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Fixed point theorems for multivalued contractions in Kasahara spaces

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ABSTRACT. In this paper we present some fixed point results for multivalued α-contractions. Our results are obtained in a more general setting, the so called Kasahara space. Some of them are generalizations of Maia's type fixed point result for multivalued α-contractions. As application, a fixed point theorem for integral inclusions is given.

1. INTRODUCTION AND PRELIMINARIES

The contraction type mappings were defined on complete metric spaces as generalizations of the well known Banach's contraction principle. Many fixed point theorems for contractions were proved. If we carefully examine their proofs by the iteration method, we can see that in some cases, the metric properties, in particular the axiom of triangle inequality, are not essentials. The same remark can be made also for the fixed point results involving multivalued contractions defined on complete metric spaces.

In this paper we present some fixed point results for multivalued α-contractions. Our results are obtained in a more general setting, the so called Kasahara space. Some of them are generalizations of Maia's type fixed point result for multivalued α-contractions. As application, a fixed point theorem for integral inclusions is given.

Let us recall some notions and notations which will be used in our results.

Definition 1.1 (M. Fréchet [2]). Let X be a nonempty set. Let

$$s(X) := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\}.$$

Let $c(X) \subset s(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$ be an operator. By definition, the triple $(X, c(X), Lim)$ is called an L-space if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

By definition, an element $(x_n)_{n\in\mathbb{N}}$ of $c(X)$ is a convergent sequence and $x = Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence. We shall write $x_n \to x$ as $n \to \infty$.

We denote an L-space by $(X, \to)$.

Example 1.1 (I. A. Rus [8]). In general, an L-space is any set endowed with a structure implying a notion of convergence for sequences. For example, Hausdorff topological spaces, metric spaces, generalized metric spaces in Perov' sense (i.e. $d(x, y) \in \mathbb{R}^m_+$), generalized metric spaces in Luxemburg' sense (i.e. $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$), K-metric spaces

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(i.e. $d(x,y) \in K$, where K is a cone in an ordered Banach space), gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces, are such L-spaces. For more details in this sense, we have the paper of I. A. Rus [8] and the references therein.

Definition 1.2 (I. A. Rus [9]). Let $(X, \to)$ be an L-space and $d : X \times X \to \mathbb{R}_+$ be a functional. The triple $(X, \to, d)$ is a Kasahara space if and only if we have the following compatibility condition between $\to$ and d:

$$x_n \in X,\ \sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < +\infty \ \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to).$$

Example 1.2 (The trivial Kasahara space). Let (X, d) be a complete metric space. Let $\overset{d}{\to}$ be the convergence structure induced by d on X. Then $(X, \overset{d}{\to}, d)$ is a Kasahara space.

Example 1.3 (S. Kasahara [4]). Let X denote the closed interval $[0, 1]$ and $\to$ be the usual convergence structure on $\mathbb{R}$. Let $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x,y) = \begin{cases} |x-y|, & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 1, & \text{otherwise .} \end{cases}$$

Then $(X, \to, d)$ is a Kasahara space.

Example 1.4 (I. A. Rus [9]). Let (X, ρ) be a complete quasimetric space, with $\rho : X \times X \to \mathbb{R}_+$ a quasimetric. Let $d : X \times X \to \mathbb{R}_+$ be a functional such that there exists $c > 0$ with $\rho(x,y) \leq c \cdot d(x,y)$, for all $x, y \in X$. Then $(X, \overset{\rho}{\to}, d)$ is a Kasahara space.

We give next some notions and notations concerning multivalued operators.

Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. We consider $P(X) = \{A \subset X \mid A \neq \emptyset\}$ and $P_{cp}(X) = \{A \in P(X) \mid A \text{ is compact}\}$.

We define:

(i) the gap functional $D_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ by $D_d(A,B) = \inf_{a\in A,\ b\in B} d(a,b)$, for all $A, B \in P(X)$.
Note that $D_d(x, B)$, where $x \in X$, will be understood as $D_d(\{x\}, B)$.

(ii) the delta functional $\delta_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ by $\delta_d(A,B) = \sup_{a\in A,\ b\in B} d(a,b)$, for all $A, B \in P(X)$.

(iii) the excess functional $e_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ by $e_d(A,B) = \sup_{a\in A} D_d(a,B)$, for all $A, B \in P(X)$.

(iv) the general Pompeiu-Hausdorff functional $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ by $H_d(A,B) = \max\{e_d(A,B), e_d(B,A)\}$, for all $A, B \in P(X)$.

Definition 1.3. Let $(X, \to, d)$ be a Kasahara space and let $x \in X$. A set $A \in P(X)$ is said to be d-closed in X if for any sequence $(x_n)_{n\in\mathbb{N}} \subset A$ with $d(x_n, x) \to 0$ as $n \to \infty$, we have that $x \in A$.

We define $P^d_{cl}(X) := \{A \in P(X) \mid A \text{ is } d\text{-closed in } X\}$.

Remark 1.1. Let $(X, \to, d)$ be a Kasahara space. Let $A \in P^d_{cl}(X)$ and $x \in X$. Then $D_d(A, x) = 0 \Rightarrow x \in A$.

Indeed, let $x \in X$ such that $D_d(A, x) = 0$, i.e., $\inf_{a\in A} d(a, x) = 0$. Then there exists a sequence $(a_n)_{n\in\mathbb{N}} \subset A$ such that $d(a_n, x) \to 0$ as $n \to \infty$. Since A is d-closed in X, it follows that $x \in A$.

Lemma 1.1 (S. Kasahara [3]). *If $A, B \in P^d_{cl}(X)$ then $H_d(A, B) = 0$ if and only if $A = B$.*

Let us recall also the following notions:

Let $(X, \to)$ be an L-space and $F : X \to P(X)$ be a multivalued operator.

An element $x \in X$ is called a fixed point for F if $x \in F(x)$. For the sake of the simplicity, we will denote $F(x)$ by Fx. Let $Fix(F) = \{x \in X \mid x \in Fx\}$ be the set of all fixed points of F.

Let $Graph(F) = \{(x, y) \in X \times X \mid y \in Fx\}$ be the graph of F. The multivalued operator F is called closed if $Graph(F)$ is a closed subset of $X \times X$, i.e., if $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset X$ and $y_n \in Fx_n$, for all $n \in \mathbb{N}$ with $x_n \to x^* \in X$ and $y_n \to y^* \in X$ as $n \to \infty$, then $y^* \in Fx^*$.

A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called sequence of successive approximations for F starting from $(x_0, x_1) \in Graph(F)$ if and only if $x_{n+1} \in Fx_n$, for all $n \in \mathbb{N}$. The set-valued operator F is called multivalued weakly Picard operator ($MWPO$) if for each $x_0 \in X$ and any $x_1 \in Fx_0$, there exists a sequence of successive approximations for F starting from (x_0, x_1), it is convergent in $(X, \to)$ and its limit is a fixed point for F. If, in addition, $Card(Fix(F)) = 1$, then F is a multivalued Picard operator (MPO).

In this paper we consider the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying the property that for all $x, y \in X$, $d(x, y) = 0 \Rightarrow x = y$.

2. Nadler's fixed point theorem

The study of fixed point theorems for multivalued mappings has been initiated by J. T. Markin [5] and S. B. Nadler [6]. The following result, usually referred as Nadler's fixed point theorem, extends Banach-Caccioppolli's contraction principle from single-valued maps to set-valued contractive maps.

Theorem 2.1. *Let (X, d) be a complete metric space and $F : X \to P_{b,cl}(X)$ be a set-valued α-contraction, i.e., a mapping for which there exists a constant $\alpha \in]0, 1[$ such that $H_d(F(x), F(y)) \leq \alpha \cdot d(x, y)$, for all $x, y \in X$. Then F has at least one fixed point.*

In the above result, $P_{b,cl}(X)$ stands for the set of all bounded and closed subsets of X.

We remark also that Nadler's fixed point theorem is given in the context of metric spaces. We adapt this result into the context of Kasahara spaces.

First we prove the following lemma:

Lemma 2.2. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $A, B \in P^d_{cl}(X)$ and a real number $q > 1$. Then for every $a \in A$, there exists $b \in B$ such that $d(a, b) \leq q \cdot H_d(A, B)$.*

Proof. If $A = B$ then, by Lemma 1.1 we have $H_d(A, B) = 0$. Hence $d(a, b) = 0 \Rightarrow a = b$. So, for every $a \in A$, there exists $b := a \in B$ such that the conclusion holds.

Now let $A, B \in P^d_{cl}(X)$ such that $A \neq B$. By Lemma 1.1 we get that $H_d(A, B) > 0$.

Supposing contrary: there exists $q > 1$ and there exists $a \in A$ such that for every $b \in B$, $d(a, b) > q \cdot H_d(A, B)$. By taking the $\inf\limits_{b\in B}$ in the above inequality, we get that $H_d(A, B) \geq D_d(a, B) \geq q \cdot H_d(A, B)$. Hence $q \leq 1$ which is a contradiction. □

Theorem 2.2. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $F : X \to P^d_{cl}(X)$ be a multivalued operator. Assume that $Graph(F)$ is closed in $(X, \to)$ and there exists $\Lambda : \mathbb{R}_+ \to [0, 1[$ with $\limsup\limits_{s\to t^+}\Lambda(s) < 1$, for all $t \in \mathbb{R}_+$ such that $H_d(Fx, Fy) \leq \Lambda(d(x, y)) \cdot d(x, y)$, for all $x, y \in X$.*

Then F is a multivalued weakly Picard operator.

Proof. Let $q > 1$. Let $x_0 \in X$ and $x_1 \in Fx_0$.

If $x_0 = x_1$ then $x_0 \in Fix(F)$ and the proof is complete. If $x_0 \neq x_1$ then by Lemma 2.2, there exists $x_2 \in Fx_1$ such that $d(x_1, x_2) \leq q \cdot H_d(Fx_0, Fx_1) \leq q \cdot \Lambda(d(x_0, x_1)) \cdot d(x_0, x_1)$. For $x_2 \in Fx_1$, we have the following cases:

If $x_1 = x_2$ then $x_1 \in Fix(F)$ and the proof is complete. If $x_1 \neq x_2$ then by Lemma 2.2, there exists $x_3 \in Fx_2$ such that $d(x_2, x_3) \leq q \cdot H_d(Fx_1, Fx_2) \leq q \cdot \Lambda(d(x_1, x_2)) \cdot d(x_1, x_2) \leq q^2 \cdot \Lambda(d(x_1, x_2)) \cdot \Lambda(d(x_0, x_1)) \cdot d(x_0, x_1)$.

By induction, we get that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for F which starts from $(x_0, x_1) \in Graph(F)$ with $x_{n+1} \in Fx_n$, for all $n \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq q^n \cdot \prod_{k=0}^{n-1} \Lambda(d(x_k, x_{k+1})) \cdot d(x_0, x_1).$$

Let $M = \max_{k=\overline{0,n-1}} \{\Lambda(d(x_k, x_{k+1}))\} < 1$. Then $d(x_n, x_{n+1}) \leq (qM)^n \cdot d(x_0, x_1),$ for all $n \in \mathbb{N}$. We take $q > 1$ such that $\theta := qM < 1$ and hence, $d(x_n, x_{n+1}) \leq \theta^n \cdot d(x_0, x_1),$ for all $n \in \mathbb{N}$.

We have next the following estimations:

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) \leq \sum_{n\in\mathbb{N}} \theta^n \cdot d(x_0, x_1) = \frac{1}{1-\theta} d(x_0, x_1) < +\infty.$$

Since $(X, \to, d)$ is a Kasahara space, we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. So, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. In addition, since $Graph(F)$ is closed, we have that $x^* \in Fix(F)$. □

Remark 2.2. Note that if in the above theorem, $\Lambda(s) = \alpha$, for all $s \in \mathbb{R}_+$, then we get Nadler's fixed point theorem for multivalued α-contractions in Kasahara spaces.

Remark 2.3. Due to the context of Kasahara spaces, some Maia type fixed point results, obtained for multivalued contractions defined on a nonempty set X endowed with two metrics, can be generalized. Usually, in these results, the set X is equipped with a complete metric ρ and another metric d used in the contractive condition satisfied by a singlevalued or multivalued operator. In the following result, we can see that d must not necessarily be a metric.

Corollary 2.1. *Let X be a nonempty set and $\rho : X \times X \to \mathbb{R}_+$ be a complete metric on X. Let $d : X \times X \to \mathbb{R}_+$ be a functional with the property that for all $x, y \in X$, $d(x, y) = 0 \Rightarrow x = y$. Let $F : X \to P_{cl}^d(X)$ be a multivalued operator. We assume that:*

(i) *there exists $\alpha \in [0, 1[$ such that $H_d(Fx, Fy) \leq \alpha \cdot d(x, y)$, for all $x, y \in X$;*

(ii) *$Graph(F)$ is closed in $(X, \stackrel{\rho}{\to})$;*

(iii) *there exists $c > 0$ such that $\rho(x, y) \leq c \cdot d(x, y)$, for all $x, y \in X$.*

Then $Fix(F) \neq \emptyset$ and there exists $\theta \in [0, 1[$ such that

$$\rho(x_n, x^*) \leq c\frac{\theta^n}{1-\theta} d(x_0, x_1), \text{ for all } n \in \mathbb{N}, \tag{2.1}$$

where $x^* \in Fix(F)$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for F starting from $(x_0, x_1) \in Graph(F)$.

Proof. By (i) and by following the proof of Theorem 2.2 with $\Lambda = \alpha$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for F starting from $(x_0, x_1) \in Graph(F)$ such that $x_{n+1} \in Fx_n$ and $d(x_n, x_{n+1}) \leq \theta^n \cdot d(x_0, x_1)$, for all $n \in \mathbb{N}$. By (iii), there exists $c > 0$ such that $\rho(x_n, x_{n+1}) \leq c \cdot d(x_n, x_{n+1}) \leq c \cdot \theta^n \cdot d(x_0, x_1),$ for all $n \in \mathbb{N}$.

Let $p \in \mathbb{N}$, $p > 0$. Since ρ is a metric, we have that

$$\begin{aligned}\rho(x_n, x_{n+p}) &\leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+p-1}, x_{n+p}) \\ &\leq c \cdot \theta^n \cdot d(x_0, x_1) + c \cdot \theta^{n+1} \cdot d(x_0, x_1) + \ldots + c \cdot \theta^{n+p-1} \cdot d(x_0, x_1).\end{aligned}$$

So, the following estimation holds

$$(2.2) \qquad \rho(x_n, x_{n+p}) \leq c \cdot \theta^n \cdot \frac{1-\theta^p}{1-\theta} \cdot d(x_0, x_1), \text{ for all } n \in \mathbb{N} \text{ and all } p \in \mathbb{N} \text{ with } p > 0.$$

By letting $n \to \infty$, we get that $\rho(x_n, x_{n+p}) \to 0$, so $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space (X, ρ). Therefore $(x_n)_{n\in\mathbb{N}}$ is convergent in (X, ρ), so there exists $x^* \in X$ such that $x_n \xrightarrow{\rho} x^*$. By (ii), it follows that $x^* \in Fix(F)$. By letting $p \to \infty$ in (2.2), we get the estimation (2.1). □

Remark 2.4. Corollary 2.1 generalizes the Maia type fixed point result for multivalued contractions in complete metric spaces, given by A. Petruşel and I.A. Rus [7] (Theorem 2.1).

Corollary 2.2. *Let X be a nonempty set and $\rho : X \times X \to \mathbb{R}_+$ be a complete metric on X. Let $d : X \times X \to \mathbb{R}_+$ be a functional with the property that for all $x, y \in X$, $d(x, y) = 0 \Rightarrow x = y$. Let $F : X \to P_{cl}^d(X)$ be a multivalued operator. We assume that:*

(i) there exists $\alpha \in [0, 1[$ such that $H_d(Fx, Fy) \leq \alpha \cdot d(x, y)$, for all $x, y \in X$;

(ii) $Graph(F)$ is closed in $(X, \xrightarrow{\rho})$;

(iii) there exists $c > 0$ such that $\delta_\rho(Fx, Fy) \leq c \cdot d(x, y)$, for all $x, y \in X$.

Then $Fix(F) \neq \emptyset$ and there exists $\theta \in [0, 1[$ such that $\rho(x_n, x^*) \leq c\frac{\theta^{n-1}}{1-\theta} d(x_0, x_1)$, for all $n \in \mathbb{N}$, where $x^* \in Fix(F)$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for F starting from $(x_0, x_1) \in Graph(F)$.

Proof. By (i) and by following the proof of Theorem 2.2 with $\Lambda = \alpha$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for F starting from $(x_0, x_1) \in Graph(F)$ such that $x_n \in Fx_{n-1}$ and $d(x_{n-1}, x_n) \leq \theta^{n-1} \cdot d(x_0, x_1)$, for all $n \in \mathbb{N}$, $n \geq 1$. By (iii), we have $\rho(x_n, x_{n+1}) \leq \delta_\rho(Fx_{n-1}, Fx_n) \leq c \cdot d(x_{n-1}, x_n) \leq c \cdot \theta^{n-1} d(x_0, x_1)$, for all $n \in \mathbb{N}$, $n \geq 1$.

Let $p \in \mathbb{N}$, $p > 0$.

Then $\rho(x_n, x_{n+p}) \leq \sum_{i=0}^{p-1} \rho(x_{n+i}, x_{n+i+1}) \leq \sum_{i=0}^{p-1} c \cdot \theta^{n+i-1} d(x_0, x_1) \leq c\dfrac{\theta^{n-1}}{1-\theta} d(x_0, x_1)$.

By letting $n \to \infty$, we get $\rho(x_n, x_{n+p}) \to 0$. So $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space (X, ρ). Therefore, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By (ii), it follows that $x^* \in Fix(F)$.

On the other hand, by letting $p \to \infty$, we get the desired estimation. □

3. Integral inclusions

In this section we present an application to Corollary 2.2, concerning the existence of fixed points for a Fredholm-type integral inclusion.

Let $C([a, b], \mathbb{R}^n) = \{x : [a, b] \to \mathbb{R}^n \mid x \text{ is a continuous function on } [a, b]\}$ endowed with the following metrics

- $\rho : C[a, b] \times C[a, b] \to \mathbb{R}_+$, $\rho(x, y) = \max_{t\in[a,b]} |x(t) - y(t)|$, for all $x, y \in C[a, b]$;
- $d : C[a, b] \times C[a, b] \to \mathbb{R}_+$, $d(x, y) = \left(\int_a^b |x(t) - y(t)|^2 dt\right)^{\frac{1}{2}}$, for all $x, y \in C[a, b]$.

Let $x_0 \in C([a,b],\mathbb{R}^n)$ and $r > 0$.

We denote by $B_\rho(x_0,r) = \{x \in C([a,b],\mathbb{R}^n) \mid \rho(x_0,x) < r\}$ the open ball centered in x_0 with radius r.

Let $P_{cp,cv}(\mathbb{R}^n) := \{Y \in P(\mathbb{R}^n) \mid Y$ is compact and convex$\}$.

Theorem 3.3. *We consider the Fredholm-type integral inclusion*

$$x(t) \in \int_a^b K(t,s,x(s))ds + g(t), \text{ for all } t \in [a,b]. \tag{3.3}$$

Assume that:

(i) *$K : [a,b] \times [a,b] \times \mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ is lower semi-continuous;*

(ii) *$g \in C([a,b],\mathbb{R}^n)$;*

(iii) *for all $\varepsilon > 0$ and $t \in [a.b]$, there exists some $\eta(t,\varepsilon) > 0$ such that for all $u, v \in \mathbb{R}^n$ and all $s \in [a,b]$, $|u-v|_{\mathbb{R}^n} < \eta(t,\varepsilon) \Rightarrow H_\rho(K(t,s,u),K(t,s,v)) < \varepsilon$;*

(iv) *$H_{|\cdot|_{\mathbb{R}^n}}(K(t,s,u),K(t,s,v)) \le l(t,s)\cdot|u-v|_{\mathbb{R}^n}$, for all $t,s \in [a,b]$, $u,v \in \mathbb{R}^n$, where $l \in C[a,b]$ and $\int_a^b\int_a^b l^2(t,s)ds\,dt < 1$.*

Then the integral inclusion (3.3) *has a solution in $C[a,b]$.*

Proof. Let $A : C[a,b] \to P(C[a,b])$ defined by

$$Ax = \left\{ v \in C[a,b] \;\middle|\; v(t) \in \int_a^b K(t,s,x(s))ds + g(t),\ t \in [a,b] \right\}.$$

We prove successively that:

(a) $Ax \ne \emptyset$, for all $x \in C([a,b],\mathbb{R}^n)$;

(b) $Ax \in P_{cl}^d(C[a,b])$, for all $x \in C([a,b],\mathbb{R}^n)$;

(c) $Graph(A)$ is closed in $(C[a,b],\xrightarrow{\rho})$;

(d) there exists $c > 0$ such that for each $x,y \in C([a,b],\mathbb{R}^n)$ and each $u \in Ax$ there exists $v \in Ay$ such that $\rho(u,v) \le c\cdot d(x,y)$;

(e) there exists $\alpha \in [0,1[$ such that $H_d(Ax,Ay) \le \alpha\cdot d(x,y)$, for all $x,y \in C([a,b],\mathbb{R}^n)$.

Then by Corollary 2.2 it follows that there exists $x^* \in Ax^*$, i.e., the integral inclusion (3.3) has a solution in $C[a,b]$.

(a). Let $x \in C([a,b],\mathbb{R}^n)$. The multivalued operator K_x defined by $K_x(t,s) = K(t,s,x(s))$ being lower semi-continuous, it has a continuous selection, say $k(t,s) \in K_x(t,s)$, for all $t,s \in [a,b]$. Let $v(t) = \int_a^b k(t,s)ds + g(t) \in \int_a^b K(t,s,x(s))ds + g(t)$. It is clear that $v \in Ax$ and thus $Ax \ne \emptyset$.

(b). Let $(x_n)_{n\in\mathbb{N}} \subset Ax$ such that $d(x_n,\tilde{x}) \to 0$ as $n \to \infty$. We show that $\tilde{x} \in Ax$.
Since $(x_n)_{n\in\mathbb{N}} \subset (C[a,b],\mathbb{R}^n)$ and $(C[a,b],\mathbb{R}^n)$ is a closed space with respect to d, it follows that $\tilde{x} \in (C[a,b],\mathbb{R}^n)$. On the other hand, for all $t \in [a,b]$, $x_n(t) \in \int_a^b K(t,s,x(s))ds + g(t)$, and since the set $\int_a^b K(t,s,x(s))ds$ is compact and $d(x_n,\tilde{x}) \to 0$ as $n \to \infty$, it follows that $\tilde{x}(t) \in \int_a^b K(t,s,x(s))ds + g(t)$. So $\tilde{x} \in Ax$.

(c). Suppose that $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset C([a,b],\mathbb{R}^n)$ such that $x_n \xrightarrow{\rho} \tilde{x} \in C([a,b],\mathbb{R}^n)$, $y_n \xrightarrow{\rho} \tilde{y} \in C([a,b],\mathbb{R}^n)$ as $n \to \infty$ and $y_n \in Ax_n$, for all $n \in \mathbb{N}$. We show that $\tilde{y} \in A\tilde{x}$.

Fix $t \in [a,b]$ and let $m \in \mathbb{Z}_+$ be arbitrary. Since $x_n \xrightarrow{\rho} \tilde{x}$ as $n \to \infty$, there exists some $n_m \in \mathbb{N}$ such that for all $n \ge n_m$, $\rho(x_n,\tilde{x}) < \eta(t,\frac{1}{m})$. It follows that $|x_n(t)-\tilde{x}(t)|_{\mathbb{R}^n} \le \max\limits_{t\in[a,b]}|x_n(t)-\tilde{x}(t)|_{\mathbb{R}^n} < \eta(t,\frac{1}{m})$. By (iii), we get $H_\rho(K(t,s,x_n(s)),K(t,s,\tilde{x}(s))) < \frac{1}{m}$, so $K(t,s,x_n(s)) \subseteq B_\rho(K(t,s,\tilde{x}(s)),\frac{1}{m})$, for all $n \ge n_m$.

Since $y_n \in Ax_n$ we have that $y_n(t) \in \int_a^b K(t,s,x_n(s))ds + g(t)$ and therefore $y_n(t) = \int_a^b K_n(t,s)ds + g(t)$, where $K_n(t,s) \in K(t,s,x_n(s))$, for all $s \in [a,b]$. It follows that $K_n(t,s) \in B_\rho(K(t,s,\tilde{x}(s)), \frac{1}{m})$ for all $s \in [a,b]$.

Now, we consider the multivalued function $G(s) = \overline{B_\rho(K_n(t,s), \frac{1}{m})} \cap K(t,s,\tilde{x}(s))$ for t fixed. Since $K(t,s,\tilde{x}(s))$ is closed and convex valued for t fixed, it follows that $K(t,s,\tilde{x}(s))$ is Carathéodory in s and $\tilde{x}(s)$, i.e., measurable in $\tilde{x}(s)$ and continuous in s. By Theorem 8.2.8. in [1], we get that $K(t,s,\tilde{x}(s))$ is measurable in s. Also $\overline{B_\rho(K_n(t,s), \frac{1}{m})}$ is measurable with closed images. Hence $G(s)$ is measurable with closed images. By Theorem 8.1.3 in [1], there exists a measurable selection $K^{(m)}(t,s) \in G(s)$ (measurable in s) such that $|K_n(t,s) - K^{(m)}(t,s)| < \frac{1}{m}$, for all $s \in [a,b]$.

We have the following estimations $\left|\tilde{y}(t) - \left(\int_a^b K^{(m)}(t,s)ds + g(t)\right)\right| \leq |\tilde{y}(t) - y_n(t)| + \left|y_n(t) - \left(\int_a^b K^{(m)}(t,s)ds + g(t)\right)\right| = |\tilde{y}(t) - y_n(t)| + \left|\int_a^b \left(K_n(t,s) - K^{(m)}(t,s)\right)ds\right| \leq |\tilde{y}(t) - y_n(t)| + \int_a^b \left|K_n(t,s) - K^{(m)}(t,s)\right|ds \leq \rho(y_n, \tilde{y}) + \frac{b-a}{m}$, for all $n \geq n_m$.

Now, let $\varepsilon > 0$ be arbitrary and choose m such that $\frac{b-a}{m} < \frac{\varepsilon}{2}$ and choose $\tilde{n} \geq n_m$ such that $\rho(y_n, \tilde{y}) < \frac{\varepsilon}{2}$ for all $n > \tilde{n}$. Then, if $n > \tilde{n}$ we have $\left|\tilde{y}(t) - \left(\int_a^b K^{(m)}(t,s)ds + g(t)\right)\right| < \varepsilon$ for t fixed. Since K is convex valued and integrably bounded, $\int_a^b K(t,s,\tilde{x}(s))ds + g(t)$ is closed (by Theorem 8.6.4. of [1]). Since $\tilde{y}(t)$ is arbitrary close to the set $\int_a^b K(t,s,\tilde{x}(s))ds + g(t)$, it must be a limit point and hence (since the set is closed), $\tilde{y}(t) \in \int_a^b K(t,s,\tilde{x}(s))ds + g(t)$. Since $t \in [a,b]$ was arbitrary, this shows that $\tilde{y} \in A\tilde{x}$.

(d). Let $x, y \in C([a,b], \mathbb{R}^n)$ and $u \in Ax$. Then $u(t) \in \int_a^b K(t,s,x(s))ds + g(t)$, for all $t \in [a,b]$. It follows that there exists a continuous selection $k_1(t,s) \in K_x(t,s) = K(t,s,x(s))$ such that $u(t) = \int_a^b k_1(t,s)ds + g(t)$, for all $t \in [a,b]$.

By (iv) it follows that $H_{\mathbb{R}^n}(K(t,s,x(s)), K(t,s,y(s))) \leq l(t,s) \cdot |x(s) - y(s)|_{\mathbb{R}^n}$ and hence, there exists a continuous selection $v \in K_y(t,s) = K(t,s,y(s))$ such that $|k_1(t,s) - v|_{\mathbb{R}^n} \leq l(t,s) \cdot |x(s) - y(s)|_{\mathbb{R}^n}$.

Let $U : [a,b] \times [a,b] \to P(\mathbb{R}^n)$, $U(t,s) = \{v \in \mathbb{R}^n \mid |k_1(t,s) - v|_{\mathbb{R}^n} \leq l(t,s) \cdot |x(s) - y(s)|_{\mathbb{R}^n}\}$. Since the multivalued operator $V(t,s) = U(t,s) \cap K(t,s,y(s))$ is lower semi-continuous, there exists $k_2(t,s) \in K_y(t,s) = K(t,s,y(s))$ a continuous selection for V and

$$|k_1(t,s) - k_2(t,s)|_{\mathbb{R}^n} \leq l(t,s) \cdot |x(s) - y(s)|_{\mathbb{R}^n}, \text{ for all } t, s \in [a,b]. \tag{3.4}$$

Let $v(t) = \int_a^b k_2(t,s)ds + g(t)$, for all $t \in [a,b]$.

By (3.4), we have the following estimations: $|u(t) - v(t)|_{\mathbb{R}^n} = \int_a^b |k_1(t,s) - k_2(t,s)|_{\mathbb{R}^n} ds \leq \int_a^b l(t,s) \cdot |x(s) - y(s)|_{\mathbb{R}^n} ds \overset{\text{Hölder's inequality}}{\leq} \left(\int_a^b l^2(t,s)ds\right)^{\frac{1}{2}} \left(\int_a^b |x(s) - y(s)|^2_{\mathbb{R}^n} ds\right)^{\frac{1}{2}} = \alpha(t) \cdot d(x,y)$, where $\alpha(t) := \left(\int_a^b l^2(t,s)ds\right)^{\frac{1}{2}}$, for all $t \in [a,b]$. In the above estimations, we take next $\max\limits_{t \in [a,b]}$ and we get that $\rho(u,v) \leq c \cdot d(x,y)$, where $c := \max\limits_{t \in [a,b]} \alpha(t)$, for all $t \in [a,b]$.

(e). We consider Hölder's inequality obtained previously:

$$|u(t) - v(t)|_{\mathbb{R}^n} \leq \left(\int_a^b l^2(t,s)ds\right)^{\frac{1}{2}} \left(\int_a^b |x(s) - y(s)|^2_{\mathbb{R}^n} ds\right)^{\frac{1}{2}} = \alpha(t) \cdot d(x,y), \text{ for all } t \in [a,b].$$

We have next $d(u,v) = \left(\int_a^b |u(t) - v(t)|^2_{\mathbb{R}^n} dt\right)^{\frac{1}{2}} \leq \left(\int_a^b \alpha^2(t) \cdot d^2(x,y)dt\right)^{\frac{1}{2}} = \left(\int_a^b \alpha^2(t)dt\right)^{\frac{1}{2}} \cdot d(x,y)$. So, there exists $\alpha := \left(\int_a^b \alpha^2(t)dt\right)^{\frac{1}{2}} < 1$, such that $d(u,v) \leq \alpha d(x,y)$. Similarly, by interchanging the roles of x and y, we get that $d(v,u) \leq \alpha d(y,x)$, for $u \in Ax$ and $v \in Ay$. It follows that $H_d(Ax, Ay) \leq \alpha \cdot d(x,y)$, for all $x, y \in C([a,b], \mathbb{R}^n)$. $\square$

Example 3.5. In the above result, let us consider $n = 1$, $[a, b] = [\frac{1}{2}, 1]$ and $K : [\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ be defined by $K(t, s, u) := t \cdot s \cdot [u, u + 1]$, for all $t, s \in [\frac{1}{2}, 1]$ and all $u \in \mathbb{R}$.

It is clear that K is continuous on $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \times \mathbb{R}$, so K is also lower semi-continuous on its definition domain. Hence, the assumption (i) of Theorem 3.3 is satisfied.

Let $\varepsilon > 0$ and $t \in [\frac{1}{2}, 1]$. There exists $\eta(t, \varepsilon) := \frac{\varepsilon}{t+1} > 0$ such that if $|u - v| < \eta(t, \varepsilon)$ for all $u, v \in \mathbb{R}$, we have

$$H_\rho(K(t, s, u), K(t, s, v)) = H_\rho(ts[u, u + 1], ts[v, v + 1]) =$$
$$= ts \cdot \max\{|v - u|, |(v + 1) - (u + 1)|\} = ts|u - v| < ts\frac{\varepsilon}{t + 1} \leq \frac{t}{t + 1}\varepsilon < \varepsilon,$$

for all $s \in [\frac{1}{2}, 1]$, so the assumption (iii) of Theorem 3.3 is satisfied.

Finally, let $t, s \in [\frac{1}{2}, 1]$ and $u, v \in \mathbb{R}$. Consider the continuous function $l : [\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \to \mathbb{R}$, be defined by $l(t, s) = ts + \frac{1}{t+1}$, for all $t, s \in [\frac{1}{2}, 1]$. Then $H_\mathbb{R}(K(t, s, u), K(t, s, v)) = ts|u - v| \leq (ts + \frac{1}{t+1})|u - v| = l(t, s)|u - v|$, for all $t, s \in [\frac{1}{2}, 1]$ and $u, v \in \mathbb{R}$. In addition, it can be shown that $\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} l^2(t, s)\ dt\ ds < 1$, so the assumption (iv) of Theorem 3.3 is satisfied.

Applying Theorem 3.3 it follows that the Fredholm-type integral inclusion

$$x(t) \in \int_{\frac{1}{2}}^{1} t \cdot s \cdot [x(s), x(s) + 1]ds + g(t)$$

for all $t \in [\frac{1}{2}, 1]$ and any function $g \in C([\frac{1}{2}, 1], \mathbb{R})$, has a solution in $C[\frac{1}{2}, 1]$.

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