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Extensions of Perov theorem

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ABSTRACT. [Perov, A. I., *On Cauchy problem for a system of ordinary diferential equations*, (in Russian), Priblizhen. Metody Reshen. Difer. Uravn., **2** (1964), 115-134] used the concept of vector valued metric space and obtained a Banach type fixed point theorem on such a complete generalized metric space. In this article we study fixed point results for the new extensions of Banach's contraction principle to cone metric space, and we give some generalized versions of the fixed point theorem of Perov. As corollaries some results of [Zima, M., *A certain fixed point theorem and its applications to integral-functional equations*, Bull. Austral. Math. Soc., **46** (1992), 179–186] and [Borkowski, M., Bugajewski, D. and Zima, M., *On some fixed-point theorems for generalized contractions and their perturbations*, J. Math. Anal. Appl., **367** (2010), 464–475] are generalized for a Banach cone space with a non-normal cone. The theory is illustrated with some examples.

1. INTRODUCTION

There exist many generalizations of the concept of metric spaces in the literature. Perov [16] used the concept of vector valued metric space, and obtained a Banach type fixed point theorem on such a complete generalized metric space. After that, fixed point results of Perov type in vector valued metric spaces were studied by many other authors (see e.g., [9], [11], [18], [19], [21] for some works in this line of research). Let us point out that Perov theorem and related results have many applications in coincidence problems, coupled fixed point problems and systems of semilinear differential inclusions. In this article we study fixed point results for the new extensions of Banach's contraction principle to cone metric space, and we give some generalized versions of the fixed point theorem of Perov. As corollaries we generalized some results of Zima [24] and Borkowski, Bugajewski and Zima [6] for a Banach space with a non-normal cone. The theory is illustrated with some examples.

Consistent with [12] (see, e.g., [1], [2], [3], [7], [10], [13], [14], [20], [22] for more details and recent results), the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if:
i) P is closed, nonempty and $P \neq \{0\}$;
ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subseteq E$, we define the partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$ (interior of P).

There exist two kinds of cones: normal and non-normal ones.

The cone P in a real Banach space E is called normal if

$$\inf\{\|x + y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0 \tag{1.1}$$

or, equivalently, if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K \|y\| . \tag{1.2}$$

The least positive number satisfying (2.2) is called the normal constant of P. It is clear that $K \geq 1$.

Definition 1.1. [12] Let X be a nonempty set, and let P be a cone on a real ordered Banach space E. Suppose that the mapping $d : X \times X \mapsto E$ satisfies:
(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

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Then d is called a cone metric on X and (X,d) is called a cone metric space.

It is known that the class of cone metric spaces is bigger than the class of metric spaces.

Example 1.1. Let $E = l^1, P = \left\{\{x_n\}_{n\geq 1} \in E : x_n \geq 0, \text{ for all } n\right\}, (X,\rho)$ be a metric space and $d : X \times X \mapsto E$ defined by $d(x,y) = \left\{\frac{\rho(x,y)}{2^n}\right\}_{n\geq 1}$. Then (X,d) is a cone metric space.

Example 1.2. Let $X = \mathbb{R}, E = \mathbb{R}^n$ and $P = \{(x_1,...,x_n) \in \mathbb{R}^n : x_i \geq 0\}$. It is easy to see that $d : X\times X \mapsto E$ defined by $d(x,y) = (|x-y|, k_1|x-y|, \ldots, k_{n-1}|x-y|)$ is a cone metric on X, where $k_i \geq 0$ for all $i \in \{1,...,n-1\}$.

Example 1.3. [7] Let $E = C^1[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E : x(t) \geq 0$ on $[0,1]\}$. This cone is not normal. Consider, for example,

$$x_n(t) = \frac{1-\sin nt}{n+2} \quad \text{and} \quad y_n(t) = \frac{1+\sin nt}{n+2}.$$

Since, $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| = \frac{2}{n+2} \to 0$, it follows by (1.1) that P is non-normal.

Let X be a nonempty set and $n \in \mathbb{N}$.

Definition 1.2. A mapping $d : X \times X \mapsto \mathbb{R}^n$ is called a *vector-valued metric* on X if the following statements are satisfied for all $x,y,z \in X$.

(d1) $d(x,y) \geq 0_n$ and $d(x,y) = 0_n$ if and only if $x = y$, $0_n = (0,\ldots,0) \in \mathbb{R}^n$;
(d2) $d(x,y) = d(y,x)$;
(d3) $d(x,y) \leq d(x,z) + d(z,y)$.

If $x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n) \in \mathbb{R}^n$, then $x \leq y$ means that $x_i \leq y_i$, $i = 1,\ldots,n$. This partial order determines normal cone $P = \{x = (x_1,\ldots,x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1,2,\ldots,n\}$ on $\mathbb{R}^n$, with the normal constant $K = 1$. A nonempty set X with a vector-valued metric d is called a *generalized metric space*.

Throughout this paper we denote by $\mathcal{M}_{n,n}$ the set of all $n\times n$ matrices, by $\mathcal{M}_{n,n}(\mathbb{R}^+)$ the set of all $n \times n$ matrices with nonegative elements. It is well known that if $A \in \mathcal{M}_{n,n}$, then $A(P) \subseteq P$ if and only if $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$. We write Θ for the zero $n\times n$ matrix and I_n for the identity $n \times n$ matrix. For the sake of simplicity we will identify row and column vector in $\mathbb{R}^n$.
A matrix $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ is said to be convergent to zero if $A^n \to \Theta$ as $n \to \infty$.

Theorem 1.1. (Perov [16], [17]) *Let (X,d) be a complete generalized metric space, $f : X \mapsto X$ and $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ is a matrix convergent to zero, such that*

$$d(f(x),f(y)) \leq A(d(x,y)), \quad x,y \in X.$$

Then:

i) f has a unique fixed point $x^* \in X$;

ii) the sequence of successive approximations $x_n = f(x_{n-1}), n \in \mathbb{N}$ converges to x^* for all $x_0 \in X$;

iii) $d(x_n,x^*) \leq A^n(I_n - A)^{-1}(d(x_0,x_1)),\ n \in \mathbb{N}$;

iv) if $g : X \mapsto X$ satisfies the condition $d(f(x),g(x)) \leq c$ for all $x \in X$ and some $c \in \mathbb{R}^n$, then by considering the sequence $y_n = g^n(x_0), n \in \mathbb{N}$, one has

$$d(y_n,x^*) \leq (I_n - A)^{-1}(c) + A^n(I_n - A)^{-1}(d(x_0,x_1)),\ n \in \mathbb{N}.$$

For completeness of the paper and convenience of the reader, we collect some basic definitions and facts which are applied in subsequent sections. In the following we suppose that E is a Banach space, P is a cone in E with int$P \neq \emptyset$ and $\leq$ is the partial order on E with respect to P.

Let $\{x_n\}$ be a sequence in X, and $x \in X$. If for every c in E with $0 \ll c$, there is n_0 such that for all $n > n_0$, $d(x_n,x) \ll c$, then it is said that $\{x_n\}$ converges to x, and we denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \to x, n \to \infty$. If for every c in E with $0 \ll c$, there is n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Let us recall [12] that if P is a normal cone then $\{x_n\} \subseteq X$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$, $n \to \infty$. Further, $\{x_n\} \subseteq X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$, $n, m \to \infty$.

Let (X, d) be a cone metric space. Then the following properties are often used (particulary when dealing with cone metric spaces in which the cone need not be normal):

($\mathbf{p}_1$) If $u \leq v$ and $v \ll w$ then $u \ll w$.

($\mathbf{p}_2$) If $0 \leq u \ll c$ for each $c \in \operatorname{int} P$ then $u = 0$.

($\mathbf{p}_3$) If $a \leq b + c$ for each $c \in \operatorname{int} P$ then $a \leq b$.

($\mathbf{p}_4$) If $0 \leq x \leq y$, and $a \geq 0$, then $0 \leq ax \leq ay$.

($\mathbf{p}_5$) If $0 \leq x_n \leq y_n$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $0 \leq x \leq y$.

($\mathbf{p}_6$) If $0 \leq d(x_n, x) \leq b_n$ and $b_n \to 0$, then $x_n \to x$.

($\mathbf{p}_7$) If E is a real Banach space with a cone P and if $a \leq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = 0$.

($\mathbf{p}_8$) If $c \in \operatorname{int} P$, $0 \leq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.

From ($\mathbf{p}_8$) it follows that the sequence $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$ and $\{x_n\}$ is a Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. In the situation with a non-normal cone we have only one part of Lemmas 1 and 4 from [12]. Also, in this case the fact that $d(x_n, y_n) \to d(x, y)$ if $x_n \to x$ and $y_n \to y$ is not applicable.

We write $\mathcal{B}(E)$ for the set of all bounded linear operators on E and $\mathcal{L}(E)$ for the set of all linear operators on E. $\mathcal{B}(E)$ is a Banach algebra, and if $A \in \mathcal{B}(E)$ let $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_n \|A^n\|^{1/n}$ be the spectral radius of A. Let us remark that if $r(A) < 1$, then the series $\sum_{i=0}^{\infty} A^n$ is absolutely convergent, $I - A$ is invertible in $\mathcal{B}(E)$ and $\sum\limits_{i=0}^{\infty} A^n = (I - A)^{-1}$.

2. Main results

In this section we prove our main results. We start with some auxiliary results.

Lemma 2.1. *Let (X, d) be a cone metric space. Suppose that $\{x_n\}$ is a sequence in X and that $\{bx_n\}$ is a sequence in E. If $0 \leq d(x_n, x_m) \leq b_n$ for $m > n$ and $b_n \to 0, n \to \infty$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. For every $c \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $b_n \ll c, n > n_0$. It follows that $0 \leq d(x_n, x_m) \ll c, m > n > n_0$, i.e., $\{x_n\}$ is a Cauchy sequence. □

Lemma 2.2. *Let E be Banach space, $P \subseteq E$ cone in E and $A : E \mapsto E$ a linear operator. The following conditions are equivalent:*

i) A is increasing, i.e., $x \leq y$ implies $A(x) \leq A(y)$.

ii) A is positive, i.e., $A(P) \subseteq P$.

Proof. If A is monotonically increasing and $p \in P$, then, by definitions, it follows $p \geq 0$ and $A(p) \geq A(0) = 0$. Thus, $A(p) \in P$, and $A(P) \subseteq P$.
To prove the other implication, let us assume that $A(P) \subseteq P$ and $x, y \in E$ are such that $x \leq y$. Now $y - x \in P$, and so $A(y - x) \in P$. Thus $A(x) \leq A(y)$. □

The results in the next theorem are applied to the cone metric spaces in the case when cone is not necessary normal, and Banach space should not be finite dimensional. This extends the results of Perov for matrices and also as a corollary we generalize Theorem 1 of Zima [24].

Theorem 2.2. *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, $f : X \mapsto X$, $A \in \mathcal{B}(E)$, with $r(A) < 1$ and $A(P) \subseteq P$, such that*

$$d(f(x), f(y)) \leq Ad(x, y), \quad x, y \in X. \tag{2.3}$$

Then:

i) f has a unique fixed point $z \in X$;

ii) For any $x_0 \in X$ *the sequence* $x_n = f(x_{n-1}),\ n \in \mathbb{N}$ *converges to* z *and*

$$d(x_n, z) \leq A^n(I - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N};$$

iii) Suppose that $g : X \mapsto X$ *satisfies the condition* $d(f(x), g(x)) \leq c$ *for all* $x \in X$ *and some* $c \in P$. *Then if* $y_n = g^n(x_0), n \in \mathbb{N}$, *we have*

$$d(y_n, z) \leq (I - A)^{-1}(c) + A^n(I - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

Proof. $i)$ For $n, m \in \mathbb{N}, m > n$, we have

$$0 \leq d(x_n, x_m) \leq \sum_{i=n}^{m-1} A^i(d(x_0.x_1)) \leq \sum_{i=n}^{\infty} A^i(d(x_0, x_1)).$$

Now, $r(A) < 1$, implies

$$\|\sum_{i=n}^{\infty} A^i(d(x_0, x_1))\| \leq \sum_{i=n}^{\infty} \|A^i\| \cdot \|(d(x_0, x_1))\| \to 0, \quad n \to \infty.$$

Thus $a_n = \sum_{i=n}^{\infty} A^i(d(x_0, x_1)) \to 0,\ n \to \infty$, and by Lemma 2.1 $\{x_n\}$ is a Cauchy sequence. Since X is a complete cone metric space, we know that there exists $z \in X$ such that $x_n \to z, n \to \infty$.

Let us prove that $f(z) = z$. Set $p = d(z, f(z))$, and suppose that $c \gg 0$ and $\epsilon \gg 0$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$d(z, x_n) \ll c \quad \text{and} \quad d(z, x_n) \ll \epsilon \quad \text{for all} \quad n \geq n_0.$$

Therefore, $p = d(z, f(z)) \leq d(z, x_{n+1}) + d(x_{n+1}, f(z)) \leq d(z, x_{n+1}) + A(d(z, x_n)) \leq c + A(\epsilon)$ for $n \geq n_0$. Thus, $p \leq c + A(\epsilon)$ for each $c \gg 0$, and so $p \leq A(\epsilon)$. Now, for $\epsilon = \epsilon/n$, $n = 1, 2, \ldots$, we get

$$0 \leq p \leq A\left(\frac{\epsilon}{n}\right) = \frac{A(\epsilon)}{n}, \quad n = 1, 2, \ldots.$$

Because $\frac{A(\epsilon)}{n} \to 0, n \to \infty$, this shows that $p = 0$, i.e., $z = f(z)$.

If $f(y) = y$, for some $y \in X$, then $d(z, y) \leq A(d(z, y))$. Thus, $d(z, y) \leq A^n(d(z, y))$, for each $n \in \mathbb{N}$. Furthermore, $r(A) < 1$ implies

$$\|A^n(d(z, y)\| \leq \|A^n\| \cdot \|(d(z, y)\| \to 0, \quad n \to \infty,$$

so, $d(z, y) = 0$, i.e., $z = y$.

$ii)$ By $i)$, for arbitrary $n \in \mathbb{N}$, we have

$$d(x_n, z) \leq A(d(x_{n-1}, z)) \leq \cdots \leq A^n(d(x_0, z)). \tag{2.4}$$

On the other hand,

$$\begin{aligned} d(x_0, z) &\leq d(x_0, x_n) + d(x_n, z) \\ &\leq \sum_{i=0}^{n-1} d(x_i, x_{i+1}) + A^n(d(x_0, x_1)) + A^n(d(x_1, z)) \\ &\leq \sum_{i=0}^{i=n} A^i(d(x_0, x_1)) + A^n(d(x_1, z)). \end{aligned}$$

Because $A^n(d(x_1, z)) \to 0,\ n \to \infty$, we get

$$d(x_0, z) \leq \sum_{i=0}^{\infty} A^i(d(x_0, x_1)) = (I - A)^{-1}(d(x_0, x_1)).$$

That implies $d(x_n, z) \leq A^n(I - A)^{-1}(d(x_0, x_1))$.

$iii)$ Let us remark that for any $n \in \mathbb{N}$,$d(y_n, z) \leq d(y_n, x_n) + d(x_n, z)$, and $ii)$ imply

$$d(y_n, z) \leq d(y_n, x_n) + A^n(I - A)^{-1}(d(x_0, x_1)).$$

Now,

$$\begin{aligned} d(y_n, x_n) &\le d(y_n, f(y_{n-1})) + d(f(y_{n-1}), x_n) \\ &\le c + A(d(y_{n-1}, x_{n-1})) \\ &\le c + A\bigg(d(y_{n-1}, f(y_{n-2})) + d(f(y_{n-2}), x_{n-1})\bigg) \\ &\le c + A(c) + A^2(d(y_{n-2}, x_{n-2})) \le \ldots \le \sum_{i=0}^{n-1} A^i(c) \le (I-A)^{-1}(c), \end{aligned}$$

implies $iii)$. □

Investigations of the existence of fixed points of set-valued contractions in metric spaces were initiated by S. B. Nadler [15]. The following theorem is motivated by Nadler's results and also generalizes the well-known Banach contraction theorem in several ways. Furthermore, it is a generalization of the recent result Theorem 3.2 of Borkowski, Bugajewski and Zima [6] for a Banach space with a non-normal cone.

Theorem 2.3. *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, and let T be a set-valued d–Perov contractive mapping (i.e. there exists $A \in \mathcal{B}(E)$, such that $r(A) < 1$, $A(P) \subseteq P$ and for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$ there is $y_2 \in Tx_2$ with $d(y_1, y_2) \le A(d(x_1, x_2))$ from X into itself such that for any $x \in X$, Tx is a nonempty closed subset of X. Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$, i.e., x_0 is a fixed point of T.*

Proof. Suppose that $u_0 \in X$ and $u_1 \in Tu_0$. Then there exists $u_2 \in Tu_1$ such that $d(u_1, u_2) \le A(d(u_0, u_1))$. Thus, we have a sequence $\{u_n\}_{n\ge 1}$ in X such that $u_{n+1} \in Tu_n$ and $d(u_n, u_{n+1}) \le A(d(u_{n-1}, u_n))$ for every $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have,

$$d(u_n, u_{n+1}) \le A(d(u_{n-1}, u_n)) \le \ldots \le A^n(d(u_0, u_1)). \tag{2.5}$$

So for $m > n$,

$$\begin{aligned} d(u_n, u_m) &\le d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \ldots + d(u_{m-1}, u_m) \\ &\le (A^n + A^{n+1} + \ldots + A^{m-1})(d(u_0, u_1)) \\ &\le A^n(I-A)^{-1}(d(u_0, u_1)) \to 0, \quad n \to \infty. \end{aligned}$$

Thus, $\{u_n\}_{n\ge 1}$ is a Cauchy sequence in X. Since X is a complete space there exists $v_0 \in X$ such that $u_n \to v_0$ as $n \to \infty$. Furthermore, for any $\epsilon \gg 0$ there exists $m_0 \in \mathbb{N}$ such that $d(u_m, v_0) \ll \epsilon$, $\quad m \ge m_0$. Thus, we have (for $m \ge \max\{n, m_0\}$)

$$\begin{aligned} d(u_n, v_0) &\le d(u_n, u_m) + d(u_m, v_0) \\ &\le A^n(1-A)^{-1}(d(u_0, u_1)) + \epsilon, \quad \text{for} \quad n \ge 1. \end{aligned}$$

Hence,

$$d(u_n, v_0) \le A^n(1-A)^{-1}(d(u_0, u_1)), \quad \text{for} \quad n \ge 1. \tag{2.6}$$

Let us define $w_n \in Tv_0$ such that $d(u_n, w_n) \le A(d(u_{n-1}, v_0))$, for $n \ge 1$. So, for any $n \in \mathbb{N}$,

$$d(u_n, w_n) \le A(d(u_{n-1}, v_0)) \le A^n(I-A)^{-1}(d(u_0, u_1)). \tag{2.7}$$

Moreover,

$$d(w_n, v_0) \le d(w_n, u_n) + d(u_n, v_0) \le 2 \cdot A^n(I-A)^{-1}(d(u_0, u_1)).$$

Thus, $\{w_n\}$ converges to v_0. Since Tv_0 is closed, it follows $v_0 \in Tv_0$. □

Example 2.4. Let $X = E$ and E be with non normal cone P as in Example 1.3. Let us define cone metric $d : X \times X \mapsto E$ by

$$d(f, g) = f + g, \quad \text{for} \quad f \ne g; \quad d(f, f) = 0, \quad f, g \in X.$$

If $T : X \mapsto X$ is defined by $T(f) = f/2$, $f \in X$, then

$$d(T(f), T(g)) \le A(d(f, g)), \quad f, g \in X,$$

where $A : E \mapsto E$ is a bounded linear operator defined by $A(f) = f/2$, $f \in E$. Clearly, $\|A\| = 1/2$, and all the assumptions from Theorem 2.3 are satisfied. Hence, T has a unique fixed point $f = 0 \in X$.

Remark 2.1. Let us remark that the initial assumption $A \in \mathcal{M}_{n,n}(\mathbb{R}_+)$, in Perov theorem, is unnecessary. The latest remark will be illustrated by the following example.

Example 2.5. Let

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$X = \left\{ \begin{bmatrix} x_1 \\ 1 \\ x_3 \end{bmatrix} \mid x_1, x_3 \in \mathbb{R} \right\}$ and $f : X \mapsto X$, $f\left(\begin{bmatrix} x_1 \\ 1 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} \frac{x_1+1}{2} \\ 1 \\ \frac{x_3+2}{3} \end{bmatrix}$. Set $\|x\| = \max\{|x_1|, |x_2|, |x_3|\}$ for $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $x_i \in \mathbb{R}, i = 1,2,3$.

For arbitrary $x \in X$, we have

$$\begin{aligned} \|Ax\| &= \max\left\{ |\frac{1}{2}x_1 - \frac{1}{4}x_2|, |\frac{1}{4}x_1 - \frac{1}{2}x_2|, |\frac{1}{2}x_3| \right\} \\ &\leq \max\left\{ \frac{1}{2}\|x\| + \frac{1}{4}\|x\|, \frac{1}{4}\|x\| + \frac{1}{2}\|x\|, \frac{1}{2}\|x\| \right\} = \frac{3}{4}\|x\|. \end{aligned}$$

Thus, $\|A\| \leq \frac{3}{4}$. If $x = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\|x\| = 1$, then $\|Ax\| = \frac{3}{4}$. Hence, $\|A\| = \frac{3}{4}$.

Now $r(A) \leq \|A\| = 3/4$ and $d(f(x), f(y)) \leq A(d(x,y))$, $x, y \in X$. Clearly, $A(P) \not\subseteq P$, and $(1,1,1)$ is a unique fixed point of f in X.

Based on the previous comments, we obtain the next result, where we do not suppose that $A(P) \subset P$.

Theorem 2.4. *Let (X,d) be a complete cone metric space, $d : X \times X \mapsto E$, P a normal cone with normal constant K, $A \in \mathcal{B}(E)$ and $K\|A\| < 1$. If the condition (2.3) holds for a mapping $f : X \mapsto X$, then f has a unique fixed point $z \in X$ and the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$, converges to z for any $x_0 \in X$.*

Proof. Let $x_0 \in X$ be arbitrary, $x_n = f(x_{n-1})$, $n \in \mathbb{N}$. Inequality

$$d(x_n, x_{n+1}) \leq A(d(x_{n-1}, x_n)), \quad n \in \mathbb{N},$$

implies

$$\begin{aligned} \|d(x_n, x_{n+1})\| &\leq K\|A(d(x_{n-1}, x_n))\| \leq K\|A\|\|d(x_{n-1}, x_n)\| \\ &\leq K^2\|A\|^2\|d(x_{n-2}, x_{n-1})\| \leq \ldots \leq K^n\|A\|^n\|d(x_0, x_1)\|. \end{aligned}$$

If $n, m \in \mathbb{N}, n < m$, then

$$\|d(x_n, x_m)\| \leq \sum_{i=n}^{m-1} \|d(x_i, x_{i+1})\| \leq \sum_{i=n}^{m-1} K^i\|A\|^i\|d(x_0, x_1)\|.$$

Clearly, $K\|A\| < 1$, implies that the series $\sum_{i=0}^{\infty} K^i\|A\|^i$ is convergent. Hence, $\|d(x_n, x_m)\| \to 0$, as $n, m \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence, and so there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Let us prove that $f(z) = z$. From $d(f(z), x_{n+1}) \leq A(d(z, x_n))$, we get

$$\|d(f(z), x_{n+1})\| \leq K\|A(d(z, x_n))\| \leq K\|A\|\|d(z, x_n)\|.$$

Thus, $\lim_{n\to\infty} x_n = f(z)$, and so $f(z) = z$.
It remains to show that z is a unique fixed point of f.
If $f(y) = y$, $y \in X$, then $d(z,y) = d(f(z), f(y)) \leq A(d(z,y))$ it follows $\|d(z,y)\| \leq K\|A\|\|d(z,y)\|$. Now, $K\|A\| < 1$ implies $d(z,y) = 0$, i.e., $z = y$. $\square$

Following the work of Berinde ([4], [5]), in the next theorem we investigate the weak contraction of Perov type.

Theorem 2.5. *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, $f : X \mapsto X$, $A \in \mathcal{B}(E)$, with $r(A) < 1$ and $A(P) \subseteq P$, $B \in \mathcal{L}(E)$ with $B(P) \subseteq P$, such that*

$$d(f(x), f(y)) \leq A(d(x, y)) + B(d(x, f(y))), \quad x, y \in X. \tag{2.8}$$

Then

i) $f : X \mapsto X$ has a fixed point in X and for any $x_0 \in X$ the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$, converges to a fixed point of f.
ii) If, under the previous conditions,

$$B \in \mathcal{B}(E) \quad \textit{and} \quad r(A + B) < 1, \tag{2.9}$$

or

$$d(f(x), f(y)) \leq Ad(x, y) + B(d(x, f^{n_0}(x))), \quad x, y \in X, \textit{for some } n_0 \in \mathbb{N}, \tag{2.10}$$

then f has a unique fixed point.

Proof. $i)$ For an arbitrary $x_0 \in X$ observe $x_n = f(x_{n-1})$, $n \in \mathbb{N}$. Since

$$\begin{aligned} d(x_n, x_{n+1}) &\leq A(d(x_{n-1}, x_n)) + B(d(x_n, f(x_{n-1}))) = A(d(x_{n-1}, x_n)) \\ &\leq A^2(d(x_{n-2}, x_{n-1})) \leq \ldots \leq A^n(d(x_0, x_1)), \end{aligned}$$

then, as in the proof of Theorem 2.3, we conclude that $\{x_n\}$ converges to some $z \in X$.

Let us prove that $f(z) = z$. Set $p = d(z, f(z))$, and suppose that $c \gg 0$ and $\epsilon \gg 0$. Hence, there exists $n_0 \in \mathbb{N}$ such that $d(z, x_n) \ll c$ and $d(z, x_n) \ll \epsilon$ for all $n \geq n_0$. Now

$$\begin{aligned} p = d(z, f(z)) &\leq d(z, x_{n+1}) + d(x_{n+1}, f(z)) \\ &\leq d(z, x_{n+1}) + A(d(z, x_n)) + B(d(z, x_{n+1})) \\ &\leq c + A(\epsilon) + B(\epsilon), \quad n \geq n_0. \end{aligned}$$

Thus, $p \leq c + A(\epsilon) + B(\epsilon)$ for each $c \gg 0$, and so $p \leq A(\epsilon) + B(\epsilon)$. For $\epsilon = \epsilon/n$, $n = 1, 2, \ldots$, we get

$$0 \leq p \leq A\left(\frac{\epsilon}{n}\right) + B\left(\frac{\epsilon}{n}\right) = \frac{A(\epsilon)}{n} + \frac{B(\epsilon)}{n}, \quad n = 1, 2, \ldots.$$

Because $A(\epsilon)/n + B(\epsilon)/n \to 0$, $n \to \infty$, this shows that $p = 0$, i.e., $z = f(z)$.

$ii)$ If $f(y) = y$, for some $y \in X$, then

$$d(z, y) \leq A(d(z, y)) + B((d(z, y)) = (A + B)((d(z, y)).$$

Thus $d(z, y) \leq (A + B)^n(d(z, y))$, for each $n \in \mathbb{N}$. Moreover, $r(A + B) < 1$ implies

$$\|(A + B)^n(d(z, y)\| \leq \|(A + B)^n\| \cdot \|(d(z, y)\| \to 0, \quad n \to \infty.$$

Hence, $d(z, y) = 0$, i.e., $z = y$. Let us remark that (2.10) implies $d(z, y) \leq A(d(z, y))$, and so

$$d(z, y) \leq A^n(d(z, y)), \quad \text{for each} \quad n \in \mathbb{N}.$$

The rest of the proof follows from the proof of Theorem 2.3 $i)$. □

Remark 2.2. Let us remark that in the works of [8] and [23] the authors have studied (2.3) with more general approach where A is a nonlinear operator and $A(P) \subseteq P$. Their results are given for the case where cone P is a normal cone. For example, the "policeman lemma" is essential in their results (see p.p. 369 of [8]) while the policeman lemma is not true it the case where P is non normal cone. Furthermore, we do not suppose that $A(P) \subseteq P$ where cone P is normal. If A is a linear operator and obeys (2.3) then results in [23] are given under special assumptions on A and on a cone P (such that X is a sequentially complete (in the Weierstrass sense)) and in our results we do not need such assumptions. Thus, our results and results from ([8], [23]) are independent from each other.

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