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Uniform approximation of functions by Bernstein-Stancu operators

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ABSTRACT. For the class of bounded and continuous functions on $(0,1)$ we give a characterization of the functions which can be uniformly approximated by Bernstein-Stancu operators. We also study the possibility of uniform approximation of unbounded functions by Bernstein-Stancu operators in weighted spaces with Jacobi weights.

1. INTRODUCTION

Uniform approximation of continuous functions by using polynomials was a problem studied by K. Weierstrass [13]. A construction of a sequence of polynomials uniformly converging to every continuous function defined on the compact $[0,1]$ was obtained by S. N. Bernstein [1] in 1912. The operators introduced by Bernstein were generalized in 1969 by D. D. Stancu [10] and now these operators are called Bernstein-Stancu operators. They are defined by

$$P_{n,\alpha,\beta}(f,x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k+\alpha}{n+\beta}\right),\quad x\in[0,1],\ n\geq 1 \tag{1.1}$$

where $0\leq\alpha\leq\beta$. For $\alpha=\beta=0$ we obtain the classical Bernstein operators. They approximate uniformly every continuous function f defined on the compact $[0,1]$, i.e.

$$\|P_{n,\alpha,\beta}f-f\|=\sup_{x\in[0,1]}|P_{n,\alpha,\beta}(f,x)-f(x)|\to 0,\quad \text{when } n\to\infty.$$

One problem studied in this paper is the following: if we restrict to the class of bounded and continuous functions defined on $(0,1)$ does the uniform approximation property of Bernstein-Stancu operators still hold? It is possible to uniformly approximate $\sin\frac{1}{x}$? We give in Theorem 3.1 the characterization of the functions from this class which can be uniformly approximated by $P_{n,\alpha,\beta}$.

The second problem studied is whether we can uniformly approximate continuous and unbounded functions defined on $(0,1)$ by using Bernstein-Stancu operators. We use the Jacobi weights $\rho(x)=x^{-\gamma}(1-x)^{-\delta}$ with $\gamma,\delta\geq 0$ to approximate functions with singularities located at the endpoints of the interval $(0,1)$. Let $B_{\gamma,\delta}$ be the space of all functions $f\colon(0,1)\to\mathbb{R}$ with the property that there exists a constant $M>0$ such that $|f(x)|\leq M\rho(x)$, for every $x\in(0,1)$- a space which can be endowed with the norm

$$\|f\|_{\gamma,\delta}=\sup_{x\in(0,1)}x^{\gamma}(1-x)^{\delta}|f(x)|.$$

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We consider also the subspaces

$$C_{\gamma,\delta}(0,1) = C_{\gamma,\delta} = \{ f \in B_{\gamma,\delta} \mid f \text{ is continuous on } (0,1) \}$$
$$C_{\gamma,\delta}[0,1] = \{ f \in C_{\gamma,\delta} \mid f/\rho \text{ has finite limits at } x=0 \text{ and } x=1 \}$$
$$C_{\gamma,\delta,0}[0,1] = \left\{ f \in C_{\gamma,\delta}[0,1] \mid \lim_{x\to 0} f(x)/\rho(x) = 0 \text{ and } \lim_{x\to 1} f(x)/\rho(x) = 0 \right\}$$

This paper is motivated by the article [9], where the author studies the weighted uniform approximation by a class of Bernstein-type operators using the weight ρ, but only for $\gamma, \delta \in (0,1)$. The author uses K-functionals to estimate the error of approximation. We will use a different approach. Another article is [12], where the author gave pointwise approximation results using Bernstein operators.

The main results of this paper for the weighted approximation are Theorems 3.2 and 3.3 and Remark 3.2, which prove that only the functions which belong to $C_{\gamma,\delta,0}$ can be uniformly approximated. We use in the proof a generalization of the modulus of continuity, which is something "between" the local and the global modulus of continuity. Other related results were obtained in [2, 5, 6, 7, 8].

2. Preliminaries

The operators $P_{n,\alpha,\beta}$ have the properties (see [10])

$$P_{n,\alpha,\beta}(1,x) = 1,$$
$$P_{n,\alpha,\beta}(t,x) = \frac{nx+\alpha}{n+\beta},$$
$$P_{n,\alpha,\beta}(t^2,x) = \frac{(nx+\alpha)^2 + nx(1-x)}{(n+\beta)^2}.$$

From these relations we easily deduce the relation

$$P_{n,\alpha,\beta}\left((t-x)^2, x\right) = \frac{nx(1-x) + (\beta x - \alpha)^2}{(n+\beta)^2} \le \frac{C}{n+\beta}, \quad \text{for every } x \in [0,1], \tag{2.2}$$

where C is a constant independent on x and n.

Denote by $p_{n,k}$ the polynomials $\binom{n}{k} x^k (1-x)^{n-k}$.

Lemma 2.1 (Hoeffding's inequalities [4]). *For $a \in (0,1)$ and $x \in (0, 1-a)$ we have*

$$\sum_{\frac{k}{n} - x > a} p_{n,k}(x) \le e^{-2na^2}, \tag{2.3}$$

$$\sum_{\left|\frac{k}{n} - x\right| > a} p_{n,k}(x) \le 2e^{-2na^2}, \tag{2.4}$$

Lemma 2.2. *For $\alpha > 0$, for every $x \in (0,1]$ and for every $\gamma \ge 0$, we have*

$$P_{n,\alpha,\beta}\left(\frac{1}{t^\gamma}, x\right) \le \frac{C_0}{x^\gamma}. \tag{2.5}$$

where $C_0 = \lfloor \gamma + 1 \rfloor ! (\beta+1)^\gamma \max(1, \alpha^{-\gamma})$.

Proof. Starting from the inequalities

$$\frac{1}{k+\alpha} \le \max\left(1, \frac{1}{\alpha}\right) \cdot \frac{i+1}{k+i+1} \quad \text{and} \quad n+\beta \le (n+i+1)(\beta+1),$$

which are true for every nonnegative integers k and i, we deduce that

(2.6) $$p_{n+i,k+i}(x)\cdot\frac{n+\beta}{k+\alpha}\le p_{n+i+1,k+i+1}(x)\cdot\frac{(i+1)(\beta+1)\max(1,\alpha^{-1})}{x}$$

Consider $m=\lfloor\gamma\rfloor$. Using (2.6) we have

$$\begin{aligned}P_{n,\alpha,\beta}\left(\frac{1}{t^\gamma},x\right)&=\sum_{k=0}^{n}p_{n,k}(x)\frac{(n+\beta)^m}{(k+\alpha)^m}\cdot\frac{(n+\beta)^{\gamma-m}}{(k+\alpha)^{\gamma-m}}\\&\le\frac{C_m}{x^m}\sum_{k=0}^{n}p_{n+m,k+m}(x)\cdot\frac{(n+\beta)^{\gamma-m}}{(k+\alpha)^{\gamma-m}}\end{aligned}$$

where $C_m=m!(\beta+1)^m\max\left(1,\frac{1}{\alpha^m}\right)$.

If $\gamma-m=0$, then, using the inequality $\sum_{k=0}^{n}p_{n+m,k+m}(x)\le 1$, we obtain inequality (2.5). Otherwise, $\gamma-m\in(0,1)$. Setting $p=1/(\gamma-m)>1$ and considering $q>1$ such that $1/p+1/q=1$, we can apply Hölder inequality and obtain

$$P_{n,\alpha,\beta}\left(\frac{1}{t^\gamma},x\right)\le\frac{C_m}{x^m}\left(\sum_{k=0}^{n}p_{n+m,k+m}(x)\cdot\frac{n+\beta}{k+\alpha}\right)^{\frac{1}{p}}\cdot\left(\sum_{k=0}^{n}p_{n+m,k+m}(x)\right)^{\frac{1}{q}}$$

Applying one more time the inequality (2.6) we get

$$P_{n,\alpha,\beta}\left(\frac{1}{t^\gamma},x\right)\le\frac{C_m}{x^m}\left(\frac{(m+1)(\beta+1)\max(1,\alpha^{-1})}{x}\sum_{k=0}^{n}p_{n+m+1,k+m+1}(x)\right)^{\gamma-m}$$

which implies (2.5). □

Remark 2.1. Inspecting, the proof of Lemma 2.2 more carefully, we can notice that the upper bound for $P_{n,\alpha,\beta}(t^{-\gamma},x)=\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{n+\beta}{k+\alpha}\right)^\gamma$ is

$$\frac{C_0}{x^\gamma}\left(\sum_{k=0}^{n}p_{n+\lfloor\gamma\rfloor+1,k+\lfloor\gamma\rfloor+1}(x)\right)^{\gamma-\lfloor\gamma\rfloor}\left(\sum_{k=0}^{n}p_{n+\lfloor\gamma\rfloor,k+\lfloor\gamma\rfloor}(x)\right)^{1-\gamma+\lfloor\gamma\rfloor}.$$

This inequality remains true if the summation is taken on a subset of $\{0,1,\dots,n\}$.

Lemma 2.3. *For $\beta-\alpha>0$ and for every $\delta\ge 0$ we have*

(2.7) $$P_{n,\alpha,\beta}\left(\frac{1}{(1-t)^\delta},x\right)\le\frac{C_1}{(1-x)^\delta},\qquad \text{for every } x\in[0,1).$$

where $C_1=(\beta+1)^\delta\max\left(1,(\beta-\alpha)^{-\delta}\right)\cdot\lfloor\delta+1\rfloor!$.

Proof. Using the relation

$$P_{n,\alpha,\beta}\left(\frac{1}{(1-t)^\delta},x\right)=P_{n,\beta-\alpha,\beta}\left(\frac{1}{t^\delta},1-x\right).$$

we can use the same argument as in the proof of relation (2.5). □

Lemma 2.4. *For $0<\alpha<\beta$ and for every $\gamma,\delta\ge 0$ we have*

$$P_{n,\alpha,\beta}\left(\frac{1}{t^\gamma(1-t)^\delta},x\right)\le\frac{C}{x^\gamma(1-x)^\delta},\qquad \text{for every } x\in(0,1),$$

where C is a constant depending only on $\alpha,\beta,\gamma,\delta$.

Proof. Using the Hölder inequality for positive linear operators for $p = \frac{\gamma+\delta}{\gamma}$ and $q = \frac{\gamma+\delta}{\delta}$ and using the relations (2.5) and (2.7) we obtain the result. □

Corollary 2.1. *For every $f \in C_{\gamma,\delta}$ the polynomials $P_{n,\alpha,\beta} f$ belong to $C_{\gamma,\delta}$.*

Proof.

$$|P_{n,\alpha,\beta}(f(t), x)| \le P_{n,\alpha,\beta}(|f(t)|, x) \le P_{n,\alpha,\beta}(\|f\|_{\gamma,\delta}\, \rho(t), x) \le C \|f\|_{\gamma,\delta} \cdot \rho(x).$$

□

In the proof of the main results we will use the following moduli of continuity introduced in [11]. Consider the interval $I \subseteq \mathbb{R}$, the function $f : I \to \mathbb{R}$ and the nonempty set $J \subseteq I$. The modulus of continuity of f with respect to J is defined by

$$\omega(f, I, J, \delta) = \omega(f, J, \delta) = \sup_{\substack{|t-x|\le\delta \\ t\in I,\ x\in J}} |f(t) - f(x)|, \qquad \delta \ge 0.$$

If $J = \{ a \}$ we obtain the local modulus of continuity $\omega(f, a, \delta)$. If $J = I$ we obtain the global modulus of continuity $\omega(f, \delta)$.

This modulus has the property of monotony with respect to the set J: if $J \subseteq J' \subseteq I$ then $\omega(f, J, \delta) \le \omega(f, J', \delta)$.

3. Main results

Theorem 3.1. *If $f \in C_{0,0}$ then*

$$\|P_{n,\alpha,\beta} f - f\| \to 0, \text{ when } n \to \infty$$

if and only if

$$f \text{ is uniformly continuous on } (0, 1).$$

Proof. Suppose $\|P_{n,\alpha,\beta} f - f\| \to 0$, when $n \to \infty$. We prove that f must be uniformly continuous on $(0, 1)$.

We have (see [3, p. 305]) $p'_{n,k}(x) = n\, [p_{n-1,k-1}(x) - p_{n-1,k}(x)]$. So,

$$P'_{n,\alpha,\beta}(f, x) = n \sum_{k=1}^{n} p_{n-1,k-1}(x) f\left(\frac{k+\alpha}{n+\beta}\right) - n \sum_{k=0}^{n-1} p_{n-1,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right).$$

Because $\sum_{k=0}^{n} p_{n,k}(x) = 1$, we deduce that $\left|P'_{n,\alpha,\beta}(f, x)\right| \le 2n \|f\|$. Using the properties of the global modulus of continuity (see [3]) we have

$$\begin{aligned}
\omega(f, \delta_n) &\le \omega(f - P_{n,\alpha,\beta} f, \delta_n) + \omega(P_{n,\alpha,\beta} f, \delta_n) \\
&\le 2 \|f - P_{n,\alpha,\beta} f\| + \sup_{\substack{|t-x|\le\delta_n \\ t,x\in(0,1)}} |P_{n,\alpha,\beta}(f, t) - P_{n,\alpha,\beta}(f, x)| \\
&\le 2 \|f - P_{n,\alpha,\beta} f\| + \delta_n \sup_{c\in(0,1)} \left|P'_{n,\alpha,\beta}(f, c)\right| \le 2 \|f - P_{n,\alpha,\beta} f\| + 2 \|f\|\, n\, \delta_n.
\end{aligned}$$

If we choose the sequence δ_n such that $\delta_n \cdot n$ tends to zero, we deduce from the above inequality that $\omega(f, \delta_n) \to 0$ when $n \to \infty$. This proves that f is uniformly continuous on $(0, 1)$.

For the converse part, if f is uniformly continuous on $(0, 1)$ then the one-sided limits at 0 and 1 exist and so f can be extended to a continuous function on $[0, 1]$ and we can apply [10, Theorem 2] and deduce that $\|P_{n,\alpha,\beta} f - f\| \to 0$ tends to zero, when n tends to infinity. □

Example 3.1. The functions $f(x) = \sin\frac{1}{x}$ and $g(x) = \ln x$ for $x \in (0,1)$ cannot be uniformly approximated by the Bernstein-Stancu operators in the norm of $C_{0,0}$.

Theorem 3.2. *For every $f \in C_{\gamma,\delta}$ with the property*

$$\lim_{x\to 0} x^\gamma (1-x)^\delta f(x) = \ell_0 \in \mathbb{R}\setminus\{0\} \text{ or}$$
$$\lim_{x\to 1} x^\gamma (1-x)^\delta f(x) = \ell_1 \in \mathbb{R}\setminus\{0\},$$

the sequence $(\|P_{n,\alpha,\beta}f - f\|_{\gamma,\delta})_{n\geq 1}$ does not tend to zero when n tends to infinity.

Proof. This follows easily from the inequalities:

$$\begin{aligned}\|P_{n,\alpha,\beta}f - f\|_{\gamma,\delta} &= \sup_{x\in(0,1)} x^\gamma(1-x)^\delta \left|P_{n,\alpha,\beta}(f,x) - f(x)\right| \\ &\geq \lim_{x\to 0} x^\gamma(1-x)^\delta \left|P_{n,\alpha,\beta}(f,x) - f(x)\right| = |\ell_0| > 0, \\ \|P_{n,\alpha,\beta}f - f\|_{\gamma,\delta} &\geq \lim_{x\to 1} x^\gamma(1-x)^\delta \left|P_{n,\alpha,\beta}(f,x) - f(x)\right| = |\ell_1| > 0.\end{aligned}$$

□

Remark 3.2. If one of the limits ℓ_0 or ℓ_1 from Theorem 3.2 does not exist, we can apply $\limsup$ and deduce the imposibility of uniform approximation of such functions.

Theorem 3.3. *If γ and δ are positive real numbers and $f \in C_{\gamma,\delta}$ is a function having the properties*

$$\lim_{x\to 0} x^\gamma(1-x)^\delta f(x) = 0 \quad \text{and} \quad \lim_{x\to 1} x^\gamma(1-x)^\delta f(x) = 0,$$

then $\|P_{n,\alpha,\beta}f - f\|_{\gamma,\delta} \to 0$, when $n \to \infty$.

Proof. We have

$$\|P_{n,\alpha,\beta}f - f\|_{\gamma,\delta} = \sup_{x\in(0,1)} x^\gamma(1-x)^\delta \left|P_{n,\alpha,\beta}(f,x) - f(x)\right| \leq \max(A_n, B_n, C_n),$$

where the sequences (A_n), (B_n) and (C_n) are defined by

$$\begin{aligned}A_n &= \sup_{x\in(0,\varepsilon_n)} x^\gamma(1-x)^\delta \left|P_{n,\alpha,\beta}(f,x) - f(x)\right| \\ B_n &= \sup_{x\in(\varepsilon_n,1-\varepsilon_n)} x^\gamma(1-x)^\delta \left|P_{n,\alpha,\beta}(f,x) - f(x)\right| \\ C_n &= \sup_{x\in(1-\varepsilon_n,1)} x^\gamma(1-x)^\delta \left|P_{n,\alpha,\beta}(f,x) - f(x)\right|,\end{aligned}$$

for a suitable sequence (ε_n). We prove that (A_n), (B_n) and (C_n) converge to 0.

Consider the function

$$f^\star(x) = \begin{cases} x^\gamma(1-x)^\delta f(x), & x \in (0,1) \\ 0, & x = 0 \text{ and } x = 1. \end{cases}$$

From the properties of f given in the statement of the theorem we deduce that $f^\star \in C[0,1]$.

We have

$$A_n \leq \sup_{x\in(0,\varepsilon_n)} x^\gamma(1-x)^\delta \left|P_{n,\alpha,\beta}(f,x)\right| + \sup_{x\in(0,\varepsilon_n)} x^\gamma(1-x)^\delta \left|f(x)\right|. \tag{3.8}$$

The second term from the right-hand side of the above inequality can be evaluated using the local modulus of continuity

$$\sup_{x\in(0,\varepsilon_n)} x^\gamma(1-x)^\delta |f(x)| \leq \sup_{x\in(0,\varepsilon_n)} |f^\star(x) - f^\star(0)| = \omega(f^\star, 0, \varepsilon_n).$$

This quantity will converge to 0 if we choose (ε_n) to be a sequence converging to 0.

The first term of the right-hand side of (3.8) can be evaluated as follows:

$$\begin{aligned}
|P_{n,\alpha,\beta}(f,x)| &\le \sum_{k=0}^{N} p_{n,k}(x)\left|f\left(\frac{k+\alpha}{n+\beta}\right)\right| + \sum_{k=N+1}^{n} p_{n,k}(x)\left|f\left(\frac{k+\alpha}{n+\beta}\right)\right| \\
&\le \max_{0\le k\le N}\left|f^\star\left(\frac{k+\alpha}{n+\beta}\right)\right| \cdot \sum_{k=0}^{N} p_{n,k}(x)\cdot\rho\left(\frac{k+\alpha}{n+\beta}\right) \\
&\quad + \|f^\star\| \sum_{k=N+1}^{n} p_{n,k}(x)\cdot\rho\left(\frac{n+\beta}{k+\alpha}\right) \\
&\le \sup_{x\in(0,\frac{N+\alpha}{n+\beta})} |f^\star(x)|\cdot P_{n,\alpha,\beta}(\rho,x) + \|f^\star\| \sum_{k=N+1}^{n} p_{n,k}(x)\rho\left(\frac{k+\alpha}{n+\beta}\right)
\end{aligned}$$

Using Lemma 2.4 and the Remark 2.1 we get

$$x^\gamma(1-x)^\delta\,|P_{n,\alpha,\beta}(f,x)| \le C\cdot\omega\left(f^\star,0,\frac{N+\alpha}{n+\beta}\right) + C\,\|f\|_{\gamma,\delta}\,S_n,$$

where S_n is the product of

$$\left(\sum_{k=N+1}^{n} p_{n+m+1,k+m+1}(x)\right)^{\frac{a\gamma}{\gamma+\delta}} \cdot \left(\sum_{k=N+1}^{n} p_{n+m,k+m}(x)\right)^{\frac{(1-a)\gamma}{\gamma+\delta}}$$

with

$$\left(\sum_{k=N}^{n} p_{n+m+1,k}(x)\right)^{\frac{a\delta}{\gamma+\delta}} \cdot \left(\sum_{k=N+1}^{n} p_{n+m,k}(x)\right)^{\frac{(1-a)\delta}{\gamma+\delta}},$$

where $m=\lfloor\gamma+\delta\rfloor$ and $a=\gamma+\delta-m$. Choosing $N=\lfloor(n+m)(x+\varepsilon_n)\rfloor$ and using the relation (2.3) we obtain

$$S_n \le \left(\sum_{k=N+1}^{n} p_{n+m,k}(x)\right)^{1-a} \le \left(\sum_{(n+m)(x+\varepsilon_n)<k\le n+m} p_{n+m,k}(x)\right)^{1-a} \le e^{-2(1-a)(n+m)\varepsilon_n^2}.$$

Choosing $n\ge 2m+\beta$ we obtain $(N+\alpha)\le 3\varepsilon_n(n+\beta)$. All these give us the bound for A_n:

$$A_n \le (3C+1)\cdot\omega(f^\star,0,\varepsilon_n) + C\,\|f\|_{\gamma,\delta}\cdot e^{-2(1-a)(n+m)\varepsilon_n^2}.$$

If we choose $\varepsilon_n\in(0,\frac12)$ such that $\varepsilon_n\to 0$ and $\varepsilon_n^2\cdot n\to\infty$ then $A_n\to 0$, when $n\to\infty$.

Next we prove that (C_n) is convergent to 0. This can be done using the above argument and the following representation of C_n:

$$\begin{aligned}
C_n &= \sup_{x\in(1-\varepsilon_n,1)} x^\gamma(1-x)^\delta\,|P_{n,\alpha,\beta}(f(t),x)-f(x)| \\
&= \sup_{y\in(0,\varepsilon_n)} (1-y)^\gamma y^\delta\,|P_{n,\alpha,\beta}(f(t),1-y)-f(1-y)| \\
&= \sup_{y\in(0,\varepsilon_n)} (1-y)^\gamma y^\delta\,|P_{n,\beta-\alpha,\beta}(g(t),y)-g(y)|, \qquad \text{where } g(t)=f(1-t).
\end{aligned}$$

We obtain

$$\begin{aligned} C_n &\le (3C+1)\cdot\omega(g^\star,0,\varepsilon_n)+C\,\|g\|_{\gamma,\delta}\cdot e^{-2(1-a)(n+m)\varepsilon_n^2} \\ &\le (3C+1)\cdot\omega(f^\star,1,\varepsilon_n)+C\,\|f\|_{\gamma,\delta}\cdot e^{-2(1-a)(n+m)\varepsilon_n^2}, \end{aligned}$$

where $g^\star(t)=f^\star(1-t)$. The same conditions on (ε_n) ($\varepsilon_n\to 0$ and $\varepsilon_n^2\cdot n\to\infty$) assure that $C_n\to 0$, when n goes to the infinity.

It remains to prove that (B_n) is convergent to 0. Consider the intervals $J_n=(\varepsilon_n,1-\varepsilon_n)$ and denote by

$$M_n=\sup_{x\in J_n} x^\gamma(1-x)^\delta \sum_{\left|\frac{k+\alpha}{n+\beta}-x\right|\le\eta_n} p_{n,k}(x)\left|f\left(\frac{k+\alpha}{n+\beta}\right)-f(x)\right|$$

$$T_n=\sup_{x\in J_n} x^\gamma(1-x)^\delta \sum_{\left|\frac{k+\alpha}{n+\beta}-x\right|>\eta_n} p_{n,k}(x)\left|f\left(\frac{k+\alpha}{n+\beta}\right)-f(x)\right|$$

the middle part and the tail part of B_n. Then, $B_n\le M_n+T_n$.

If we consider (η_n) a sequence converging to 0 such that $\eta_n<\varepsilon_n$ and denote the intervals $I_n=(\varepsilon_n-\eta_n,1-\varepsilon_n+\eta_n)$, the inclusion $J_n\subset I_n$ is true. Using the relation

$$|f(t)-f(x)|\le\frac{|f(t)|}{\rho(t)}\cdot|\rho(t)-\rho(x)|+\rho(x)\left|\frac{f}{\rho}(t)-\frac{f}{\rho}(x)\right|$$

we obtain

$$M_n\le\|1/\rho\|_{J_n}\|f^\star\|\cdot\omega(\rho,I_n,J_n,\eta_n)+\omega(f^\star,I_n,J_n,\eta_n).$$

The second term from the right-hand side of the above inequality converges to 0 because is less than the global modulus of continuity of $f^\star$ on $[0,1]$ and because the function $f^\star$ is continuous on the compact $[0,1]$.

The generalized modulus of continuity of ρ can be evaluated using the Mean Value Theorem:

$$\omega(\rho,I_n,J_n,\eta_n)\le\eta_n\cdot\max\left(|\rho'(\varepsilon_n-\eta_n)|,|\rho'(1-\varepsilon_n+\eta_n)|\right)\le\frac{C\eta_n}{(\varepsilon_n-\eta_n)^{\max(\gamma,\delta)+1}}.$$

Choosing $\eta_n=\varepsilon_n^{\max(\gamma,\delta)+2}$ we obtain that $\omega(\rho,I_n,J_n,\eta_n)\to 0$ when $n\to\infty$ and so M_n converges to 0.

It remains to prove that the tail part T_n converges to 0. We have

$$\begin{aligned} T_n &\le \sup_{x\in J_n} x^\gamma(1-x)^\delta \sum_{\left|\frac{k+\alpha}{n+\beta}-x\right|>\eta_n} p_{n,k}(x)\left(\left|f\left(\frac{k+\alpha}{n+\beta}\right)\right|+|f(x)|\right) \\ &\le \sup_{x\in J_n}\frac{\|f^\star\|}{\rho(x)}\sum_{\left|\frac{k+\alpha}{n+\beta}-x\right|>\eta_n} p_{n,k}(x)\cdot\rho\left(\frac{k+\alpha}{n+\beta}\right)+\|f^\star\|\sum_{\left|\frac{k+\alpha}{n+\beta}-x\right|>\eta_n} p_{n,k}(x). \end{aligned}$$

The inequality $\left|\frac{k+\alpha}{n+\beta}-x\right|>\eta_n$ implies $\left|\frac{k}{n}-x\right|>\eta_n-\frac{\beta-\alpha}{n}$. Using the inequality (2.4) we obtain

$$\sum_{\left|\frac{k+\alpha}{n+\beta}-x\right|>\eta_n} p_{n,k}(x)\le\sum_{\left|\frac{k}{n}-x\right|>\eta_n-\frac{\beta-\alpha}{n}} p_{n,k}(x)\le e^{-\frac{n}{2}(\eta_n-\frac{\beta-\alpha}{n})^2}.$$

This converges to 0 if $n\cdot\eta_n^2$ tends to 0. But this condition is satisfied if we choose (ε_n) to be a sequence that converges to 0 and $\varepsilon_n^{2\max(\gamma,\delta)+4}\cdot n\to\infty$.

The inequality $\left|\frac{k+\alpha}{n+\beta}-x\right|>\eta_n$ implies also $\left|\frac{k}{n+m}-x\right|>\eta_n-\frac{\beta-\alpha}{n+m}$. Using Lemma 2.4 and the Remark 2.1 we obtain as in the evaluation of S_n:

$$\sup_{x\in J_n}\frac{\|f^\star\|}{\rho(x)}\sum_{\left|\frac{k+\alpha}{n+\beta}-x\right|>\eta_n}p_{n,k}(x)\cdot\rho\left(\frac{k+\alpha}{n+\beta}\right)\le C\,\|f\|_{\gamma,\delta}\left(\sum_{\left|\frac{k}{n+m}-x\right|>\eta_n-\frac{\beta-\alpha}{n+m}}p_{n,k}(x)\right)^{\frac{(1-a)\delta}{\gamma+\delta}}.$$

Using the inequality (2.4) we obtain

$$\left(\sum_{\left|\frac{k}{n+m}-x\right|>\eta_n-\frac{\beta-\alpha}{n+m}}p_{n,k}(x)\right)^{\frac{(1-a)\delta}{\gamma+\delta}}\le e^{-\frac{2(1-a)\delta}{\gamma+\delta}n\left(\eta_n-\frac{\beta-\alpha}{n+m}\right)^2},$$

and this converges to 0.

We have proved that T_n converges to 0 and so the proof is complete. □

Example 3.2. The functions $f(x)=\sin\frac{1}{x}$ and $g(x)=\ln x$ for $x\in(0,1)$ can be uniformly approximated by the Bernstein-Stancu operators in the norm of $C_{\gamma,0}$ for an arbitrary $\gamma>0$.

References

[1] Bernstein, S. N., *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités*, Commun. Soc. Math. Kharkow (2), **13** (1912-1913), 1–2

[2] Della Vecchia, B, Mastroianni, G., and Szabados, J., *Weighted approximation of functions with endpoint or inner singularities by Bernstein operators*, Acta Math. Hungar., **103** (2004), 19–41

[3] DeVore, R. A., and Lorentz, G. G., *Constructive Approximation*, Springer-Verlag, Berlin-Heidelberg-New York, 1993

[4] Hoeffding, W., *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc., **58** (1963), 13–30

[5] Holhoş, A., *Uniform approximation in weighted space using some positive linear operators*, Stud. Univ. Babeş-Bolyai Math., **56** (2011), No. 2, 413–422

[6] Holhoş, A., *Uniform weighted approximation by positive linear operators*, Stud. Univ. Babeş-Bolyai Math., **56** (2011), No. 3, 135–146

[7] Holhoş, A., *Uniform approximation of functions by Meyer-König and Zeller operators*, J. Math. Anal. Appl., **393** (2012), 33–37

[8] Ivanov, K. G. and Parvanov, P. E., *Weighted Approximation by Baskakov-Type Operators*, Acta Math. Hungar., **133** (2011), No. 1-2, 33–57

[9] Parvanov, P. E., *Weighted Approximation by a Class of Bernstein-Type Operators*, Math. Balkanica (N.S.), **25** (2011), fasc. 1-2, 31–38

[10] Stancu, D. D., *Asupra unei generalizări a polinoamelor lui Bernstein*, Stud. Univ. Babeş-Bolyai Math., **14** (1969), 31–45 (in Romanian)

[11] Száz, Á., *Moduli of continuity and spaces of functions*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., **8** (1997), 44–51

[12] Weba, M., *On the approximation of unbounded functions by Bernstein polynomials*, East J. Approx., **9** (2003), 351–356

[13] Weierstrass, K., *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen*, Sitzungsber. Akad. Berlin (1885), 633–639, 789–805

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