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# Existence of three solutions for a three-point boundary value problem via a three-critical-point theorem

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ABSTRACT. In this paper, we study the existence of at least three solutions for a three-point boundary value problem. By constructing and showing an appropriate separable and reflexive Banach space, a new multiplicity result of the three-point boundary value problem is established. Our main tool is based upon variational method and three-critical-point theorem.

## 1. INTRODUCTION

The aim of this paper is to establish a multiplicity result for the following three-point boundary value problem

$$\begin{cases} u'' + \lambda f(t,u) = 0, \\ u(0) = 0, u(1) = \alpha u(\eta), \end{cases} \tag{1.1}$$

where $\alpha \in R, \eta \in (0,1)$, $\lambda$ is a positive parameter and $f : [0,1] \times R \to R$ is a continuous function.

In [12-14], Ricceri proposed and developed an innovative minimal method for the study of nonlinear eigenvalue problems. After that, Bonannao [1] gave an application of the method to the two point boundary value problem

$$\begin{cases} u'' + \lambda f(u) = 0, \\ u(0) = u(1) = 0. \end{cases} \tag{1.2}$$

Candito [3] extended the main results of [1] to the nonautonomous case

$$\begin{cases} u'' + \lambda f(t,u) = 0, \\ u(a) = u(b) = 0. \end{cases} \tag{1.3}$$

In [6], He and Ge extended the main results of [1,3] to the quasilinear problem

$$\begin{cases} (\varphi_p(u'))' + \lambda f(t,u) = 0, \\ u(a) = u(b) = 0. \end{cases} \tag{1.4}$$

Livrea [8] extended the main results of [1,3] to the following boundary value problem

$$\begin{cases} u'' + \lambda h(u') f(t,u) = 0, \\ u(0) = u(1) = 0. \end{cases} \tag{1.5}$$

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Du, Lin and Tisdell [5] discussed a two point boundary value problem with p-Lapacian operator via critical points theorem. Bonannao and Livrea [2] studied the following Dirichlet problems involving the p-Lapacian

$$\begin{cases} \Delta_p u + \lambda f(t,u) = 0, \text{ in } \Omega, \\ u(0) = 0, \text{ on } \partial\Omega. \end{cases} \tag{1.6}$$

In the above papers, in order to use the three-critical-point theorem obtained by Ricceri [12] to study the above two-point boundary value problems, the authors all used the Sobolve space $W_0^{1,2}((0,1))$ or $W_0^{1,p}((0,1))(p>1)$ since it satisfies the condition in the three-critical-point theorem.

However, if the boundary condition is $u(0) = 0, u(1) = \alpha u(\eta)$, then the Sobolev space $W_0^{1,2}((0,1))$ is not valid. In order to solve the difficulty, we need to construct an appropriate space $W_1^{1,2}((0,1)) = \{u \in W^{1,2}((0,1)) : u(0) = 0, u(1) = \alpha u(\eta)\}$ and show it is a separable and reflexive real Banach space (i.e.,Theorem 3.2).

In the past few years, there has been increasing interest in studying three-point boundary value problems (1.1); to identify a few, we refer the reader to [4, 7, 9, 10]. The methods and techniques employed in these papers are the Leray-Schauder continuation theorem, upper and lower solutions method, the nonlinear alternative of Leray-Schauder, the coincidence degree theory, or some fixed point theorem, for example, Du et al. [4] studied the following second-order three-point boundary value problem

$$\begin{cases} u'' + f(t,u,u') = 0, \\ u(0) = 0, u(1) = \alpha u(\eta). \end{cases} \tag{1.7}$$

They assumed that boundary value problem (1.7) exists two pairs of lower and upper solutions to ensure the existence of at least three solutions under $0 < \alpha$ and the non-resonance case $0 < \alpha\eta < 1$. He and Ge [7] discussed the existence of at least three solutions to boundary value problem (1.1) when $\lambda = 1$ under the non-resonance case $0 < \alpha\eta < 1$ by using the the Leggett-Williams fixed-point theorem. Ma [9] considered the the following second-order three-point boundary value problem

$$\begin{cases} u'' + a(t)f(u) = 0, \\ u(0) = 0, u(1) - \alpha u(\eta) = b, \end{cases} \tag{1.8}$$

and required the assumption that $f$ is sublinear or superlinear to obtained the existence of at least one positive solutions for boundary value problem (1.8) under $0 < \alpha$ and the non-resonance case $0 < \alpha\eta < 1$. Ma [10] dealt with the multiplicity results for boundary value problem (1.1) when $\lambda = 1$ and satisfying the resonance case $\alpha\eta = 1$ by employing the methods of lower and upper solutions by the connectivity properties of the solution set of parameterized families of compact vector fields.

In this paper, to show the existence of solutions an open interval $\Lambda \subseteq (0, +\infty)$ such that for every $\lambda \in \Lambda$, the problem (1.1) has at least three solutions (i.e., Theorem 4.3), we will use a three-critical-point- theorem of B. Ricceri, which is different from the above-mentioned references [4, 7, 9, 10]. Our results are new and different from those of [4, 7, 9, 10], we do not require the assumption that $f$ is sublinear or superlinear and the non-resonance case $0 < \alpha\eta < 1$ or the resonance case $\alpha\eta = 1$. In our work, we shall remove the restriction $0 < \alpha\eta < 1$ or $\alpha\eta = 1$.

## 2. PRELIMINARY

First, we recall the three-critical-point theorem [12] and some definitions [11, 15] which shall help us to obtain our main results.

**Theorem 2.1.** ([12]) *Let $X$ be a separable and reflexive real Banach space, $\Phi : X \to R$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$, and $\Psi : X \to R$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

$$\lim_{\|u\|\to\infty} \Phi(u) + \lambda\Psi(u) = +\infty$$

*for all $\lambda \in [0, +\infty)$, and that there exists a continuous concave function $h : [0, +\infty) \to R$ such that*

$$\sup_{\lambda\geq 0} \inf_{u\in X} (\Phi(u) + \lambda\Psi(u) + h(\lambda)) < \inf_{u\in X} \sup_{\lambda\geq 0} (\Phi(u) + \lambda\Psi(u) + h(\lambda)).$$

*Then there exists an open interval $\Lambda \subseteq (0, +\infty)$ and a positive real number $q$ such that, for each $\lambda \in \Lambda$, the equation*

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

*has at least three solutions in $X$ whose norms are less than $q$.*

**Proposition 2.1.** ([13]) *Let $X$ be a nonempty set and $\Phi$, $\Psi$ two real functions on $X$. Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that*

$$\Phi(x_0) = \Psi(x_0) = 0, \ \Phi(x_1) > r, \sup_{x\in\Phi^{-1}((-\infty,r])} \Psi(x) < r\frac{\Psi(x_1)}{\Phi(x_1)}.$$

*Then, for each $\rho$ satisfying*

$$\sup_{x\in\Phi^{-1}((-\infty,r])} (\Psi(u)) < \rho < r\frac{\Psi(x_1)}{\Phi(x_1)},$$

*one has*

$$\sup_{\lambda\geq 0} \inf_{x\in X} (\Phi(x) + \lambda(\rho - \Psi(x))) < \inf_{x\in X} \sup_{\lambda\geq 0} (\Phi(x) + \lambda(\rho - \Psi(x))).$$

**Definition 2.1.** Let $(E, \rho)$ and $(E_1, \rho_1)$ be metric spaces, the operator $\varphi : E \longrightarrow E_1$ is called an isometric isomorphic mapping if

(i) the mapping $\varphi$ is surjective;

(ii) $\rho(x, y) = \rho_1(\varphi x, \varphi y)$, for all $x, y \in E$.

The metric space $(E, \rho)$ is said to be isometric isomorphic to $(E_1, \rho_1)$.

**Remark 2.1.** From the condition (ii) in the definition 2.1, $\varphi$ is also a single-valued mapping.

**Definition 2.2.** A normed space $E$ is uniformly convex, provided that for any $\varepsilon \in (0, 2)$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|x\| = \|y\| = 1$, $\|x - y\| \geq \varepsilon$ imply

$$\|\frac{x+y}{2}\| \leq 1 - \delta(\varepsilon).$$

Here and in the sequel, we take

$$X = W_1^{1,2}((0,1)) = \{u \in W^{1,2}((0,1)) : u(0) = 0, u(1) = \alpha u(\eta)\}$$

endowed with the norm $\|u\| = (\int_0^1 |u'|^2 dt)^{\frac{1}{2}}$.

## 3. $X$ IS A SEPARABLE AND REFLEXIVE REAL BANACH SPACE

**Theorem 3.2.** $W_1^{1,2}((0,1))$ *is a separable and reflexive real Banach space.*

We prove this result via the following lemmas.

**Lemma 3.1.** *$X$ is a Banach space.*

*Proof.* Clearly, $X$ is a normed linear space. Suppose sequence of $\{u_n\} \in X$ is an arbitrary cauchy sequence. Since $\{u_n\}$ is also a cauchy sequence of $W^{1,2}((0,1))$ and $W^{1,2}((0,1))$ is a Banach space, then there exists a $u \in W^{1,2}((0,1))$ with $\lim_{n\longrightarrow+\infty} u_n = u$. From $\lim_{n\longrightarrow+\infty} u_n(0) = u(0) = 0$, $\lim_{n\longrightarrow+\infty} u_n(1) = u(1)$, $\lim_{n\longrightarrow+\infty} \alpha u_n(\eta) = \alpha x(\eta)$, $u_n(1) = \alpha u_n(\eta)$, we obtain $u(0) = 0$, $u(1) = \alpha u(\eta)$. Hence $X$ is a Banach space. $\square$

**Lemma 3.2.** *$X$ is separable.*

*Proof.* Owing to $W^{1,2}((0,1))$ be separable, there exists an enumerable subset $A \subset W^{1,2}((0,1))$ such that $\forall x \in W^{1,2}((0,1))$, $\exists\{y_n\} \subset A$ satisfying $y_n \to x$, as $n \to +\infty$. For $\forall y \in A$, we define $\phi : A \longrightarrow X$ as

$$(\phi y)(t) = \begin{cases} y(t) - y(0)\frac{\eta-t}{\eta}, & 0 \le t \le \eta, \\ y(t) + [\alpha y(\eta) - y(1)]\frac{t-\eta}{1-\eta}, & \eta \le t \le 1. \end{cases}$$

Let $z = \phi(y)$, then $z \in X$. Thus the set $A_1 = \{z = \phi(y) : y \in A\}$ is an enumerable subset of $X$.

Now we shall prove that $\forall x \in X$, there exists $\{z_n\} \subset A_1$ such that $\|z_n - x\| \longrightarrow 0$, as $n \to +\infty$. By $x \in W^{1,2}((0,1))$, then there exists $\exists\{y_n\} \subset A$ satisfying

$$\|y_n - x\| \longrightarrow 0, \text{ as } n \to +\infty. \tag{3.9}$$

Since

$$\|z_n - x\| \le \|z_n - y_n\| + \|y_n - x\|. \tag{3.10}$$

Let $z_n = \phi(y_n)$, one has

$$\begin{aligned} \|z_n - y_n\| &= \Big(\int_0^1 |(z_n - y_n)'|^2 dt\Big)^{\frac{1}{2}} \\ &= \Big(\int_0^\eta \Big(\frac{y_n(0)}{\eta}\Big)^2 dt + \int_\eta^1 \Big(\frac{\alpha y_n(\eta) - y_n(1)}{1-\eta}\Big)^2 dt\Big)^{\frac{1}{2}} \\ &= \Big(\frac{y_n^2(0)}{\eta} + \frac{(\alpha y_n(\eta) - y_n(1))^2}{1-\eta}\Big)^{\frac{1}{2}} \\ &\longrightarrow \Big(\frac{x^2(0)}{\eta} + \frac{(\alpha x(\eta) - x(1))^2}{1-\eta}\Big)^{\frac{1}{2}} = 0, \ n \to +\infty. \end{aligned}$$

i.e.,

$$\|z_n - y_n\| \longrightarrow 0, \ n \to +\infty. \tag{3.11}$$

Then from (3.9), (3.10) and (3.11), we show $\|z_n - x\| \longrightarrow 0$, as $n \to +\infty$. Therefore $X$ is separable. $\square$

**Lemma 3.3.** (Milman Theorem [11]) *A uniformly convex Banach space is reflexive.*

**Lemma 3.4.** (Clakson Inequality [11]) *For $u, v \in L^p((0,1))$, $2 \le p < ++\infty$, then*

$$\Big\|\frac{u+v}{2}\Big\|^p + \Big\|\frac{u-v}{2}\Big\|^p \le \frac{1}{2}(\|u\|^p + \|v\|^p).$$

**Lemma 3.5.** *$W_1^{1,2}((0,1))$ is reflexive.*

*Proof.* We define the operator $J : W_1^{1,2}((0,1)) \longrightarrow L^2((0,1))$ as follows

$$\text{for every } u \in W_1^{1,2}((0,1)),\ J : u \longmapsto u' \in L^2((0,1)).$$

Let

$$W_1(1,2) = \{u' | u \in W_1^{1,2}((0,1))\}.$$

Then $W_1(1,2)$ is a subspace of $L^2((0,1))$.

Now we show the operator $J$ is a isometric isomorphic mapping onto $W_1^{1,2}((0,1))$ to $W_1(1,2)$. It is obvious that the mapping $J$ is surjective, we only need to show the mapping $J$ is isometric. In the space $W_1(1,2)$, we define the norm $\|v\| = (\int_0^1 |v|^2 dt)^{\frac{1}{2}}$. From the mapping

$$J: \begin{array}{l} W_1^{1,2}((0,1)) \longrightarrow W_1(1,2), \\ u \longmapsto u' = v, \end{array}$$

we obtain $\|u\| = \|Ju\|$. Then the operator $J : W_1^{1,2}((0,1)) \longrightarrow W_1(1,2)$ is a isometric isomorphic mapping and the space $W_1^{1,2}((0,1))$ is isometric isomorphic to the space $W_1(1,2)$.

We shall show $W_1(1,2)$ is uniformly convex. for any $\varepsilon > 0$, $u, v \in W_1(1,2)$, such that $\|u\| = \|v\| = 1$, $\|u - v\| \geq \varepsilon$, we choose $\delta(\varepsilon) = (\frac{\varepsilon}{2})^2$. In Lemma 3.4, selecting $p = 2$, one has

$$\|\frac{u+v}{2}\|^2 \leq \frac{1}{2}(\|u\|^2 + \|v\|^2) - \|\frac{u-v}{2}\|^2 \leq 1 - \delta(\varepsilon)$$

which implies that $W_1(1,2)$ is uniformly convex. Then $W_1^{1,2}((0,1))$ is also uniformly convex since $W_1^{1,2}((0,1))$ is isometric isomorphic to the space $W_1(1,2)$. $\square$

The proof of Theorem 3.2 is now an easy consequence of the above lemmata.

## 4. Triple solutions results

Let the positive constant

$$M = \frac{1 - \eta + 2\alpha^2\eta}{\eta(1-\eta)},$$

and define the real function $g(t,\xi)$ by

$$g(t,\xi) = \int_0^\xi f(t,u)du, \quad \text{for all } (t,\xi) \in [0,1] \times R,$$

and $f : [0,1] \times R \to R$ is a continuous function.

Our main results fully depend on the following lemma.

**Lemma 4.6.** *We assume that there exist two positive constants $d, c$, with $c < \frac{1}{2}\sqrt{M}d$, such that*

*(i) $g(t,\xi) \geq 0$ for each $(t,\xi) \in [0,\eta] \cup [\frac{1+\eta}{2}, 1] \times [0,d]$;*

*(ii)* $\displaystyle \max_{(t,\xi)\in[0,1]\times[-c,c]} g(t,\xi) < \frac{4}{M}(\frac{c}{d})^2 \int_\eta^{\frac{1+\eta}{2}} g(t,d)dt.$

*Then there exist $r > 0$ and $u \in X$ such that $2r < \|u\|^2$ and*

$$\max_{(t,\xi)\in[0,1]\times[-c,c]} g(t,\xi) < 2r \frac{\int_0^1 g(t,u(t))dt}{\|u\|^2}.$$

*Proof.* Define the function $u(t)$ as

$$u(t) = \begin{cases} \dfrac{1}{\eta}dt, & \text{for } t \in [0,\eta], \\ d, & \text{for } t \in [\eta, \frac{1+\eta}{2}], \\ d(\dfrac{\alpha(1+\eta)}{\eta-1} - \dfrac{2\alpha}{\eta-1}t), & \text{for } t \in [\frac{1+\eta}{2}, 1]. \end{cases}$$

Selecting $r = 2c^2$. It is clear that $u \in X$ and

$$\|u\|^2 = \int_0^1 |u'(t)|^2 dt = \int_0^\eta (\frac{d}{\eta})^2 dt + \int_{\frac{1+\eta}{2}}^1 (\frac{2\alpha d}{\eta-1})^2 dt = \frac{1-\eta+2\alpha^2\eta}{\eta(1-\eta)}d^2 = Md^2.$$

By the assumption $c < \frac{1}{2}\sqrt{M}d$, the we obtain $2r < \|u\|^2$. From assumption (ii), we have

$$\begin{aligned} \frac{\int_0^1 g(t,u(t))dt}{\|u\|^2}2r &\geq \frac{4c^2}{\|u\|^2}\int_\eta^{\frac{1+\eta}{2}} g(t,d)dt = \frac{4}{M}(\frac{c}{d})^2 \int_\eta^{\frac{1+\eta}{2}} g(t,d)dt \\ &> \max_{(t,\xi)\in[0,1]\times[-c,c]} g(t,\xi). \end{aligned}$$

□

Our main result is the following theorem.

**Theorem 4.3.** *Suppose that there exist four positive constants $c, d, \mu, l$ with $l < 2$ and $c < \frac{1}{2}\sqrt{M}d$ such that*

*(i) $g(t,\xi) \geq 0$ for each $(t,\xi) \in [0,\eta] \cup [\frac{1+\eta}{2}, 1] \times [0,d]$;*

*(ii) $\max_{(t,\xi)\in[0,1]\times[-c,c]} g(t,\xi) < \frac{4}{M}(\frac{c}{d})^2 \int_\eta^{\frac{1+\eta}{2}} g(t,d)dt$;*

*(iii) $g(t,\xi) \leq \mu(1+|\xi|^l)$ for each $t \in [0,1]$ and $\xi \in R$.*

*Then there exist an open interval $\Lambda \subseteq (0, ++\infty)$ and a positive real number $q$ such that, for each $\lambda \in \Lambda$, problem (1.1) has at least three solutions belonging to $X$ whose norms in $W_1^{1,2}((0,1))$ are less than $q$.*

*Proof.* For each $u \in X$, we define

$$\Phi(u) = \frac{1}{2}\|u\|^2 \text{ and } \Psi(u) = -\int_0^1 (\int_0^{u(t)} f(t,x)dx)dt, \ J(u) = \Phi(u) + \lambda\Psi(u).$$

It is well known that the critical points of $J$ are the generalized solutions of (1.1). So, our end is to verify that $\Phi$ and $\Psi$ satisfy the assumptions of Theorem 2.1. It is easy to see that $\Phi$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$, and $\Psi$ is a continuous Gâteaux differentiable functional whose Gâteaux derivative is compact.

Moreover, thanks to (iii) and to Poincaré inequality, one has

$$\lim_{\|u\|\to++\infty} (\Phi(u) + \lambda\Psi(u)) = ++\infty,$$

for all $\lambda \in (0, ++\infty)$.

we claim that there exist $r > 0$ and $u \in X$ such that

$$\sup_{u\in\Phi^{-1}(-+\infty,r]} (-\Psi(u)) < r\frac{(-\Psi(u))}{\Phi(u)}.$$

Now, taking into account that

$$\max_{0\le t\le 1}|u(t)|\le \|u\|, \text{ for each } u\in X.$$

It follows that for each $r>0$

$$\Phi^{-1}(-+\infty,r]\subseteq\{u\in X: |u(t)|<\sqrt{2r}, \text{ and every } t\in[0,1]\}.$$

On the other hand , we have

$$\sup_{u\in\Phi^{-1}(-+\infty,r]}(-\Psi(u))=\sup_{\|u\|^2\le 2r}\int_0^1 g(t,u(t))dt\le \max_{(t,\xi)\in[0,1]\times[-\sqrt{\frac{r}{2}},\sqrt{\frac{r}{2}}]} g(t,\xi).$$

Now, owing to Lemma 4.6, there exists $r>0$ and $u\in X$ such that

$$\max_{(t,\xi)\in[0,1]\times[-\sqrt{\frac{r}{2}},\sqrt{\frac{r}{2}}]} g(t,\xi)<2r\frac{\int_0^1 g(t,u(t))dt}{\|u\|^2}=r\frac{(-\Psi(u))}{\Phi(u)}.$$

Finally, owing to Proposition 2.1, choosing $h(\lambda)=\rho\lambda$, we obtain

$$\sup_{\lambda\ge 0}\inf_{x\in X}(\Phi(x)+\lambda\Psi(x)+h(\lambda))<\inf_{x\in X}\sup_{\lambda\ge 0}(\Phi(x)+\lambda\Psi(x)+h(\lambda)).$$

Hence, by Theorem 2.1 □

Now, our conclusion follows from Theorem 4.3.

Let $a\in C[0,1]$ and $h\in C(R)$ be two nonnegative functions. Put

$$A(t)=\int_0^t a(\tau)d\tau, H(\xi)=\int_0^\xi h(x)dx.$$

We consider the special case of problem (1.1)

$$\begin{cases} u''+\lambda a(t)h(u)=0,\\ u(0)=0, u(1)=\alpha u(\eta).\end{cases}\tag{4.12}$$

**Corollary 4.1.** *Suppose that there exist four positive constants $c,d,\sigma,l$ with $l<2$ and $c<\frac{1}{2}\sqrt{M}d$ such that*

*(i)* $\max_{t\in[0,1]} a(t)\le\frac{4}{M}(\frac{c}{d})^2\frac{H(d)}{H(c)}[A(\frac{1+\eta}{2})-A(\eta)]$,

*(ii)* $H(\xi)\le\sigma(1+|\xi|^l)$ *for each* $\xi\in R$.

*Then there exists an open interval $\Lambda\subseteq(0,++\infty)$ and a positive real number $q$ such that, for each $\lambda\in\Lambda$, problem (4.12) has at three solutions belonging to $X$, whose norms in $W_1^{1,2}((0,1))$ are less than $q$.*

*Proof.* In order to apply Theorem 4.3, we choose

$$f(t,u)=a(t)h(u), \text{ for each } (t,u)\in[0,1]\times R.$$

Then we have

$$\max_{(t,\xi)\in[0,1]\times[-c,c]} g(t,\xi)=\max_{(t,\xi)\in[0,1]\times[-c,c]}\int_0^\xi f(t,x)dx=\max_{t\in[0,1]} a(t)H(c).$$

Taking $\mu=\sigma\max_{t\in[0,1]} a(t)$, it is easy to verify that all the assumptions of Theorem 4.3 are satisfied. So the proof is finished. □

Finally, we give an example as application.

**Example 4.1.** We consider the following problem

$$(4.13)\qquad \begin{cases} u'' + \lambda f(t,u) = 0, \\ u(0) = 0, u(1) = \frac{1}{4}u(\frac{1}{2}), \end{cases}$$

where $\alpha = \frac{1}{4}, \eta = \frac{1}{2}$, $f(t,u) = th(u)$, and let $d = 10$

$$h(u) = \begin{cases} e^u, & u \leq d, \\ u + e^d - d, & u > d. \end{cases}$$

In this case one has $A(t) = \dfrac{t^2}{2}$, and

$$H(\xi) = \begin{cases} e^\xi - 1, & \xi \leq d, \\ \frac{1}{2}\xi^2 + (e^d - d)\xi + \frac{1}{2}d^2 + e^d(1-d) - 1, & \xi > d. \end{cases}$$

It is easy to verify that with $M = \frac{9}{4}$, $c = 1, \sigma = e^d, l = \frac{3}{2}$, all conditions of Corollary 4.1 are satisfied. Therefore, there exist an open interval $\Lambda \subseteq (0, +\infty)$ and a positive real number $q$ such that, for each $\lambda \in \Lambda$, BVP (4.13) has at three solutions belonging to $X$, whose norms in $W_1^{1,2}((0,1))$ are less than $q$.

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