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Bounds of the second degree cumulative frontier gaps of functions with generalized convexity

GABRIELA CRISTESCU, MUHAMMAD ASLAM NOOR and MUHAMMAD UZAIR AWAN

ABSTRACT. We prove that the set of second degree cumulative frontier gaps, via fractional integrals of positive orders, of twice differentiable functions having generalized convexity at the level of the second derivative is upper bounded. A sharp Hermite-Hadamard type inequality via fractional integrals leads to an evaluation of this upper bound, which does not depend on the order of the fractional integration. Six particular generalized convexity properties are investigated from the point of view of this boundary property.

1. INTRODUCTION

In a recent paper [9] by G. Maksa and Z. Páles, the problem of existence of the generalized affine functions corresponding to various concepts of generalized convex functions of segmental type (see [4]) is solved. They consider a real or complex topological vector space X, T a nonempty set, $\alpha, \beta, a, b : T \to \mathbb{R}$ and $D \subset X$ a nonempty (α, β)-convex open set, that is $\alpha(t)x + \beta(t)y \in D$, whenever $x, y \in D$ and $t \in T$. They identify those functions $f : D \to \mathbb{R}$ that make sharp the functional inequality

$$f(\alpha(t)x + \beta(t)y) \leq a(t)f(x) + b(t)f(y), \quad \forall x, y \in D, t \in T. \tag{1.1}$$

The functions satisfying (1.1) are called (α, β, a, b)-convex and the functions, which make sharp (1.1) are called (α, β, a, b)-affine.

The framework of the present paper is a particular case of (α, β, a, b)-convex functions, in which $X = \mathbb{R}$ the set of all real numbers, $D = I \subseteq \mathbb{R}$ is an open interval, $T = [0, 1]$, $\alpha(t) = t$ and $\beta(t) = 1 - t$, $t \in T$. We denote by $h_1(t) = a(t)$ and $h_2(t) = b(t)$, $t \in T$, and obtain the concept of (h_1, h_2)-convexity for functions defined over a classically convex domain. In this paper we study some properties of twice differentiable functions, which have generalized convexity properties of this type at the level of their second derivative. We define and study the cumulative frontier second degree gap of a function on a closed subinterval of its domain, which represents the behavior of the function and its derivatives in the inner neighborhood of the extremities. The main result is that the set of all the cumulative frontier second degree gaps of such a function is upper bounded. This result is obtained by deriving a sharp Hermite-Hadamard type inequality via Riemann-Liouville fractional integrals within the above described class of functions. This kind of inequalities are useful in applied statistics in social sciences, in physics and in engineering.

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Corresponding author: Muhammad Uzair Awan; awan.uzair@gmail.com

The boundary results derived in this paper are used, in the last section, to get special sharp inequalities of Hermite-Hadamard type for twice differentiable functions having their second derivative within the framework of five types of convexity properties: classical convexity, P-functions defined by S. S. Dragomir, J. E. Pečarić and L. E. Persson [5], W. W. Breckner's s-convexity in the second sense [1] and the h-convexity defined in 2007 by S. Varošanec [13]. Also, the Godunova-Levin class of functions [7] provides us with an example of unbounded set of cumulative frontier second degree gaps. Other generalized convexity properties of segmental type (see [4]), to which our results do not apply are W. Orlicz's s-convexity in the first sense [11], and also can be found in [6] and in [10].

2. Hermite-Hadamard inequalities and upper bounds of cumulative frontier second degree gap

Let $\mathbb{R}$ be the set of real numbers and $I \subseteq \mathbb{R}$ an open interval. Throughout the paper $[a,b] \subseteq I \subset \mathbb{R}$ is an interval unless otherwise specified and $L_1[a,b]$ denotes the set of all the functions $f : I \to \mathbb{R}$, which are Lebesque integrable over $[a,b]$. Let us consider two non-negative functions $h_1, h_2 : [0,1] \to \mathbb{R}$. In this section we focus on the class of (h_1,h_2)-convex functions, defined as it follows.

Definition 2.1. A function $f : I \to (0,\infty)$ is said to be (h_1,h_2)-convex if

$$f\left(tx+(1-t)y\right) \le h_1(t)f(x)+h_2(t)f(y), \quad \forall x,y \in I, t \in [0,1]. \tag{2.2}$$

The functions that transform (2.2) into an identity are called (h_1,h_2)-affine. The following characterization will be useful in the proofs of our main results, since it has consequences on the integrability of the (h_1,h_2)-convex functions, whether h_1 and h_2 are integrable.

Lemma 2.1. *If a function $f : I \to \mathbb{R}$ then the following statements are equivalent:*

(1) f is (h_1,h_2)-convex on I;
(2) The function $\mu_{x,y} : [0,1] \to \mathbb{R}$ defined by $\mu_{x,y}(t) = f(tx+(1-t)y)$ is (h_1,h_2)-convex on $[0,1]$, whenever $x,y \in I$.

Proof. Suppose that f is (h_1,h_2)-convex on I. Let us consider $t_1,t_2 \in [0,1]$ and $\alpha \in [0,1]$. Then for $x,y \in I$ one has:

$$\mu_{x,y}(\alpha t_1+(1-\alpha)t_2) = f((\alpha t_1+(1-\alpha)t_2)x+(1-\alpha t_1-(1-\alpha)t_2)y)$$

$$= f(\alpha t_1 x+\alpha(1-t_1)y+(1-\alpha)t_2)x+(1-\alpha)(1-t_2)y)$$

$$\le h_1(\alpha)f(t_1x+(1-t_1)y)+h_2(\alpha)f(t_2x+(1-t_2)y)$$

$$= h_1(\alpha)\mu_{x,y}(t_1)+h_2(\alpha)\mu_{x,y}(t_2),$$

which proves the (h_1,h_2)-convexity of $\mu_{x,y}$.
Conversely, supposing that $\mu_{x,y}$ is h-convex, one has

$$f(\alpha x+(1-\alpha)y) = \mu_{x,y}(\alpha) = \mu_{x,y}(1\alpha+(1-\alpha)0)$$

$$\le h_1(\alpha)\mu_{x,y}(1)+h_2(\alpha)\mu_{x,y}(0) = h_1(\alpha)f(x)+h_2(\alpha)f(y),$$

which means that function f is (h_1,h_2)-convex on I, as required. □

Definition 2.2. Let $a \geq 0$ and $f \in L_1[a,b]$. Then Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} x^{\alpha-1} dx$ is the Gamma function.

In what follows we consider that I is such an interval as it is possible to find $0 \leq a < b$ and $[a,b] \subset I$.

Definition 2.3. Let f a differentiable function. Then the second degree cumulative frontier (α, n)-gap of function f on $[a,b]$ is the number

$$\begin{aligned}(2.3)\qquad \mathcal{G}^{(2)}_{\alpha,n}(a,b)(f) = &\frac{(n+1)^{\alpha}\Gamma(\alpha+2)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)-} f(a) + J^{\alpha}_{\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)+} f(b) \right] \\ &- \frac{b-a}{n+1} \left[f'\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right) - f'\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right) \right] \\ &- (\alpha+1) \left[f\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right) + f\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right) \right].\end{aligned}$$

First we derive few preliminary properties, which are useful in proving our next results.

Lemma 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function on (a,b) with $a < b$. If $f" \in L_1[a,b]$ and $n \in \mathbb{N}^*$, then the following equality for fractional integrals holds*

$$\begin{aligned}(2.4)\qquad &\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f) \\ &= \frac{(b-a)^2}{(n+1)^2} \int_0^1 (1-t)^{\alpha+1} \left[f"\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) + f"\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right) \right] dt.\end{aligned}$$

Proof. For proving this identity, one split the right side into two integrals and compute these integrals by two successive integrations by parts. □

The following Pachpatte type inequality (see [12], [3]) will be useful in the proofs of the main result in this paper.

Theorem 2.1. *Let $h_1, h_2, h_3, h_4 : [0,1] \to \mathbb{R}$ be non-negative functions, $f : I \to \mathbb{R}$ a non-negative (h_1,h_2)-convex function and $g : I \to \mathbb{R}$ a non-negative (h_3,h_4)-convex function. If $h_k \in L_1[0,1]$, $k \in \{1,2,3,4\}$, and $f, g \in L_1[a,b]$, then the following inequality holds:*

$$\begin{aligned}(2.5)\qquad \frac{1}{b-a}\int_a^b f(x)g(x)dx \leq\ & f(a)g(a)H_{1,3}(t,t) + f(b)g(b)H_{2,4}(t,t) \\ &+ f(b)g(a)H_{2,3}(t,t) + f(a)g(b)H_{1,4}(t,t),\end{aligned}$$

whenever $a, b \in I$, $a < b$. Here

$$(2.6)\qquad H_{i,j}(u(t), v(t)) = \int_0^1 h_i(u(t)) h_j(v(t)) dt,$$

with $u, v : [0,1] \to [0,1]$ continuous functions.

Proof. The convexity properties of functions f and g mean that whenever $a, b \in I$, $a < b$ and $t \in [0,1]$ the following inequalities are valid:

$$f(ta + (1-t)b) \leq h_1(t)f(a) + h_2(t)f(b),$$

$$g(ta + (1-t)b) \leq h_3(t)g(a) + h_4(t)g(b).$$

Multiplying side by side these inequalities, one gets:

$$f(ta + (1-t)b)g(ta + (1-t)b) \leq h_1(t)h_3(t)f(a)g(a) + h_2(t)h_4(t)f(b)$$

$$+h_1(t)h_4(t)f(a)g(b) + h_2(t)h_3(t)f(b)g(a).$$

Due to Lemma 2.1 and the hypotheses, it is possible to integrate this inequality side by side over $[0,1]$, getting:

$$\int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt \leq f(a)g(a)H_{1,3}(t,t) + f(b)g(b)H_{2,4}(t,t)$$

$$+f(b)g(a)H_{2,3}(t,t) + f(a)g(b)H_{1,4}(t,t),$$

which becomes formula (2.5) after the change of variable $x = ta + (1-t)b$ in the left side integral. $\square$

Theorem 2.2. *Let $h_1, h_2, h_3, h_4 : [0,1] \to \mathbb{R}$ be non-negative functions, $h_3(t) = t$, $h_4(t) = 1-t$, $t \in [0,1]$, and $f : I \to \mathbb{R}$ a twice differentiable function. If $a, b \in I$, $a < b$, $f" \in L_1[a,b]$, $|f"|$ is a (h_1, h_2)-convex function on $[a,b]$ and $h_1, h_2 \in L_1[0,1]$ then the set*

$$\{|\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f)| | \alpha \geq 0\} \tag{2.7}$$

is upper bounded as

$$|\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f)| \leq \mathcal{S}^{(2)}_n(a,b)(f) \tag{2.8}$$

$$= \left(\frac{b-a}{n+1}\right)^2 |f"(a)| \left[h_1\left(\frac{n}{n+1}\right) H_{2,4}(t,t) + h_1\left(\frac{1}{n+1}\right) H_{1,3}(t,t) + H_{1,4}(t,t)\right]$$

$$+ \left(\frac{b-a}{n+1}\right)^2 |f"(b)| \left[h_2\left(\frac{1}{n+1}\right) H_{2,4}(t,t) + h_2\left(\frac{n}{n+1}\right) H_{1,3}(t,t) + H_{2,3}(t,t)\right],$$

for any $n \in \mathbb{N}$.

Proof. Due to Lemma 2.2 it follows that it is enough to evaluate the right side of formula (2.4). First, let us transform the two integrals, $I_1 = \int_0^1 (1-t)^{\alpha+1} f"\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) dt$ and $I_2 = \int_0^1 (1-t)^{\alpha+1} f"\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right) dt$. By the change of variable $x = \frac{(n+t)a+(1-t)b}{n+1}$ performed in I_1 and also, denoting by $x = \frac{(1-t)a+(n+t)b}{n+1}$ in I_2, one gets
$I_1 = \left(\frac{n+1}{b-a}\right)^{\alpha+2} \int_a^{\frac{na+b}{n+1}} (x-a)^{\alpha+1} f"(x)dx$ and $I_2 = \left(\frac{n+1}{b-a}\right)^{\alpha+2} \int_{\frac{a+nb}{n+1}}^b (b-x)^{\alpha+1} f"(x)dx$.
As consequence,

$$|\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f)| \leq \left(\frac{n+1}{b-a}\right)^{\alpha} \left[\int_a^{\frac{na+b}{n+1}} (x-a)^{\alpha+1}|f"(x)|dx + \int_{\frac{a+nb}{n+1}}^b (b-x)^{\alpha+1}|f"(x)|dx\right].$$

Let us notice that the classical convexity property, written in generalized terms, means the (h_3, h_4)-convexity. Then the function $g_1(x) = (x-a)^{\alpha+1}$ is (h_3, h_4)-convex on $\left[a, \frac{na+b}{n+1}\right]$.

Also, $g_2(x) = (b-x)^{\alpha+1}$ is (h_3, h_4)-convex on $\left[\frac{a+nb}{n+1}, b\right]$. So, we use the previous Pachpatte type inequality (2.5) into the above written inequality and obtain:

$$|\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f)| \leq \left(\frac{b-a}{n+1}\right)^2 \left[|f"\left(\frac{na+b}{n+1}\right)|H_{2,4}(t,t) + |f"(a)|H_{1,4}(t,t)\right]$$
$$+ \left(\frac{b-a}{n+1}\right)^2 \left[|f"\left(\frac{a+nb}{n+1}\right)|H_{1,3}(t,t) + |f"(b)|H_{2,3}(t,t)\right].$$

Since $|f"|$ is (h_1, h_2)-convex, this inequality becomes, after using (2.2), the following estimation:

$$|\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f)| \leq \left(\frac{b-a}{n+1}\right)^2 [|f"(a)|H_{1,4}(t,t) + |f"(b)|H_{2,3}(t,t)]$$
$$+ \left(\frac{b-a}{n+1}\right)^2 \left[h_1\left(\frac{n}{n+1}\right)|f"(a)| + h_2\left(\frac{1}{n+1}\right)|f"(b)|\right] H_{2,4}(t,t)$$
$$+ \left(\frac{b-a}{n+1}\right)^2 \left[h_1\left(\frac{1}{n+1}\right)|f"(a)| + h_2\left(\frac{n}{n+1}\right)|f"(b)|\right] H_{1,3}(t,t).$$

After conveniently grouping the terms of the right side of this inequality one obtains that

$$|\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f)| \leq \mathcal{S}^{(2)}_n(a,b)(f),$$

for any $\alpha \geq 0$, as required. □

Theorem 2.3. *There are two non-negative integrable functions $h_1, h_2 : [0,1] \to \mathbb{R}$, $h_1, h_2 \in L_1[a,b]$, and a twice differentiable function $f : I \to \mathbb{R}$, such as $f" \in L_1[a,b]$ and $|f"|$ is a (h_1,h_2)-convex function on $[a,b] \subseteq I$, which make the inequality (2.8) sharp.*

Proof. In order to find an example of context, in which (2.8) is sharp we think about some known particular cases of (h_1,h_2)-affine functions, identified in [9]. We take $0 < s \leq 1$, $h_1(t) = t^s$, $h_2(t) = (1-t)^s$, $t \in [0,1]$ and obtain the s-convexity in the second sense [1]. The s-affine in the second sense functions are constant, as proved in [9]. So, we should have $|f"| \equiv c$ for some $c > 0$. Due to the continuity properties of the s-convex in the second sense functions and to the fact that $f"$ should have Darboux property, it follows that $f" \equiv 0$. As consequence, the right side of (2.8) is zero for all $|f"|$ s-affine functions. It means that one should have $f(x) = cx + d$, with $c, d \in \mathbb{R}$, whenever $x \in [a,b]$. By direct calculus one can deduce that $\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f) = 0$, for every $\alpha > 0$. So, the (2.8) is sharp. □

3. Applications to particular generalized convexity cases

In this section we study the cumulative frontier behavior of the Riemann-Liouville fractional integrals of a twice differentiable function $f : I \to \mathbb{R}$, in the

$$\left[a, a + \frac{b-a}{n+1}\right] \bigcup \left[b - \frac{b-a}{n+1}, b\right]$$

neighborhood of the extremities of an interval $[a,b] \subseteq I$, when some particular convexity-type properties are fulfilled.

Case 1. Classic convexity

Function f is convex (i.e. classically convex) if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

whenever $x, y \in I$ and $t \in [0,1]$.

Corollary 3.1. *If $f : I \to \mathbb{R}$ is twice differentiable, $|f"|$ is convex on I and $[a, b] \subseteq I$, then*

$$(3.9)\qquad |\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f)| \le \frac{(b-a)^2}{2(n+1)^2}\left(|f"(a)| + |f"(b)|\right)$$

for any $\alpha \ge 0$.

Proof. The inequality (3.9) is a particular case of (2.8). Taking $h_1, h_2 : [0,1] \to \mathbb{R}$ defined by $h_1(t) = t$ and $h_2(t) = 1 - t$, $t \in [0,1]$ one gets $H_{2,4}(t,t) = H_{1,3}(t,t) = \frac{1}{3}$, $H_{1,4}(t,t) = H_{2,3}(t,t) = \frac{1}{6}$. Replacing these formulas in (2.8) one obtains the required result by elementary calculus. □

Case 2. P-functions

As in [5], a nonnegative function $f : I \to \mathbb{R}$ is called a P-function (or a function of P-type) if

$$(3.10)\qquad f(tx + (1-t)y) \le f(x) + f(y), \quad \forall x, y \in I, t \in [0,1].$$

Corollary 3.2. *If $f : I \to \mathbb{R}$ is twice differentiable, $|f"|$ is a P-function on I and $[a, b] \subseteq I$, then*

$$(3.11)\qquad |\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f)| \le \frac{3}{2}\frac{(b-a)^2}{(n+1)^2}\left(|f"(a)| + |f"(b)|\right)$$

for any $\alpha \ge 0$.

Proof. The inequality (3.11) is a particular case of (2.8), obtained taking $h_1, h_2 : [0,1] \to \mathbb{R}$ defined by $h_1(t) = h_2(t) = 1$, for any $t \in [0,1]$. By elementary calculus one gets $H_{2,4}(t,t) = H_{1,3}(t,t) = H_{1,4}(t,t) = H_{2,3}(t,t) = \frac{1}{2}$ and replacing in (2.8) one obtains (3.11). □

Case 3. Breckner convexity

As in [1], suppose that $0 < s \le 1$. A function $f : I \to \mathbb{R}$ is said to be s-convex in the second sense (or Breckner-convex, since it originates in [1]) if

$$(3.12)\qquad f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y), \quad \forall x, y \in I, t \in [0,1].$$

Corollary 3.3. *If $f : I \to \mathbb{R}$ is twice differentiable, $|f"|$ is a Breckner-convex function on I and $[a, b] \subseteq I$, then*

$$(3.13)\qquad |\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f)| \le \frac{(b-a)^2}{(s+2)(n+1)^{s+2}}\left(n^s + 1 + \frac{(n+1)^s}{s+1}\right)\left(|f"(a)| + |f"(b)|\right),$$

for any $\alpha \ge 0$. The inequality (3.13) is sharp if and only if $f(x) = cx + d$, with $c, d \in \mathbb{R}$.

Proof. The inequality (3.13) is a particular case of (2.8). Taking $h_1, h_2 : [0,1] \to \mathbb{R}$ defined by $h_1(t) = t^s$, $h_2(t) = (1-t)^s$, for any $t \in [0,1]$, one has $H_{2,4}(t,t) = H_{1,3}(t,t) = \frac{1}{s+2}$, $H_{1,4}(t,t) = \frac{1}{s+1} - \frac{1}{s+2}$ and $H_{2,3}(t,t) = \frac{1}{(s+1)(s+2)}$. Replacing these results in (2.8) and conveniently arranging the result one gets (3.13).
The sharpness of (3.13) was discussed within the proof of Theorem 2.3. The maximality of the sharpness class is a consequence of the continuity properties of the s-convex functions in the second sense [2] and the sharpness condition of (3.12) for constant functions only, proved in [9], Corollary 3.3. □

Remark 3.1. If $s = 1$ the inequality (3.13) becomes that from the classical convexity case (3.9).

Case 4. Godunova-Levin class of functions

A function $f : I \to \mathbb{R}$ is said to belong to the Godunova-Levin class of functions (see [7]) if

$$f(tx + (1-t)y) \le \frac{f(x)}{t} + \frac{f(y)}{1-t}, \quad \forall x, y \in I, t \in (0,1). \tag{3.14}$$

This is an example of convexity type property, in which Theorem 2.2 does not apply in order to get information on the upper bound of $|\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f)|$. In this case $h_1, h_2 : (0,1) \to \mathbb{R}$, $h_1(t) = \frac{1}{t}$, $h_2(t) = \frac{1}{1-t}$ and, as consequence, $H_{2,4}(t,t) = H_{1,3}(t,t) = 1$ but $H_{1,4}(t,t)$ is expressible by means of a divergent improper integral.

Case 5. Varošanec h-convexity

Let us suppose, as in [13], that $h : [0,1] \to \mathbb{R}$ is a nonnegative function. A function $f : I \to \mathbb{R}$ is said to be h-convex on I if

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y), \quad \forall x, y \in I, t \in [0,1]. \tag{3.15}$$

Corollary 3.4. *If $h : [0,1] \to \mathbb{R}$ is a nonnegative integrable function, if $f : I \to \mathbb{R}$ is twice differentiable, such that $|f"|$ is a h-convex function on I, if $[a,b] \subseteq I$, $f" \in L_1[a,b]$, then*

$$|\mathcal{G}^{(2)}_{\alpha,n}(a,b)(f)| \tag{3.16}$$

$$\le \left(\frac{b-a}{n+1}\right)^2 \left[h\left(\frac{n}{n+1}\right) H_{2,4}(t,t) + h\left(\frac{1}{n+1}\right) H_{1,3}(t,t) + H_{1,4}(t,t)\right] (|f"(a)| + |f"(b)|),$$

for any $\alpha \ge 0$. Here $H_{1,3}(t,t)$, $H_{2,4}(t,t)$ and $H_{1,4}(t,t)$ are defined as in Theorem 2.2, with $h_1(t) = h(t)$ and $h_2(t) = h(1-t)$, $t \in [0,1]$.

Proof. Indeed, in case of h-convexity one has $h_1\left(\frac{n}{n+1}\right) = h_2\left(\frac{1}{n+1}\right) = h\left(\frac{n}{n+1}\right)$, and $h_1\left(\frac{1}{n+1}\right) = h_2\left(\frac{n}{n+1}\right) = h\left(\frac{1}{n+1}\right)$. On another hand, the change of variable $u = 1 - t$ yields $H_{2,3}(t,t) = H_{1,4}(t,t)$. Replacing all these in (2.8) one gets (3.16). □

Remark 3.2. If $h(t) = t^s$, $t \in [0,1]$ and $0 < s \le 1$, then the inequality (3.16) becomes (3.13) of the Breckner convexity case. If $s = 1$, then one gets the classical convexity case (3.9).

Remark 3.3. The previous remark means that the inequality (3.16) is sharp. Its sharpness class is larger than that identified in this section, depending on the function h, as described in [9].

The possibility of obtaining Hermite-Hadamard inequalities via fractional integrals for various types of generalized convexities from the class of (h_1, h_2)-convexities reduces to obtaining particular cases of (2.8), with convenient values of $n \in \mathbb{N}$.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES
"AUREL VLAICU" UNIVERSITY OF ARAD
BD. REVOLUTIEI 77, 310130-ARAD, ROMANIA
E-mail address: gabriela.cristescu@uav.ro

DEPARTMENT OF MATHEMATICS
COMSATS INSTITUTE OF INFORMATION TECHNOLOGY
PARK ROAD, ISLAMABAD, PAKISTAN
E-mail address: noormaslam@gmail.com
E-mail address: awan.uzair@gmail.com