Some examples of division symbol algebras of degree 3 **and** 5

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ABSTRACT. In this paper we provide an algorithm to compute the product between two elements in a symbol algebra of degree n and we find an octonion algebra (in general, without division) in a symbol algebra of degree three. Moreover, using MAGMA software, we will provide some examples of division symbol algebras of degree 3 and of degree 5.

1. INTRODUCTION

Let K be a field which contains a primitive n-th root of unity, with n an arbitrary positive integer such that $char(K)$ does not divide n. Let $K^* = K \backslash \{0\}$, $a, b \in K^*$ and let S be an algebra over K generated by the elements x and y, where

$$(1.1) \qquad\qquad x^n = a, y^n = b, yx = \xi xy$$

and ξ is a primitive root of order n of unity. This algebra is called a *symbol algebra* (also known as a *power norm residue algebra*) and it is denoted by $\left(\frac{a, b}{K, \xi} \right)$. In [17], J. Milnor calls this algebra "the symbol algebra" because of its connection with the K-theory and with the Steinberg symbol. Symbol algebras generalize the quaternion algebras (for $n = 2$). Quaternion algebras and symbol algebras are important not only for the theory of associative algebras. They have many applications, some of them being studied by the authors of this article: in algebra and number theory ([16], [18], [19], [22]), in representation theory ([6]) or in analysis and mechanics ([11]).

In this paper, we will study the symbol algebras from two points of view: from the theory of associative algebras and from number theory.

The study of symbol algebras of degree n involves very complicated calculations and, usually, can be hard to multiply two elements or find examples for some notions. In this paper, we will provide an easy algorithm which allows us quickly compute the product of two elements in a symbol algebra.

Since for $n = 2$ the quaternion algebras are symbol algebras, a natural question is: for $n = 3$, what is the connection between the octonion algebras, algebras of dimension 8, and symbol algebras of degree 3. An answer can be that we always can find an octonion non-division algebra in a symbol algebra of degree three. Starting from this idea and from results obtained in the paper [5], in which, using the associated trace form for a symbol algebra, the author studied some properties of such objects and gave some conditions for a symbol algebra to be with division or not, only for $n = 4k + 2$ (and not for $n \in \{3, 5\}$),

we intend to find examples of division symbol algebras of degree 3 and degree 5. Since such an example is not easy to provide, we will use MAGMA software.

In the following, we will recall some general properties and definitions.

Let A be a finite dimensional unitary algebra over a field K with a *scalar involution* $^{-}$: $A \to A, a \to \bar{a}$, i.e. a linear map satisfying the following relations: $\overline{ab} = \bar{b}\bar{a}, \bar{\bar{a}} = a$, and $a + \bar{a}, a\bar{a} \in K \cdot 1$ for all $a, b \in A$. The element $\bar{a}$ is called the *conjugate* of the element a, the linear form $t : A \to K$, $t(a) = a + \bar{a}$ and the quadratic form $n : A \to K$, $n(a) = a\bar{a}$ are called the *trace* and the *norm* of the element a.

Let $\gamma \in K$ be a fixed non-zero element. We define the following algebra multiplication on the vector space $A \oplus A : (a_1, a_2)(b_1, b_2) = (a_1b_1 + \gamma \bar{b_2}a_2, a_2\bar{b_1} + b_2a_1)$.

We obtain an algebra structure over $A \oplus A$, denoted by (A, γ) and called the *algebra obtained from A by the Cayley-Dickson process.* We have $\dim(A, \gamma) = 2 \dim A$.

Let $x \in (A, \gamma)$, $x = (a_1, a_2)$. The map $^{-} : (A, \gamma) \to (A, \gamma)$, $x \to \bar{x} = (\bar{a_1}, -a_2)$, is a scalar involution of the algebra (A, γ), extending the involution $^{-}$ of the algebra A.

If we take $A = K$ and apply this process t times, $t \geq 1$, we obtain an algebra over K, $A_t = \left(\frac{\alpha_1, ..., \alpha_t}{K}\right)$.

By induction in this algebra, the set $\{1, e_2, ..., e_n\}, n = 2^t$, generates a basis with the properties: $e_i^2 = \gamma_i 1$, $\gamma_i \in K, \gamma_i \neq 0$, $i = 2, ..., n$ and $e_i e_j = -e_j e_i = \beta_{ij} e_k$, $\beta_{ij} \in K$, $\beta_{ij} \neq 0, i \neq j, i, j = 2, ...n$, β_{ij} and e_k being uniquely determined by e_i and e_j. For $n = 2$, we obtain the quaternion algebra, for $n = 3$, we obtain the octonion algebra, etc.

For details about the above process, the reader is referred to [20] or [21].

If an algebra A is finite-dimensional, then it is a division algebra if and only if A does not contain zero divisors. (See [20])

A central simple algebra A over a field K is called *split by L* (where L is a field containing K), if $A \otimes_K L$ is a matrix algebra over L. We also can say that its class is in the Brauer group $\mathrm{Br}(K)$. (see [16]) L is called a *splitting field* for A.

Theorem 1.1. ([7]) *Let* K *be a field such that* $\xi \in K$, $\xi^n = 1, \xi$ *a primitive root, and let* $\alpha, \beta \in K^*$. *Then the following statements are equivalent:*

i) The cyclic algebra $A = \left(\frac{\alpha, \beta}{K, \xi}\right)$ *is split.*

ii) The element β *is a norm from the extension* $K \subseteq K(\sqrt[n]{\alpha})$.

Theorem 1.2. ([9]) *Let* l *be a natural number,* $l \geq 3$ *and* ξ *be a primitive root of the unity of* l-*order. If* p *is a prime natural number,* l *is not divisible with* p *and* f *is the smallest positive integer such that* $p^f \equiv 1 \mod l$, *then we have* $p\mathbb{Z}[\xi] = P_1 P_2 .... P_r$, *where* $r = \frac{\varphi(l)}{f}, \varphi$ *is the Euler's function and* P_j, $j = 1, ..., r$ *are different prime ideals in the ring* $\mathbb{Z}[\xi]$.

Theorem 1.3. ([14]) *Let* ξ *be a primitive root of the unity of* $l-$*order, where* l *is a prime natural number and let* A *be the ring of integers of the Kummer field* $Q(\xi, \sqrt[l]{\mu})$. *A prime ideal* P *in the ring* $\mathbb{Z}[\xi]$ *is in* A *in one of the situations:*

i) It is equal with the $l-$*power of a prime ideal from* A, *if the* $l-$*power character* $\left(\frac{\mu}{P}\right)_l = 0$;

ii) It is a prime ideal in A, *if* $\left(\frac{\mu}{P}\right)_l = a$ *rot of order* l *of unity, different from* 1.

iii) It decomposes in l *different prime ideals from* A, *if* $\left(\frac{\mu}{P}\right)_l = 1$.

Theorem 1.4. ([16]) *Let* K *be a finite field and let* $Br(K)$ *be the Brauer group of* K. *Then* $Br(K) = 0$.

Theorem 1.5. ([1]) *Let* L/K *be an extension of finite fields. Then the norm function* $N_{L/K} : L^* \to K^*$ *is surjective .*

Remark 1.1. ([13]) Let K be a field of characteristic $\neq p, p$ prime, and let $\xi \in K$ be a primitive root of unity of order p. For $a, b \in K^*$, the symbol algebra of degree p denoted by $A = \left(\frac{a, b}{K, \xi} \right)$ is either split or a division algebra. From here, in whole this paper, we will use the notion "no-division" instead of "split", for all symbol algebras of degree n, with n a prime number.

2. MULTIPLICATION TABLE FOR SYMBOL ALGEBRAS

In [2] , the author described how we can multiply the basis vectors in all algebras obtained by the Cayley-Dickson process. Since the quaternion algebra is an algebra obtained by this process and in the same time is a particular case of symbol algebras, we use some ideas given in this paper for multiplication of two symbol elements.

Case $n = 3$.

Let S be a symbol algebra of degree three with the basis

(2.1.) $$B = \{1, x, x^2, y, y^2, xy, x^2y^2, x^2y, xy^2\}.$$

Remark 2.2. The elements from the basis B will be denoted such as follows:
$e_0 = 1, e_1 = y, e_2 = y^2, e_3 = x, e_4 = xy, e_5 = xy^2, e_6 = x^2, e_7 = x^2y, e_8 = x^2y^2$. If we use the lexicographic order for the monomials $x^i y^j$, we have that $x^i y^j \geq x^p y^q$ if and only if $i \geq p$ or $i = p$ and $j \geq q$. Therefore the elements from the basis B are lexicographic ordered.

Remark 2.3. If we write $4 = 1 \cdot 3 + 1 = 011_3 \rightarrow e_4 = x^1 y^1; 5 = 1 \cdot 3 + 2 = 012_3 \rightarrow e_5 = x^1 y^2$
$6 = 2 \cdot 3 = 020_3 \rightarrow e_6 = x^2; 7 = 2 \cdot 3 + 1 = 021_3 \rightarrow e_7 = x^2 y^1$
$8 = 2 \cdot 3 + 2 = 022_3 \rightarrow e_8 = x^2 y^2$, where $0ij_3 = i \cdot 3 + j = k, i, j \in \{1, 2\}$ is the ternary decomposition of the natural number $k \in \{4, 5, 6, 7, 8\}$, it results that $e_k = x^i y^j$, with $k = i \cdot 3 + j = 0ij_3$. If we compute two elements of the basis B, we obtain

(2.2) $$e_i e_j = \alpha(i, j) e_{i \circ j},$$

where $\alpha(i, j)$ is a function $\alpha : \mathbb{Z}_3^3 \times \mathbb{Z}_3^3 \rightarrow K$ and $i \circ j$ represents the "sum" of i and j in the group $\mathbb{Z}_3^3$ (here i and j are in the ternary forms!). Indeed, this last sentence results from (1.1). Therefore $e_7 e_8 = \alpha(7, 8) e_3$, since $021_3 + 022_3 = 010_3 \rightarrow 3$.

General case

Using the above notations, a basis in a symbol algebra of degree n is of the form

(2.3) $$B = \{x^i y^j / 0 \leq i < n, 0 \leq j < n\}.$$

The elements from the basis B are lexicographic ordered, as in Remark 2.2. We denote an element from the basis B, given by (2.3) with $e_k = x^i y^j, 0 \leq k < n$, such that $k = 0ij_n = i \cdot n + j$, where $0ij_n$ is the $n-$ary decomposition of the natural number k. Then, using relation (1), if we compute two elements of the basis B, we obtain

(2.4) $$e_i e_j = \alpha(i, j) e_{i \circ j},$$

where $\alpha(i, j)$ is a function $\alpha : \mathbb{Z}_n^n \times \mathbb{Z}_n^n \rightarrow K$ and $i \circ j$ represents the "sum" of i and j in the group $\mathbb{Z}_n^n$ (with i and j are in the $n-$nary forms!). If $e_i = x^{i_1} y^{i_2}$ and $e_j = x^{j_1} y^{j_2}$, then we have

(2.5) $$e_i e_j = x^{i_1} y^{i_2} x^{j_1} y^{j_2} \rightarrow x^{i_1} \underbrace{yy...y}_{i_2} \underbrace{xx...x}_{j_1} y^{j_2} \xrightarrow{\xi} x^{i_1} \underbrace{yy...x}_{i_2} \underbrace{yx...x}_{j_1} y^{j_2}$$

and y will commute with x from j_1 times. Since we do this from i_2 times, we obtain the below formula for the function

(2.6)
$$\alpha\,(i,j):\left\{\begin{array}{l}\xi^{i_2 j_1},\ \text{if } i_1 + j_1 < n \text{ and } i_2 + j_2 < n \\ \xi^{i_2 j_1}a,\ \text{if } i_1 + j_1 \geq n \text{ and } i_2 + j_2 < n \\ \xi^{i_2 j_1}b,\ \text{if } i_1 + j_1 < n \text{ and } i_2 + j_2 \geq n \\ \xi^{i_2 j_1}ab,\ \text{if } i_1 + j_1 \geq n \text{ and } i_2 + j_2 \geq n\end{array}\right.\ .$$

The algorithm

Input: $n, e_i, e_j, i_1, i_2, j_1, j_2$
Step 1. Find $n-$ary decomposition i_n and j_n for the numbers i and j.
Step 2. Compute $i_n \circ j_n$ in the group $\mathbb{Z}_n^n$.
Step 3. Compute $\alpha\,(i,j)$ using formula (2.6).
Output: $e_i e_j$.

3. OCTONION ALGEBRA IN A SYMBOL ALGEBRA OF DEGREE THREE

In the following, we will prove that in all symbol algebra of degree three we can find an octonion algebra without division.

Let S be an associative algebra of degree three. For $z \in S$, let $P\,(X, z)$ be the characteristic polynomial for the element z

(3.7)
$$P\,(X, z) = X^3 - \tau\,(z)\,X^2 + \pi\,(z)\,X - \eta\,(z) \cdot 1,$$

where τ is the linear form, π is the quadratic form and η the cubic form.

Proposition 3.1. ([4], Lemma) *With the above notations, denoting by*
$z^* = z^2 - \tau\,(z)\,z + \pi\,(z) \cdot 1$, *for an associative algebra of degree three, we have:*
i) $\pi\,(z) = \tau\,(z^*)$.
ii) $z^{**} = \eta\,(z)\,z$.

An associative finite dimensional K-algebra A is *semisimple* if it can be expressed as a finite and unique direct sum of simple algebras. An associative K-algebra A is *separable* if for every field extension $K \subset L$ the algebra $A \otimes_K L$ is semisimple. We have that any central simple algebra is a separable algebra over its center (see [8] , p.463). A *Hurwitz* algebra A is a unitary (not necessarily associative) algebra over K together with a nondegenerate quadratic form n which satisfies $n\,(xy) = n\,(x)\,n\,(y)$, $x, y \in A$.

Theorem 3.6. ([10], Theorem 6.2.3) *Let A be a finite-dimensional algebra with unity over the field K and $\varphi : A \to K$ be a nondegenerate quadratic form such that $\varphi\,(xy) = \varphi\,(x)\,\varphi\,(y)$ for all $x, y \in A$.Then the algebra A has dimension $1, 2, 4$ or 8. If $\dim A \in \{4.8\}$, A is a quaternion or an octonion algebra.*

Let $S = \left(\frac{a,b}{K,\xi}\right)$ be a symbol algebra of degree n. For $n = 3$, the obtained symbol algebra has dimension 9 over the field K and, since an octonion algebra generalizes the quaternion algebra and has dimension 8 less than 9, we ask if we can find a relation between a symbol algebra of degree three and an octonion algebra.

Proposition 3.2. ([4], Theorem) *If A is an associative algebra of degree three over a field K containing the cubic root of the unity, ξ, then, using notations from Proposition 3.1, the quadratic form π permits compositions, $\pi\,(z \circ w) = \pi\,(z)\,\pi\,(w)$, on $\widetilde{A} = \{u \in A \,/\, \tau\,(u) = 0\}$ relative to the product $z \circ w = \xi zw - \xi^2 wz - \frac{2\xi+1}{3}\tau\,(zw) \cdot 1, z, w \in \widetilde{A}.$*

If A is separable over K, therefore the quadratic form π is nondegenerate and we can find a new product "$\triangledown$" on $(\widetilde{A}, \circ)$ such that $\left(\widetilde{A}, \triangledown\right)$ is a Hurwitz algebra.

Since S is separable over K it results that π is a nondegenerate quadratic form on S and it is also nondegenerate on $\widetilde{S}$, then there is an element $u \in \widetilde{S}$ such that $\pi(u) \neq 0$. Using some ideas given in [12] , let $v = \frac{u^2}{\pi(u)}$.

Proposition 3.3. *The linear maps $R_v^\circ : \widetilde{S} \to \widetilde{S}, R_v^\circ(x) = x \circ v$ and $L_v^\circ : \widetilde{S} \to \widetilde{S}, L_v^\circ(x) = x \circ v$ are bijective.*

Proof. Let $R_v^\circ : \widetilde{S} \to \widetilde{S}, R_v^\circ(x) = x \circ v$. Since $\pi(R_v^\circ(x)) = \pi(x \circ v) = \pi(x)\pi(v) = \pi(x)$, if $R_v^\circ(x) = 0$ it results that $\pi(x) = 0$, Using that π is nondegenerate, we obtain $x = 0$, therefore R_v° is bijective. $\square$

From the above proposition, on $\widetilde{S}$, we define a new multiplication
$$z \triangledown w = (R_v^{\circ-1}(z)) \circ (L_v^{\circ-1}(w)), w, z \in \widetilde{S}.$$

We have that $v \circ v$ is the unity element and $\pi(z \triangledown w) = \pi(z)\pi(w)$. Indeed, it results
$z \triangledown (v \circ v) = (R_v^{\circ-1}(z)) \circ (L_v^{\circ-1}(v \circ v)) = (R_v^{\circ-1}(z)) \circ L_v^{\circ-1}(L_v^\circ(v)) =$
$= (R_v^{\circ-1}(z)) \circ v = R_v^\circ\left(R_v^{\circ-1}(z)\right) = z = (v \circ v) \triangledown z$ and
$\pi(z \triangledown w) = \pi\left((R_v^{\circ-1}(z)) \circ (L_v^{\circ-1}(w))\right) = \pi\left(R_v^{\circ-1}(z)\right)\pi\left(L_v^{\circ-1}(w)\right) =$
$= \pi(z)\pi(w)$, since $\pi(z) = \pi\left(R_v^\circ\left(R_v^{\circ-1}(z)\right)\right) = \pi\left(R_v^{\circ-1}(z)\right)$. Therefore the algebra $\left(\widetilde{S}, \triangledown\right)$ is a Hurwitz algebra of dimension 8 and, from Theorem 3.6, we obtain that $\left(\widetilde{S}, \triangledown\right)$ is an octonion algebra with the norm π. An octonion algebra A with the norm π is a division algebra if $\pi(x) = 0$ implies $x = 0$, for $x \in A$. This algebra, in general, is a not a division algebra since. Indeed, if we can find an element $x \in \widetilde{S}$ such that $\pi(x^*) = 0$, using Proposition 3.1, it results $0 = \tau(\eta(x)x) = \tau((x^*)^*) = \pi(x^*)$, for the element $x \in \widetilde{S}$. Example of such an element $x \in \widetilde{S}$ with $\pi(x^*) = 0$ can be found in [4] , Example.

From the above, we proved the following theorem:

Theorem 3.7. *Let $S = \left(\frac{a,b}{K,\xi}\right)$ be a symbol algebra of degree 3. On the vector space $\widetilde{S}$, we define the following products: $z \circ w = \xi zw - \xi^2 wz - \frac{2\xi+1}{3}\tau(zw) \cdot 1, z, w \in \widetilde{S}$ and $z \triangledown w = (R_v^{\circ-1}(z)) \circ (L_v^{\circ-1}(w)), z, w \in \widetilde{S}$. Therefore $\left(\widetilde{S}, \triangledown\right)$ is an octonion algebra, in general without division.*

Since always we can find an octonion algebra, in general without division, in a symbol algebra of degree three, a natural question appears: if we can find some conditions which can determine when a symbol algebra of degree three is with division or not, or, more simple, if we can find examples of division symbol algebras of degree three. Such as conditions was given in [5] for symbol algebras of degrees $n = 4k + 2$ (but not for $n = 3$ or $n = 5$), in which the author found some trace form criteria to determine if a symbol algebra is with division. In the mentioned paper, the author don't provide examples, as will do in the next section.

4. EXAMPLES OF DIVISION SYMBOL ALGEBRAS OF DEGREE 3 AND 5

In this section, we determine certain class of non-division symbol algebras using Theorem 1.1 and some properties of ramification theory in algebraic number fields, for example the decomposition of a prime ideal in $\mathbb{Z}$ in the ring of integers of a cyclotomic field or

the decomposition of a prime ideal in the ring $\mathbb{Z}[\xi]$ in the ring of integers of a Kummer field (see Theorem 1.2. and Theorem 1.3.). We also provide examples of division symbol algebras of degree 3 and 5.

Proposition 4.4. *Let ϵ be a primitive root of order 3 of unity and let $K = \mathbb{Q}(\epsilon)$ be the cyclotomic field. Let $\alpha \in K^*$, p be a prime rational integers, $p \neq 3$ and let $L = K(\sqrt[3]{\alpha})$ be the Kummer field such that α is a cubic residue modulo p. Let h_L be the class number of L. Then, the symbol algebras $A = \left(\frac{\alpha, p^{h_L}}{K, \epsilon} \right)$ are non-division.*

Proof. Since p is a prime integer, $p \neq 3$, it results $p \equiv 1 \pmod 3$ or $p \equiv 2 \pmod 3$.
Case 1: $p \equiv 2 \pmod 3$.
We know that the ring of integers of K is $\mathbb{Z}[\epsilon]$ and it is a principal ring. According to Theorem 1.2 it results that p is inert in the ring $\mathbb{Z}[\epsilon]$. If we denote with $\mathcal{O}_L$ the ring of integers of the Kummer field L and knowing that the cubic residual symbol is $\left(\frac{\alpha}{p} \right)_3 = 1$, if we apply Theorem 1.3, we get $p\mathcal{O}_L = P_1 P_2 P_3$, where P_1, P_2, P_3 are conjugate prime ideals from $\mathcal{O}_L$. We obtain that $(p\mathcal{O}_L)^{h_L} = P_1^{h_L} P_2^{h_L} P_3^{h_L}$. Therefore, there exists a principal ideal I in the ring $\mathcal{O}_L$ such that $(p\mathcal{O}_L)^{h_L} = N_{L/K}(I)$. It results that there exists $x \in \mathcal{O}_L$ such that $p^{h_L} = N_{L/K}(x)$. Applying Theorem 1.1 and Remark 1.1, we get that the symbol algebras $A = \left(\frac{\alpha, p^{h_L}}{K, \epsilon} \right)$ are non-division.
Case 2: $p \equiv 1 \pmod 3$.
Applying Theorem 1.2, it results that $p\mathbb{Z}[\epsilon] = p_1 \mathbb{Z}[\epsilon] p_2 \mathbb{Z}[\epsilon]$, where p_1, p_2 are prime elements from $\mathbb{Z}[\epsilon]$.
Since α is a cubic residue modulo p, we obtain that the cubic residual symbols are $\left(\frac{\alpha}{p_1} \right)_3 = \left(\frac{\alpha}{p_2} \right)_3 = 1$. Applying Theorem 1.3, it results that $p\mathcal{O}_L = p_1 \mathcal{O}_L p_2 \mathcal{O}_L = P_{11} P_{12} P_{13} P_{21} P_{22} P_{23}$, where P_{1i} and P_{2i}, $i = \overline{1,3}$ are conjugate prime ideals from $\mathcal{O}_L$. We obtain that $(p\mathcal{O}_L)^{h_L} = (P_{11} P_{21})^{h_L} (P_{12} P_{22})^{h_L} (P_{13} P_{23})^{h_L}$. From this, as in Case 1, we obtain that the symbol algebras $A = \left(\frac{\alpha, p^{h_L}}{K, \epsilon} \right)$ are non-division. $\square$

Corollary 4.1. *Let q be an odd prime positive integer and ξ be a primitive root of order q of unity and let $K = \mathbb{Q}(\xi)$ be the cyclotomic field. Let $\alpha \in K^*$, p a prime rational integers, $p \neq 3$ and let $L = K(\sqrt[q]{\alpha})$ be the Kummer field such that α is a q power residue modulo p. Let h_L be the class number of L. Then, the symbol algebras $A = \left(\frac{\alpha, p^{h_L}}{K, \xi} \right)$ are non-division.*

Proof. The proof is similar with the proof of Proposition 4.4. $\square$

In the following, we give some examples of division symbol algebras. Using the computer algebra system MAGMA ([15] , [3]), we found some examples of division symbol algebras of degree 3 and of degree 5.
In the Example 4.1, we declare (in input) the cyclotomic field $E = \mathbb{Q}(\epsilon)$, where $\epsilon^3 = 1$, $\epsilon \neq 1$ and the Kummer field $K = \mathbb{Q}(\sqrt[3]{7}, \epsilon)$. With a simple calculation in MAGMA, we obtain that the class number of the Kummer field K is equal with 3. In output, with the procedure $NormEquation(K, 11 + a)$ we test if $11 + \epsilon$ is or not a norm of an element from the Kummer field K. The answer in negative and applying Theorem 1.1, we obtain that the algebra $\left(\frac{7, 11 + \epsilon}{\mathbb{Q}(\epsilon), \epsilon} \right)$ is a non-split symbol algebra, therefore it is a division algebra of degree 3. With the same procedure, for the element 11^3 ($NormEquation(K, 11^3)$) we obtain

that the algebra $\left(\frac{7,11^3}{\mathbb{Q}(\epsilon),\epsilon}\right)$ is a split (i.e non-division) symbol algebra of degree 3, confirming our result from Proposition 4.4.

In Examples 4.1 and 4.2 we also give another exemples of division symbol algebras or non-division symbol algebras of degree 3 or 5.

Example 4.1. *Q:=Rationals(); E:=CyclotomicField(3); a:=RootOfUnity(3); a;*
Et < t >:=PolynomialRing(E); E; f := $t^3 - 7$; K < b >:=NumberField(f); K; b^3;
NormEquation(K,11); NormEquation(K,11+a);NormEquation(K,11^3);
NormEquation(K,$(11 + a)^3$); NormEquation(K,5); NormEquation(K,5+a);
NormEquation(K,5^3); NormEquation(K,$(5 + a)^3$);
Evaluate $zeta_3$; Cyclotomic Field of order 3 and degree 2
Number Field with defining polynomial $t^3 - 7$ over E
7   false; false
true $[-11 * zeta_3 - 11]$; true $[10 * zeta_3 - 1]$ false; false
true $[5 * zeta_3]$; true $[zeta_3 + 5]$

Example 4.2. *Q:=Rationals(); F:=CyclotomicField(5); a:=RootOfUnity(5); a;*
Ft < t >:=PolynomialRing(F); F; f := $t^5 - 13$; K < b >:=NumberField(f); K; b^3;
NormEquation(K,11); NormEquation(K,11+a);
Evaluate $zeta_5$
Cyclotomic Field of order 5 and degree 4
Number Field with defining polynomial $t^3 - 13$ over F
13 false; false

From the above examples, using Theorem 1.1 and Remark 1.1, we obtain that the symbol algebras $\left(\frac{7,11}{\mathbb{Q}(\epsilon),\epsilon}\right)$, $\left(\frac{7,11+\epsilon}{\mathbb{Q}(\epsilon),\epsilon}\right)$, $\left(\frac{7,5}{\mathbb{Q}(\epsilon),\epsilon}\right)$, $\left(\frac{7,5+\epsilon}{\mathbb{Q}(\epsilon),\epsilon}\right)$ are division symbol algebras of degree 3 and $\left(\frac{13,11}{\mathbb{Q}(\xi),\xi}\right)$, $\left(\frac{13,11+\xi}{\mathbb{Q}(\xi),\xi}\right)$ are division symbol algebras of degree 5.

In the following we determine some split symbol algebras.

Proposition 4.5. *Let $n \geq 2$ be an arbitrary positive integer and let ξ be a primitive root of order n of unity. Let K be a finite field whose $char(K)$ does not divide n, $a, b \in K^*$ and let S be the symbol algebra $S = \left(\frac{a,\, b}{K(\xi),\xi}\right)$. Then S is a split algebra.*

Proof. Applying Theorem 1.5 and Theorem 1.1 we obtain that S is a split algebra. Another solution is to apply Theorem 1.4 and Remark 1.1. $\qquad\square$

Remark 4.4. For n a prime number, all symbol algebras from the above proposition are non-division algebras.

5. CONCLUSIONS

In this paper, we gave an algorithm for compute quickly the elements from the basis in a symbol algebra of degree n and we find an octonion algebra, in general without division, in a symbol algebra of degree three. We also provide some examples of division symbol algebras of degree three and five. Starting from results obtained in this paper, we intend to find, in a further research, more conditions for a symbol algebra of degree n to be with division.

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