_r)$ is empty. This would imply $\mathbf{k} \notin \mathcal{A}_r$, contradicting our assumption.

The main point of the proof is to show more than it is required. Namely, we shall show that even the superset $\mathcal{T}_1[k_1] \cap \mathcal{V}(k_1, \dots, k_r)$ is empty when k_1, k_r both surpass a certain magnitude. We do this by proving that the angle $\mathcal{V}(k_1, \dots, k_r)$ lies under $\mathcal{T}_1[k_1]$.

From (3.3) and (3.2) we know that for $k_1 \geq 2$, the line d', the bottom edge of quadrangle $\mathcal{T}_1[k_1]$ has equation $y = (x+1)/(k_1+1)$, and d'', the top edge of $\mathcal{V}(k_1, \dots, k_r)$ has equation $y = \frac{p_{r-1}(k_2,\dots,k_r)x+1}{p_r(k_1,\dots,k_r)}$. The argument has two parts. Firstly we see that in our hypotheses, the slope of d'' is greater than the slope of d' and secondly, we check the position of the points of intersection of d' and respectively d'' with the vertical line $\{x = 1\}$.

Let m', m'' be the slopes of d', d'' respectively. Then, $m'' > m'$ is equivalent to

$$k_1 p_{r-1}(k_2, \dots, k_r) > p_r(k_1, \dots, k_r)\,. \tag{5.1}$$

Here, by (R) and by the symmetry property (1.1), the right-hand side is

$$\begin{aligned} p_r(k_1, \dots, k_r) &= p_r(k_r, \dots, k_1) \\ &= k_1 p_{r-1}(k_r, \dots, k_2) - p_{r-2}(k_r, \dots, k_3) \\ &= k_1 p_{r-1}(k_2, \dots, k_r) - p_{r-2}(k_3, \dots, k_r)\,. \end{aligned} \tag{5.2}$$

Inserting (5.2) into (5.1) and reducing the terms, one finds that the inequality $m'' > m'$ is equivalent to $p_{r-2}(k_3, \dots, k_r) > 0$, which is always true for $(k_3, \dots, k_r) \in \mathcal{A}_{r-2}$.

Now, for the second part of the argument, let A and B be the points of intersection of d', d'' with $\{x = 1\}$ respectively, that is, $\{A\} = d' \cap \{x = 1\}$ and $\{B\} = d'' \cap \{x = 1\}$. It remains to show that B lies under A, which is the same as showing that

$$(k_1 + 1)\big(p_{r-1}(k_2, \dots, k_r) + 1\big) < p_r(k_1, \dots, k_r)\,. \tag{5.3}$$

In order to make apparent the influence of k_1 and k_r in this inequality, we extract them by reducing the order. This is done by using several times (R), as in (5.2). Then (5.3) reduces to the following inequality

$$\begin{aligned} k_1 p_{r-3}(k_2, \dots, k_{r-2}) &+ k_r\Big(2p_{r-3}(k_3, \dots, k_{r-1}) + p_{r-2}(k_2, \dots, k_{r-1})\Big) \\ &< k_1 k_r p_{r-2}(k_2, \dots, k_{r-1}) + p_{r-3}(k_2, \dots, k_{r-2}) + 2p_{r-4}(k_3, \dots, k_{r-2})\,. \end{aligned}$$

Here, since $k_2, \dots, k_{r-1}$ are bounded, the inequality becomes true as soon as both k_1 and k_r get larger than a certain quantity, so B lies under A.

In conclusion, an admissible tuple has at most one very large component. On the other hand there are many possible combinations that consist of small numbers that may form a subsequence of an admissible tuple. Moreover, when the components of $\mathbf{k}$ follow a regular pattern, the vertices of polygons $\mathcal{T}_r[\mathbf{k}]$ can be expressed in closed formulas. Such a pattern is $\dots, 1, 4, 1, 4, \dots$, but the meaningful example is the constant sequence of 2s that appear in the neighborhood of a large peak. The formulas recorded in the following proposition are obtained by recording the data, step by step, during an induction process that resembles the one described above.

Proposition 5.1. *Fix $s \geq 0$ and $t \geq 0$. Then there exists a positive integer k_0 depending on s and t only, such that for any integer $k \geq k_0$ the quadrangle $\mathcal{T}_{s+1+t}[\underbrace{2, \dots, 2, 1}_{s \text{ components}}, k, \underbrace{1, 2, \dots, 2}_{t \text{ components}}]$ has vertices given by:*

$$\begin{cases} \Big\{\Big(\frac{k}{k+2}, \frac{2}{k+2}\Big); \Big(\frac{k+1}{k+1}, \frac{2}{k+1}\Big); \Big(\frac{k}{k}, \frac{2}{k}\Big); \Big(\frac{k-1}{k+1}, \frac{2}{k+1}\Big)\Big\}, & \text{for s} = 0, \\ \\ \Big\{\Big(\frac{k-2s}{k+2}, \frac{k-2s+2}{k+2}\Big); \Big(\frac{k-2s+1}{k+1}, \frac{k-2s+3}{k+1}\Big); \\ \qquad \Big(\frac{k-2s}{k}, \frac{k-2s+2}{k}\Big); \Big(\frac{k-2s-1}{k+1}, \frac{k-2s+1}{k+1}\Big)\Big\}, & \text{for s} \geq 1\,. \end{cases} \tag{5.4}$$

Notice the two 'attractors' $(1, 0)$ and $(1, 1)$ of the shrinking quadrangles $\mathcal{T}_r[\mathbf{k}]$ with one component large. They are indicated by the first and second case of relation (5.4), respectively. In particular, since polygons $\mathcal{T}_r[k, *]$ are subsets of $\mathcal{T}_1[k]$, Proposition 5.1 shows that when a component k of an admissible tuple $\mathbf{k}$ is large enough, it should be followed by 1, and next, the more distant close neighbors should be 2s. By symmetry, this pattern identifies uniquely the components that precede the very large component, also, and this concludes the proof of the theorem.

Acknowledgement. The authors are grateful to the referee for his useful comments and suggestions.

References

[1] Athreya, J. S. and Cheung, Y., *A Poincaré Section for the Horocycle Flow on the Space of Lattices*, Int. Math. Res. Notices, Vol. **2014** (2014), No. 10, 2643–2690

[2] Augustin, V., Boca, F. P., Cobeli, C. and Zaharescu, A., *The h-spacing distribution between Farey points*, Math. Proc. Cambridge Philos. Soc., **131** (2001), 23–38

[3] Badziahin, D. A. and Haynes, A., *A note on Farey fractions with denominators in arithmetic progressions*, Acta Arith., **147** (2011), No. 3, 205–215

[4] Boca, F. P., *An AF algebra associated with the Farey tessellation*, Canad. J. Math., **60** (2008), No. 5, 975–1000

[5] Boca, F. P., Cobeli, C., and Zaharescu, A., *Distribution of lattice points visible from the origin*, Comm. Math. Phys., **213** (2000), 433–470

[6] Boca, F. P., Cobeli, C. and Zaharescu, A., *A conjecture of R. R. Hall on Farey points*, J. Reine. Angew. Math., **535** (2001), 207–236

[7] Boca, F. P., Cobeli, C. and Zaharescu, A., *On the distribution of the Farey sequence with odd denominators*, Michigan Math. J., **51** (2003), 557–573

[8] Boca, F. P., Gologan, R. N. and Zaharescu, A., *On the index of Farey sequences*, Q. J. Math., **53** (2002), No. 4, 377–391

[9] Boca, F. P. and Zaharescu, A., *The correlations of Farey fractions*, J. London Math. Soc. (2), **72** (2005), No. 1, 25–39

[10] Boca, F. P. and Zaharescu, A., *Farey fractions and two-dimensional tori*, Noncommutative geometry and number theory, 57–77, Aspects Math., E37, Vieweg, Wiesbaden, 2006

[11] Bruckheimer, M. and Arcavi, A., *Farey series and Pick's area theorem*, The Math. Intelligencer, **17** (1995), No. 4, 64–67

[12] Cobeli, C., Ford, K. and Zaharescu, A., *The jumping champions of the Farey series*, Acta Arith., **110** (2003), No. 3, 259–274

[13] Cobeli, C., Iordache, A. and Zaharescu, A., *The relative size of consecutive odd denominators in Farey series*, Integers Electron. J. Comb. Number Theory, **3** (2003), A7, 14 pp.

[14] Cobeli, C., Vâjâitu, M. and Zaharescu, A., *A density theorem on even Farey fractions*, Rev. Roumaine Math. Pures Appl., **55** (2010), No. 6, 447–481

[15] Cobeli, C., Vâjâitu, M. and Zaharescu, A., *On the intervals of a third between Farey fractions*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), **53 (101)** (2010), No. 3, 239–250

[16] Cobeli, C., Vâjâitu, M. and Zaharescu, A., *The distribution of rationals in residue classes*, Math. Reports, **14 (64)** (2012), No. 1, 1–19

[17] Cobeli, C., and Zaharescu, A., *The Haros-Farey sequence at two hundred years*, Acta Univ. Apulensis Math. Inform., **5** (2003), 1–38

[18] Cobeli, C., Vâjâitu, M. and Zaharescu, A., *On the Farey fractions with denominators in arithmetic progression*, J. Integer Sequences, **9** (2006), Article 06.3.4, 26 pp. (electronic)

[19] Haynes, A., *A note on Farey fractions with odd denominators*, J. Number Theory, **98** (2003), No. 1, 89–104

[20] Haynes, A., *The distribution of special subsets of the Farey sequence*, J. Number Theory, **107** (2004), No. 1, 95–104

[1] "Simion Stoilow" Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, Bucharest 70700, Romania
E-mail address: `cristian.cobeli@imar.ro`
E-mail address: `zaharesc@illinois.edu`

[2] Department of Mathematics
University of Illinois at Urbana - Champaign
1409 West Green Street Urbana IL 61801, USA
E-mail address: `zaharesc@illinois.edu`